CRITICAL SINGULAR PROBLEMS
ON UNBOUNDED DOMAINS

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We present some results of existence for the following problem: 
\[-\Delta u = a(x)g(u) + u|u|^{2^* - 2}, \]
\[x \in \mathbb{R}^N \quad (N \geq 3), \quad u \in D^{1,2} (\mathbb{R}^N), \]
where the function \(a\) is a sign-changing function with a singularity at the origin and \(g\) has growth up to the Sobolev critical exponent \(2^* = 2N/(N - 2)\).

1. Introduction

Recently many works have been devoted to the study of existence of positive solutions \(u\) of the equation
\[-\Delta u = a(|x|)g(u), \quad x \in \mathbb{R}^N \quad (N \geq 3), \quad u \in H^1 (\mathbb{R}^N),\] (1.1)
with a continuous function \(a\) and a subcritical growth function \(g\). This type of equation includes the Makutuma equation, when \(a(|x|) = 1/(1 + |x|^2)\) and \(g(s) = |s|^{p-1}s\), with \(1 < p < 2^* - 1 = (N + 2)/(N - 2)\), which appears in astrophysics and scalar curvature equations on \(\mathbb{R}^N\) (see, e.g., [15, 17, 18]).

In [16] Munyamarere and Willem obtained a result of multiplicity of nodal solutions for these equations, considering the function \(a\) nonnegative and radially symmetric. The authors worked with a subspace of radial functions of \(H^1 (\mathbb{R}^N)\) which has the compactness properties desired to handle a problem like this modelled on an unbounded domain. In the same direction, Alama and Tarantello [2] studied the following problem when \(a\) is not radially symmetric and changes sign (see also [1, 7, 21, 22]):
\[-\Delta u - \lambda u = a(x)g(u), \quad x \in \Omega \subseteq \mathbb{R}^N \quad (N \geq 3), \quad u \in H^1 (\Omega),\] (1.2)
where \(\Omega\) is a bounded domain and \(g\) behaves at infinity like a power function, \(g(s) \simeq |s|^{p-1}s\), with \(1 < p < (N + 2)/(N - 2)\) (subcritical case).

The above results on a bounded domain were extended, in part, by Costa and Tehrani in [12] for the whole space \(\mathbb{R}^N\). They considered a weighted eigenvalue problem, namely,
\[-\Delta u = \lambda h(x)u, \quad x \in \mathbb{R}^N, \quad u \in H^1 (\mathbb{R}^N) \quad (N \geq 3),\] (1.3)
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with \( 0 \leq h \in L^{N/2}(\mathbb{R}^N) \cap L^\alpha(\mathbb{R}^N), \alpha > N/2 \), which has the same properties as the eigenvalue problem for \(-\Delta\) in a bounded domain (see, e.g., [11]). With the aid of this information, they studied the problem

\[-\Delta u - \lambda h(x)u = a(x)g(u), \quad x \in \mathbb{R}^N (N \geq 3), u \in H^1(\mathbb{R}^N).\]  

(1.4)

Recently, still in the subcritical case, Tehrani in [23], considering problem (1.4) with \( h = 0 \) and \( \Omega \) an unbounded exterior domain \( \Omega = \mathbb{R}^N \setminus \overline{O} \) with \( \overline{O} \neq \emptyset \), obtained similar results to those papers above.

Our main purpose in this work is to study the problem

\[-\Delta u = a(x)g(u) + u|u|^{2^+-2}, \quad x \in \mathbb{R}^N (N \geq 3), u \in D^{1,2}(\mathbb{R}^N),\]  

(1.5)

where the Hilbert space \( D^{1,2}(\mathbb{R}^N) \) is defined as the completion of \( C^\infty_0(\mathbb{R}^N) \) endowed with the norm \( \|u\| = (\int_{\mathbb{R}^N} |\nabla u|^2 \, dx)^{1/2} \).

The above kind of problem is important since it is related to conformal deformations of Riemannian structures on noncompact manifolds (see, e.g., [14]). Also, it is a physical model that appears when one describes the dynamics of galaxies (see, e.g., [4]).

