MULTIPLE POSITIVE SOLUTIONS FOR QUASILINEAR ELLIPTIC PROBLEMS WITH SIGN-CHANGING NONLINEARITIES

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Using variational arguments, we prove some nonexistence and multiplicity results for positive solutions of a system of $p$-Laplace equations of gradient form. Then we study a $p$-Laplace-type problem with nonlinear boundary conditions.

1. Introduction

In a recent paper, [7], the authors studied the existence, multiplicity, and nonexistence of positive classical solutions of the semilinear elliptic boundary value problem

$$
-\Delta u = \lambda f(u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
$$

(1.1)

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, $\lambda > 0$ is a parameter, and $f$ is a $C^1$ sign-changing sublinear function.

They showed using sub-super solutions arguments and recent results from semipositive problems that there are $\bar{\lambda}$ and $\lambda_0$ such that (1.1) has no positive solution for $\lambda < \lambda_0$ and at least two positive solutions for $\lambda \geq \bar{\lambda}$.

More recently, in [8], the author extends these results to the quasilinear problem

$$
-\Delta_p u = \lambda f(x,u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
$$

(1.2)

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian, $1 < p < \infty$, $\lambda > 0$, and $f$ is a sign-changing Carathéodory function on $\Omega \times [0, \infty)$.

The method in [8] is variational and allowed the author to substantially relax the assumptions on $f$. More precisely, these assumptions are

$$(H_1) \quad f(x,0) = 0, \quad |f(x,t)| \leq C|t|^{p-1},$$

$$(H_2) \quad \text{there exists } \delta > 0 \text{ such that } F(x,t) \leq 0 \text{ for } 0 \leq t \leq \delta,$$

$$(H_3) \quad \int_0^1 f(x,t)\,dt = 0.$$
(H₃) there exists \( t₀ > 0 \) such that \( F(x, t₀) > 0 \),
(H₄) \( \limsup_{t \to \infty} F(x, t)/t^p \leq 0 \) uniformly in \( x \),
where \( F(x, t) = \int_{t₀}^{t} f(x, s) \, ds \).

The purpose of this article is twofold. Applying variational methods, we first extend
the results in [8] to quasilinear elliptic systems of the form
\[
-\Delta_p u = \lambda F_u(x, u, v) \quad \text{in } \Omega,
-\Delta_q v = \lambda F_v(x, u, v) \quad \text{in } \Omega,
\]
\[
u = v = 0 \quad \text{on } \partial \Omega,
\]
(1.3)
where \((F_u, F_v)\) stands for the gradient of a given potential \( F \), and second, we want to see
to what extent these variational techniques can be adapted to deal with the nonlinear
boundary condition case
\[
-\Delta_p u + |u|^{p-2}u = 0 \quad \text{in } \Omega,
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda g(x, u) \quad \text{on } \partial \Omega,
\]
where \( \partial/\partial \nu \) is the outer unit normal derivative.

Systems of the form (1.3) are usually called gradient systems and have been widely stud-
ied in the past. See, for example, [2] for a comprehensive analysis of such systems. This
gradient structure allows us to treat (1.3) variationally. Other kinds of elliptic systems
that can be treated variationally are the so-called Hamiltonian systems, see [3].

However, as far as we know, all the results for (1.3) assume, to begin with, that
\( F_u, F_v \geq 0 \) for \( u, v \geq 0 \).

For problem (1.4), in a previous paper, [4], the authors studied the problem where
the nonlinearity \( g \) was assumed to be of power type, that is, essentially the case \( g(x, t) = |t|^q-2t \) was considered, so again \( g(x, t) \geq 0 \) for \( t \geq 0 \).

The main results of this paper can be formulated as follows.

Under hypotheses similar to (H₁)–(H₄), there exists \( 0 < \lambda < \bar{\lambda} \) such that if \( 0 < \lambda < \lambda \)
problem (1.3) (or problem (1.4)) has no positive solution and if \( \lambda > \bar{\lambda} \) problem (1.3) (or
problem (1.4)) has, at least, two positive solutions.

The rest of the paper is organized as follows: in Section 2 we deal with problem (1.3)
and in Section 3 with (1.4).

2. Gradient systems
In this section, we deal with problem (1.3). First, we prove the nonexistence result. To this
end, we assume that \( F(x, u, v) \) is a Carathéodory function on \( \Omega \times [0, \infty) \times [0, \infty) \), \( F(x, \cdot, \cdot) \)
is \( C^1 \) for a.e. \( x \in \Omega \), and \( F_u, F_v \) are also Carathéodory functions satisfying
\[
F(x, 0, 0) = F_u(x, 0, 0) = F_v(x, 0, 0) = 0,
\]
\[
|uF_u(x, u, v) + vF_v(x, u, v)| \leq C(u^p + v^q),
\]
\[
|F(x, u, v)| \leq C(u^p + v^q),
\]
for some constant \( C > 0 \).
We have the following theorem.

**Theorem 2.1.** Assume (2.1) holds. Then, there is a $\lambda$ such that (1.3) has no positive solution for $\lambda < \lambda$.

For the proof we need the following observation. We denote by $\lambda_r$ the best constant in the Sobolev embedding $W^{1,r}_0(\Omega) \hookrightarrow L^r(\Omega)$. We have

$$\lambda_p \int_\Omega |u|^p dx \leq \int_\Omega |\nabla u|^p dx \quad \text{for } u \in W^{1,p}_0(\Omega),$$

$$\lambda_q \int_\Omega |v|^q dx \leq \int_\Omega |\nabla v|^q dx \quad \text{for } v \in W^{1,q}_0(\Omega),$$

so if we denote $\lambda_{p,q} = \min\{\lambda_p, \lambda_q\}$, we obtain

$$0 < \lambda_{p,q} \leq \frac{\int_\Omega |\nabla u|^p + |\nabla v|^q dx}{\int_\Omega |u|^p + |v|^q dx} \quad \text{for } u \in W^{1,p}_0(\Omega), v \in W^{1,q}_0(\Omega)$$

and, moreover, one can easily see that $\lambda_{p,q}$ is optimal.

**Proof of Theorem 2.1.** If (1.3) has a positive solution $(u,v)$, multiplying the first equation of (1.3) by $u$, the second by $v$, and integrating by parts and adding up, we get

$$\int_\Omega |\nabla u|^p + |\nabla v|^q dx = \lambda \int_\Omega F_u(x,u,v)u + F_v(x,u,v)v dx.$$  (2.4)

Thus, using (2.1), we obtain

$$\int_\Omega |\nabla u|^p + |\nabla v|^q dx \leq \lambda C \int_\Omega |u|^p + |v|^q dx$$  (2.5)

and hence $\lambda \geq \lambda_{p,q}/C$ by (2.3), proving Theorem 2.1. \qed

Now, we prove the multiplicity result. To this end, along with (2.1), we also have to assume that

- $(F_1)$ there exists $\delta > 0$ such that $F(x,u,v) \leq 0$ for $|u|^p + |v|^q \leq \delta$,
- $(F_2)$ there exists $t_0, s_0 > 0$ such that $F(x,t_0,s_0) > 0$,
- $(F_3)$ $\limsup_{|u,v| \to \infty} F(x,u,v)/(u^p + v^q) \leq 0$ uniformly in $x$.

Under these assumptions, we have the following theorem.

**Theorem 2.2.** Under the assumptions (2.1), (F1), (F2), and (F3), there is a $\lambda$ such that (1.3) has at least two positive solutions $(u_1, v_1), (u_2, v_2)$ for $\lambda \geq \lambda$.

For the proof of Theorem 2.2, we use critical point theory. Set $F(x,u,v) = 0$ for $u, v < 0$, and consider the $C^1$ functional

$$\mathcal{F}_\lambda(u,v) = \int_\Omega \frac{|\nabla u|^p}{p} + \frac{|\nabla v|^q}{q} - \lambda F(x,u,v) dx, \quad (u,v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega).$$  (2.6)
Observe that if \((u, v)\) is a critical point of \(\mathcal{F}_\lambda\), denoting by \(u^-\) and \(v^-\) the negative parts of \(u\) and \(v\), respectively,

\[
0 = (\mathcal{F}'_\lambda(u, v), (u^-, v^-)) = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla u^- + |\nabla v|^{q-2} \nabla v \cdot v^- - \lambda (F_u(x, u, v)u^- + F_v(x, u, v)v^-) dx
\]

(2.7)

hence we have that \(u, v \geq 0\). Furthermore, by [10], \(u, v \in C^{1,\alpha}(\Omega)\) and so, by Harnack inequality (see [11]), it follows that either \(u, v > 0\) or \(u \equiv v \equiv 0\). Therefore, nontrivial critical points of \(\mathcal{F}_\lambda\) are positive solutions of (1.4).