It is relevant to remark that our concern to study this type of problem with a function \( a \) changing sign comes from the following fact: if \( u \in D^{1,2}(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N) \) is a positive solution of (1.5), using a generalized Pohazev identity (see [8, Proposition 1]), we have

\[\int_{\mathbb{R}^N} (a(x)g(u)u + u^{2^*}) \, dx = 2^* \int_{\mathbb{R}^N} \left( a(x)G(u) + \frac{1}{2^*} u^{2^*} \right) \, dx,\]  

(1.6)

where \( G(t) = \int_t^1 g(s) \, ds \). Thus, for instance, if \( g(s) = |s|^{p-2}s, 2 < p < 2^* \), then \( a \) must change sign.

We would like to mention that when \( a \in L^{2^*/(2^* - 2)}(\mathbb{R}^N) \), Benci and Cerami in [6] studied the case \( a \leq 0 \) on \( \mathbb{R}^N \), while in [19] Pan treated the case \( a > 0 \), and a case when \( a \) changes sign was handled by Ben-Naoum et al. in [5].

Our contribution to the study of these problems rely on the fact that we are working with a sign-changing discontinuous function \( a \) and with nonlinearities defined on the whole space \( \mathbb{R}^N \) involving critical Sobolev exponent growth. These conditions imply a series of restrictions on the usual methods of dealing with these problems since the compactness of the Sobolev embedding is lost. In our case, a Hardy-type inequality is demanded. We would like to point out that our approach, with the corresponding changes, also works replacing \( \mathbb{R}^N \) by a bounded or unbounded domain \( \Omega \). Finally we note that our work is precisely a version of the classical result of Brézis and Nirenberg (see [10]) considered under the aforementioned conditions.
Before stating our main theorem, we have to precise the set of assumptions on the functions \( g \) and \( a \):

(i) \( g : \mathbb{R} \to \mathbb{R} \) is a continuous function satisfying

\[
g(s) = o(|s|) \quad \text{as } s \to 0,
\]

\[
\lim_{|s| \to +\infty} \frac{sg(s)}{|s|^p} = 1, \quad \text{for some } 2 < p < 2^*,
\]

\[
g(s) > 0, \quad \forall s > 0,
\]

\[
\lim_{\varepsilon \to 0} \int_{\varepsilon - 1/2}^{1/2} G\left(\frac{\varepsilon}{1+s^2}\right)^{(N-2)/2} s^{N-1} ds = \infty.
\]

(ii) \( a : \mathbb{R}^N \to \mathbb{R} \) is a sign-changing function such that

\[
a(x) = O(|x|^{-\alpha}), \quad \text{as } |x| \to 0, \text{ for some } 0 < \alpha \leq \frac{2N - p(N - 2)}{2},
\]

\[
a(x) = O(|x|^{-2}), \quad \text{as } |x| \to +\infty,
\]

\( a \) is a continuous function in \( 1 \leq |x| \leq M \) for some \( M > 0 \),

\( a \in L(\mathbb{R}^N - B_\rho(0)) \) for some \( \rho > 0 \),

\((B_r(a) \text{ denotes a ball with radius } r \text{ centered at } a)\)

\[
a(x) < 0, \quad \text{for } |x| \geq R_0,
\]

\[
a(x) > 0, \quad \text{for } |x| \leq R_0 - \delta,
\]

where \( R_0 > M \) and \( \delta > 0 \) is small.

We also require that \( \Omega^0 = \{ x \in \mathbb{R}^N; a(x) = 0 \} \) have “thick” zero measure, that is,

\[
\overline{\Omega^+} \cap \overline{\Omega^-} = \emptyset,
\]

where \( \Omega^+ = \{ x \in \mathbb{R}^N; a(x) > 0 \} \), and \( \Omega^- = \{ x \in \mathbb{R}^N; a(x) < 0 \} \).

Our main theorem is the following.

**Theorem 1.1.** Suppose that (1.7)–(1.17) hold. Then problem (1.5) has a positive solution.

**Remark 1.2.** In addition to the hypotheses of Theorem 1.1, assuming that \( g \) is odd, problem (1.5) has infinitely many solutions. This follows by applying the classical genus theory, more exactly, a critical point theorem for even functional due to Rabinowitz (see [20]).
2. Variational framework

We are going to employ the variational methods to find a nontrivial weak solution for problem (1.5). To start, we define the Euler-Lagrange functional associated to it.

Let \( \Psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R} \) be defined by

\[
\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} a(x)G(u) \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} \, dx, \tag{2.1}
\]

where \( G(s) = \int_0^s g(t) \, dt \).

In order to guarantee that \( \Psi \) is well defined, we need the following Hardy-type inequality (see [13]).

**Proposition 2.1.** For \( N \geq 2 \), there exists a constant \( C = C(N) \) such that

\[
\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} \, dx \leq C \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \tag{2.2}
\]

for all \( u \in D^{1,2}(\mathbb{R}^N) \).

We check that the functional \( \Psi \) is well defined. Hereafter, we denote by \( C \) a generic positive constant. By (1.7) and (1.8), we have that

\[
\int_{\mathbb{R}^N} a(x)G(u) \, dx \leq C \left( \int_{\mathbb{R}^N} |a(x)||u|^2 \, dx + \int_{\mathbb{R}^N} |a(x)||u|^p \, dx \right) \equiv I_1 + I_2. \tag{2.3}
\]

We check that \( I_1 \) is finite. Since \((2N - p(N - 2))/2 < 2\), we have by (1.11) and (2.2) that

\[
\int_{|x| \leq 1} |a(x)||u|^2 \, dx \leq C \int_{|x| \leq 1} \frac{|u|^2}{|x|^a} \, dx \leq C \int_{|x| \leq 1} \frac{|u|^2}{|x|^2} \, dx \leq C. \tag{2.4}
\]

By (1.12) and (2.2),

\[
\int_{|x| \geq M} |a(x)||u|^2 \, dx \leq C \int_{|x| \geq M} (|a(x)||x|^2) \left( \frac{|u|^2}{|x|^2} \right) \, dx \leq C \int_{|x| \geq M} \frac{|u|^2}{|x|^2} \, dx \leq C. \tag{2.5}
\]

Hence, by (2.4), (2.5), and (1.13), we have \( I_1 < \infty \).

Choosing \( r = 2N/(2N - p(N - 2)) \), by Hölder’s inequality and, respectively, by (1.11) and (1.12), we have

\[
\int_{|x| \leq 1} a(x)|u|^p \, dx \leq \left( \int_{|x| \leq 1} \frac{1}{|x|^ar} \, dx \right)^{1/r} \left( \int_{|x| \leq 1} |u|^{2^*} \, dx \right)^{p/2^*} \leq C, \tag{2.6}
\]

\[
\int_{|x| \geq M} a(x)|u|^p \, dx \leq \left( \int_{|x| \geq M} \frac{dx}{|x|^{2r}} \right)^{1/r} \left( \int_{|x| \geq M} |u|^{2^*} \, dx \right)^{p/2^*} \leq C. \tag{2.7}
\]
By (2.6), (2.7), and (1.13), we achieve that $I_2 < \infty$.

Therefore, $\Psi$ is well defined and under the assumptions on the nonlinearities, a straightforward computation yields that $\Psi \in C^1(D^{1,2}(\mathbb{R}^N))$ and that for $v \in D^{1,2}(\mathbb{R}^N)$, we have

$$\langle \Psi'(u), v \rangle = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx - \int_{\mathbb{R}^N} a(x) g(u) v \, dx - \int_{\mathbb{R}^N} vu|u|^{2^* - 2} \, dx. \quad (2.8)$$

Hence, the critical points of $\Psi$ are precisely the weak solutions for (1.5) and vice versa.

We also point out that with convenient hypotheses on the nonlinearities it is possible to obtain some regularization of the solutions.

3. Obtaining critical points $\Psi$

We are going to find a solution as a critical point of the functional $\Psi$. Before proceeding, we assure that the solution that we will find is indeed positive. Taking

$$\tilde{g}(u) = \begin{cases} g(u), & \text{if } u \geq 0, \\ g(-u), & \text{if } u < 0, \end{cases} \quad (3.1)$$

and using from now on the function $\tilde{g}(u)$, the critical point of $\Psi$ is such that $u \geq 0$.