By (F3) and (2.1), there is a constant \(C_\lambda > 0\) such that

\[
\lambda F(x, u, v) \leq \frac{\lambda_{p,q}}{2} \left( \frac{|u|^p}{p} + \frac{|v|^q}{q} \right) + C_\lambda
\]

(2.8)

and hence

\[
\mathcal{F}_\lambda(u) \geq \int_\Omega \frac{|\nabla u|^p}{p} + \frac{|\nabla v|^q}{q} - \frac{\lambda_{p,q}}{2} \left( \frac{|u|^p}{p} + \frac{|v|^q}{q} \right) - C_\lambda dx
\]

\[
\geq \frac{1}{2p} ||u||_{W^{1,p}_0(\Omega)}^p + \frac{1}{2q} ||v||_{W^{1,q}_0(\Omega)}^q - C_\lambda |\Omega|_N,
\]

(2.9)

where \(|\cdot|_d\) denotes the \(d\)-dimensional Lebesgue measure in \(\mathbb{R}^N\), so \(\mathcal{F}_\lambda\) is bounded from below and coercive.

Therefore, as \(\mathcal{F}_\lambda\) is weakly lower semicontinuous, we obtain a global minimizer \((u_1, v_1)\). We show that, if \(\lambda\) is big enough, this minimizer is nontrivial.

**Lemma 2.3.** There is a \(\lambda\) such that \(\inf \mathcal{F}_\lambda < 0\), and hence \((u_1, v_1) \neq (0,0)\), for \(\lambda \geq \lambda\).

**Proof.** We consider a sufficiently large compact subset \(\Omega'\) of \(\Omega\) and take functions \(u_0 \in W^{1,p}_0(\Omega)\), \(v_0 \in W^{1,q}_0(\Omega)\) such that \(u_0(x) = t_0\) on \(\Omega'\), \(0 \leq u_0(x) \leq t_0\) on \(\Omega \setminus \Omega'\), \(v_0(x) = s_0\) on \(\Omega'\), \(0 \leq v_0(x) \leq s_0\) on \(\Omega \setminus \Omega'\), where \(t_0, s_0\) are as in (F2).

Then, we obtain

\[
\int_\Omega F(x, u_0, v_0) dx \geq \int_{\Omega'} F(x, t_0, s_0) dx - C(t_0^p + s_0^q) |\Omega \setminus \Omega'|_N > 0,
\]

(2.10)

if \(\Omega'\) is big enough. Hence, \(\mathcal{F}_\lambda(u_0, v_0) < 0\) for \(\lambda\) large enough. \(\Box\)

We will obtain a critical point \((u_2, v_2)\) with \(\mathcal{F}_\lambda(u_2, v_2) > 0\) via the mountain pass lemma, which would complete the proof since \(\mathcal{F}_\lambda(u_2, v_2) > 0 > \mathcal{F}_\lambda(u_1, v_1)\).
Lemma 2.4. The origin is a strict local minimizer of $\mathcal{F}_\lambda$.

Proof. Let $U_{(u,v)} = \{ x \in \Omega : |u(x)|^p + |v(x)|^q > \delta \}$. By (F1), $F(x,u(x),v(x)) \leq 0$ on $\Omega \setminus U_{(u,v)}$, so

$$\mathcal{F}_\lambda(u,v) \geq \frac{1}{p} \| u \|_{W_{0}^{1,p}(\Omega)}^p + \frac{1}{q} \| v \|_{W_{0}^{1,q}(\Omega)}^q - \lambda \int_{U_{(u,v)}} F(x,u,v) dx.$$  \hspace{1cm} (2.11)