Now applying the maximum principle to the equation

$$-\Delta u - a^-(x)\tilde{g}(u) = a^+(x)\tilde{g}(u) + u^{2^*-1}, \quad x \in \mathbb{R}^N, u \in D^{1,2}(\mathbb{R}^N), \quad (3.2)$$

we infer that $u$ must be positive ($a^+ = \max\{a,0\}$ and $a^- = a - a^+$).

For simplicity, in what follows, the function $\tilde{g}$ will be denoted by $g$.

Returning to the functional $\Psi$, let $E = D^{1,2}(\mathbb{R}^N)$ and we firstly check that under our hypotheses, $\Psi$ has the mountain pass geometry, that is,

$$\exists \beta > 0, \rho > 0 \text{ s.t. } \Psi(u) \geq \rho \text{ if } ||u|| = \beta, \quad (3.3)$$

$$\Psi(0) = 0, \forall e \in E, \quad ||e|| > \beta \text{ s.t. } \Psi(e) \leq 0. \quad (3.4)$$

Proposition 3.1. If (1.7), (1.8), (1.11), (1.12), and (1.13) hold, then (3.3) and (3.4) also hold.

Proof of (3.3). By (1.7) and (1.8), for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, p) > 0$ such that

$$|G(s)| \leq \varepsilon |s|^2 + C|s|^p, \quad s \in \mathbb{R}. \quad (3.5)$$

Hence, by estimates (2.4), (2.5), (2.6), and (2.7), together with the last inequality, we have

$$\Psi(u) \geq \frac{||u||^2}{2} - C\left\{ \varepsilon \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx + \left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{p/2^*} \right\}. \quad (3.6)$$
By (2.2) and the Sobolev embedding, for \( \|u\| \) sufficiently small, we achieve that
\[
\Psi(u) \geq \frac{\|u\|^2}{2} - C(\varepsilon \|u\|^2 + \|u\|^p) \geq \tilde{C}\|u\|^2,
\] (3.7)
for some constant \( \tilde{C} > 0 \) and \( \varepsilon > 0 \) small enough. Therefore, (3.3) holds. \( \square \)

**Proof of (3.4).** Hypothesis (1.8) implies that
\[
0 < \theta G(s) \leq sg(s), \quad |s| \geq s_0, \text{ for some } s_0 > 0, \quad 2 < \theta < 2^*,
\] (3.8)
which, on its turn, implies that there exists \( A > 0 \) such that
\[
|G(s)| \geq A|s|^\theta \quad \text{for } |s| \geq s_0.
\] (3.9)
Therefore, if \( 0 \leq \xi \in C_0^\infty(\Omega^+) \), by (3.9), we have that
\[
\Psi(t\xi) = \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \xi|^2 \, dx - \int_{\mathbb{R}^N} a(x)G(t\xi) \, dx - \frac{t^{2^*}}{2} \int_{\mathbb{R}^N} |\xi|^{2^*} \, dx \rightarrow -\infty,
\] (3.10)
as \( t \rightarrow \infty \). \( \square \)

Since (3.3) and (3.4) hold, by the mountain pass theorem without the Palais-Smale condition ((PS) condition, for short) (see [3]), if
\[
\Gamma = \left\{ y \in C([0,1],E); y(0) = 0, y(1) = e \right\},
\] (3.11)
\[
c := \inf_{P \in \Gamma} \max_{w \in P} \Psi(w) \geq \rho,
\] (3.12)
then there exists a sequence \( (u_n) \subset E \) such that
\[
\Psi(u_n) \rightarrow c \quad \text{in } E, \quad \text{as } n \rightarrow \infty,
\] (3.13)
\[
\Psi'(u_n) \rightarrow 0 \quad \text{in } E', \quad \text{as } n \rightarrow \infty,
\] (3.14)
where \( \Psi' \) is the Frechet derivative of \( \Psi \) and \( E' \) is the dual space of \( E \).

We define
\[
S = \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{(\int_{\mathbb{R}^N} |u|^{2^*} \, dx)^{2/2^*}}.
\] (3.15)

In the following result, we are going to prove that there exists \( w \in E \) such that the constant \( c \) in (3.12) may be chosen is such a way that \( c < (1/N)S^{N/2} \).