By (2.1), Hölder’s inequality, and Sobolev embedding,

$$\int_{U_{(u,v)}} F(x,u,v) dx \leq C \int_{U_{(u,v)}} u^p + v^q dx \leq C \left( \| U_{(u,v)} \|_{W_{0}^{1,p}(\Omega)}^{1-p/r} \| u \|_{W_{0}^{1,p}(\Omega)} + \| U_{(u,v)} \|_{W_{0}^{1,q}(\Omega)}^{1-q/s} \| v \|_{W_{0}^{1,q}(\Omega)} \right),$$  \hspace{1cm} (2.12)

where $r = \frac{Np}{N-p}$ if $p < N$ and $r > p$ if $p \geq N$, and $s = \frac{Nq}{N-q}$ if $q < N$ and $s > q$ if $q \geq N$. So, in order to finish the proof we need to show that $\| U_{(u,v)} \|_N \rightarrow 0$ as $\| u \|_{W_{0}^{1,p}(\Omega)} + \| v \|_{W_{0}^{1,q}(\Omega)} \rightarrow 0$.

Now,

$$\| u \|_{W_{0}^{1,p}(\Omega)}^p + \| v \|_{W_{0}^{1,q}(\Omega)}^q \geq \lambda_{p,q} \int_{\Omega} u^p + v^q dx \geq \lambda_{p,q} \int_{U_{(u,v)}} u^p + v^q dx \geq \lambda_{p,q} \delta \| U_{(u,v)} \|_N,$$  \hspace{1cm} (2.13)

as we wanted to show. \hfill \Box

Now, we are in position to finish the proof of Theorem 2.2.

Proof of Theorem 2.2. As $\mathcal{F}_\lambda$ is coercive, every Palais-Smale sequence is bounded and hence contains a convergent subsequence as usual. Now, the mountain pass lemma gives a critical point $(u_2,v_2)$ of $\mathcal{F}_\lambda$ at the level

$$c := \inf_{y \in \Gamma} \max_{\gamma \in \Gamma} \mathcal{F}_\lambda(u,v) > 0,$$  \hspace{1cm} (2.14)

where $\Gamma = \{ \gamma \in C([0,1], W_{0}^{1,p}(\Omega) \times W_{0}^{1,q}(\Omega)) : \gamma(0) = 0, \gamma(1) = (u_1,v_1) \}$ is the class of paths joining the origin to $(u_1,v_1)$ (see [9]). \hfill \Box

3. The nonlinear boundary condition case

In this section, we deal with the nonlinear boundary condition case, problem (1.4). The main ideas and structures of the proofs are the same as in the previous section, so we only sketch them and stress the differences between the two cases.

We begin with the nonexistence result. To this end, we assume that $g$ is a Carathéodory function on $\partial \Omega \times [0,\infty)$ satisfying

$$g(x,0) = 0, \hspace{1cm} -ct^{r-1} \leq g(x,t) \leq Ct^{p-1}$$  \hspace{1cm} (3.1)

for some $1 \leq r \leq p$ and some constants $C,c > 0$. 
We have the following theorem.

**Theorem 3.1.** There is a \( \lambda \) such that (1.4) has no positive solution for \( \lambda < \lambda \).

For the proof, we need some knowledge on the following eigenvalue problem:

\[
-\Delta_p u + |u|^{p-2}u = 0 \quad \text{in } \Omega,
\]
\[
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u \quad \text{on } \partial \Omega.
\]

This problem was studied in [4, 6] (see also [5]). It was proved there that problem (3.2) has a first positive eigenvalue \( \lambda_1 \) given by

\[
\lambda_1 = \min_{u \in W^{1,p} \setminus W^{1,p}_0} \frac{\int_{\Omega} |\nabla u|^p + |u|^p \, dx}{\int_{\partial \Omega} |u|^p \, d\sigma},
\]

where \( d\sigma \) is the boundary measure. In the linear case, \( p = 2 \), problem (3.2) is known as the Steklov problem (see [1]).

**Proof of Theorem 3.1.** If (1.4) has a positive solution \( u \), multiplying (1.4) by \( u \), integrating by parts, and using (3.1) gives

\[
\int_{\Omega} |\nabla u|^p + |u|^p \, dx = \lambda \int_{\partial \Omega} f(x,u) u \, d\sigma \leq C \lambda \int_{\partial \Omega} |u|^p \, d\sigma,
\]

and hence \( \lambda \geq \lambda_1/C \) by (3.3), proving Theorem 3.1. \( \square \)

Now we prove the multiplicity result.

The assumptions in this case are as follows: let \( G(x,t) = \int_0^t g(x,s) \, ds \), and assume the following:

(\( G_1 \)) there exists \( \delta > 0 \) such that \( G(x,t) \leq 0 \) for \( 0 \leq t \leq \delta \),

(\( G_2 \)) there exists \( t_0 > 0 \) such that \( G(x,t_0) > 0 \),

(\( G_3 \)) \( \limsup_{t \to -\infty} G(x,t)/t^p \leq 0 \) uniformly in \( x \).

**Theorem 3.2.** Assume (3.1) and (\( G_1 \)), (\( G_2 \)), and (\( G_3 \)) hold. Then, there is a \( \bar{\lambda} \) such that (1.4) has at least two positive solutions \( u_1 > u_2 \) for \( \lambda \geq \bar{\lambda} \).

Observe that for problem (1.4) we can prove that the two solutions are ordered. We believe that this should hold also for (1.3), but the truncation argument used in the proof does not work because it destroys the variational structure of (1.3).

Again, set \( g(x,t) = 0 \) for \( t < 0 \), and consider the \( C^1 \) functional

\[
\mathcal{G}_\lambda(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p + |u|^p \, dx - \lambda \int_{\partial \Omega} G(x,u) \, d\sigma, \quad u \in W^{1,p}(\Omega).
\]

\[
(3.5)
\]
Arguing as before, if \( u \) is a critical point of \( \mathcal{G}_{\lambda} \), denoting by \( u^- \) the negative part of \( u \),
\[
0 = (\mathcal{G}'_{\lambda}(u), u^-) = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla u^- + |u|^{p-2} uu^- \, dx - \lambda \int_{\partial\Omega} g(x,u)u^- \, d\sigma
\]
\[
= \|u^-\|^p_{W^{1,p}(\Omega)},
\]
hence we have that \( u \geq 0 \). Furthermore, by [10], \( u \in C^{1,\alpha} \) and so, by the strong maximum principle and Hopf’s lemma (see [12]), it follows that either \( u > 0 \) in \( \Omega \) or \( u \equiv 0 \).

Therefore, nontrivial critical points of \( \mathcal{F}_{\lambda} \) are positive solutions of (1.4). Observe that in this case, the solution \( u \) is positive up to the boundary.

By (G3) and (3.1), there is a constant \( C_{\lambda} > 0 \) such that
\[
\lambda G(x,t) \leq \frac{\lambda_1}{2p} |t|^p + C_{\lambda}
\]
and hence
\[
\mathcal{G}_{\lambda}(u) \geq \frac{1}{p} \int_{\Omega} |\nabla u|^p + |u|^p \, dx - \int_{\partial\Omega} \frac{\lambda_1}{2p} |u|^p + C_{\lambda} \, d\sigma
\]
\[
\geq \frac{1}{2p} \|u\|^p_{W^{1,p}(\Omega)} - C_{\lambda} |\partial\Omega|_{N-1},
\]
so \( \mathcal{G}_{\lambda} \) is bounded from below and coercive.

Therefore, as \( \mathcal{G}_{\lambda} \) is weakly lower semicontinuous, we obtain a global minimizer \( u_1 \). Once again, if \( \lambda \) is big enough, this minimizer is nontrivial.

**Lemma 3.3.** There is a \( \lambda \) such that \( \inf \mathcal{G}_{\lambda} < 0 \), and hence \( u_1 \neq 0 \), for \( \lambda \geq \lambda \).

**Proof.** Take the constant function \( u_0 \equiv t_0 \), where \( t_0 \) is as in (G2).

Then, we obtain
\[
\int_{\partial\Omega} G(x,u_0) \, d\sigma = \int_{\partial\Omega} G(x,t_0) \, d\sigma > 0.
\]
Hence, \( \mathcal{G}_{\lambda}(u_0) < 0 \) for \( \lambda \) large enough. \( \square \)

The main difference in the arguments arrives at this point. As we mentioned before, by a truncation argument we can prove that the two solutions are ordered. In fact, fix \( \lambda \geq \lambda \).

Let
\[
\tilde{g}(x,t) = \begin{cases} g(x,t), & t \leq u_1(x), \\ g(x,u_1(x)), & t > u_1(x), \end{cases}
\]
\[
\tilde{G}(x,t) = \int_0^t \tilde{g}(x,s) \, ds.
\]

Then consider
\[
\tilde{\mathcal{G}}_{\lambda}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p + |u|^p \, dx - \lambda \int_{\partial\Omega} \tilde{G}(x,u) \, d\sigma.
\]
If \( u \) is a critical point of \( \tilde{G}_\lambda \), then \( u \geq 0 \) as before. Now,

\[
0 = (\tilde{G}_\lambda'(u) - \tilde{G}_\lambda'(u_1), (u - u_1)^+) = \int_{\Omega} [(|\nabla u|^{p-2}\nabla u - |\nabla u_1|^{p-2}\nabla u_1) \cdot \nabla (u - u_1) + (|u|^{p-2}u - |u_1|^{p-2}u_1)(u - u_1)^+] \, dx
\]

\[
- \lambda \int_{\partial\Omega} (\tilde{g}(x,u) - g(x,u_1))(u - u_1)^+ \, d\sigma
\]

\[
\geq \int_{\{u > u_1\}} [(|\nabla u|^{p-1} - |\nabla u_1|^{p-1})(|\nabla u| - |\nabla u_1|) + (|u|^{p-1} - |u_1|^{p-1})(|u| - |u_1|)] \, dx \geq 0,
\]

so \( u \leq u_1 \). Therefore, \( u \) is a solution of (1.4).

Now, as in the previous case, we will obtain the second solution as a critical point of \( \tilde{G}_\lambda, u_2 \), with \( \tilde{G}_\lambda(u_2) > 0 \) via the mountain pass lemma, which would complete the proof since \( \tilde{G}_\lambda(0) = 0 > \tilde{G}_\lambda(u_1) \).

**Lemma 3.4.** The origin is a strict local minimizer of \( \tilde{G}_\lambda \).

**Proof.** Let \( \Gamma_u = \{ x \in \partial\Omega : u(x) > \min\{u_1(x), \delta\} \} \). By (3.10) and (G1), \( \tilde{G}(x,u(x)) \leq 0 \) on \( \partial\Omega \setminus \Gamma_u \), so

\[
\tilde{G}_\lambda(u) \geq \frac{1}{p} \|u\|_{W^{1,p}(\Omega)}^p - \lambda \int_{\Gamma_u} \tilde{g}(x,u) \, d\sigma. \tag{3.13}
\]

By (3.1), Hölder’s inequality, and Sobolev trace theorem,

\[
\int_{\Gamma_u} \tilde{g}(x,u) \, d\sigma \leq C \int_{\Gamma_u} u^p \, d\sigma \leq C \|u\|_{W^{1,p}(\Omega)} \|u\|_{W^{1,p}(\Omega)}, \tag{3.14}
\]

where \( q = (N-1)p/(N-p) \) if \( p < N \) and \( q > p \) if \( p \geq N \), so in order to finish the proof we need to show that \( \|\Gamma_u\|_{N-1} \to 0 \) as \( \|u\|_{W^{1,p}(\Omega)} \to 0 \).

Let \( k = \min\{\min_{\partial\Omega} u_1; \delta\} \), where \( \delta \) is given in (G1). Then,

\[
\|u\|_{W^{1,p}(\Omega)}^p \geq C \int_{\partial\Omega} u^p \, d\sigma \geq C \int_{\Gamma_u} u^p \, d\sigma \geq Ck^p \|\Gamma_u\|_{N-1}, \tag{3.15}
\]

as we wanted to show.

Now, we are in position to finish the proof of **Theorem 3.2**.
Proof of Theorem 3.2. The same argument used for $G_\lambda$ shows that $\tilde{G}_\lambda$ is also coercive, so every Palais-Smale sequence of $\tilde{G}_\lambda$ is bounded and hence contains a convergent subsequence as usual. Now, the mountain pass lemma gives a critical point $u_2$ of $\tilde{G}_\lambda$ at the level

$$c := \inf_{\gamma \in \Gamma} \max_{u \in [0,1]} \tilde{G}_\lambda(u) > 0,$$

where $\Gamma = \{ \gamma \in C([0,1], W^{1,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = u_1 \}$ is the class of paths joining the origin to $u_1$. \hfill \Box

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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

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