**Proposition 3.2.** Suppose that (1.7)–(1.15) hold. Then there exists \( u_0 \in E \setminus \{0\} \) such that
\[
\sup_{t \geq 0} \Psi(tu_0) < \frac{1}{N} S^{N/2}
\] (3.16)
Proof. Some ideas that follow in this proof were borrowed from [10]. We present them for completeness of the work.

We have that $a(x) > 0$ in $B_{R_0 - \delta}(0)$. We choose a cutoff function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\text{supp } \varphi \subset B_{2R}(x_0) \subset (B_{R_0 - \delta}(0) - \{0\})$, $\varphi \equiv 1$ on $B_R(x_0)$ and $0 \leq \varphi \leq 1$ on $B_{2R}(x_0)$, for some convenient open ball $B_{2R}(x_0)$. For $\varepsilon > 0$, if

$$U_\varepsilon(x) = \frac{\left[ N(N - 2)\varepsilon^2 \right]^{(N-2)/4}}{\varepsilon + |x - x_0|^2}^{(N-2)/2}, \quad (3.17)$$

it is well known that

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \, dx = \int_{\mathbb{R}^N} |U_\varepsilon|^2^* \, dx = S^{N/2}, \quad (3.18)$$

$$\int_{B_{2R}(x_0)} |\nabla U_\varepsilon|^2 \, dx \leq \int_{B_{2R}(x_0)} |U_\varepsilon|^2^* \, dx. \quad (3.19)$$

If we define $\eta_\varepsilon = \varphi U_\varepsilon$, it is easy to prove that

$$\int_{\mathbb{R}^N - B_{2R}(x_0)} |\nabla \eta_\varepsilon|^2 \, dx = O(\varepsilon^{(N-2)/2}), \quad \text{as } \varepsilon \to 0. \quad (3.20)$$

To rewrite $\Psi$ in a convenient way, let

$$v_\varepsilon = \frac{\eta_\varepsilon}{\left( \int_{B_{2R}(x_0)} |\eta_\varepsilon|^{2^*} \, dx \right)^{1/2^*}}, \quad \chi_\varepsilon = \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 \, dx. \quad (3.21)$$

With this notation, it is forward to check that $\Psi$ is bounded from above and that $\lim_{t \to \infty} \Psi(t v_\varepsilon) = -\infty$, for all $\varepsilon > 0$. So there exists $t_\varepsilon \geq 0$ such that

$$\sup_{t \geq 0} \Psi(t v_\varepsilon) = \Psi(t_\varepsilon v_\varepsilon). \quad (3.22)$$

Then differentiating $\Psi(t v_\varepsilon)$, we achieve that

$$t_\varepsilon \chi_\varepsilon - t_\varepsilon^{2^*} - \int_{B_{2R}(x_0)} a(x) g(t_\varepsilon v_\varepsilon) \, dx = 0 \quad (3.23)$$

and hence that

$$t_\varepsilon \leq \chi_\varepsilon^{1/(2^* - 1)}. \quad (3.24)$$

Also note that by (3.18), (3.19), (3.20), and (3.24), it follows that

$$\chi_\varepsilon \leq S + O(\varepsilon^{(N-2)/2}). \quad (3.25)$$

On the other hand, the function $t \to t^2 \chi_\varepsilon^{2^* - 2}/2 - t^2/2^*$ is increasing on the interval $[0, t_0]$, where $t_0 = \chi_\varepsilon^{1/(2^* - 2)}$. Then assertion (3.22) together with the above inequalities implies that

$$\Psi(t_\varepsilon u_\varepsilon) \leq \frac{1}{N} S^{N/2} + O(\varepsilon^{(N-2)/2}) - \int_{B_{2R}(x_0)} a(x) G(t_\varepsilon v_\varepsilon) \, dx. \quad (3.26)$$
By (1.7) and (1.8), for all $\tau > 0$ sufficiently small, there exists $C > 0$ satisfying $|a(x)g(u)| \leq C|u| + \tau |u|^{2^*-1}$, for all $x \in B_{2R}(x_0)$.

Thus
\[
\left| \int_{B_{2R}(x_0)} \frac{a(x)g(t_\varepsilon v_\varepsilon)}{t_\varepsilon} \, dx \right| \leq \tau t_\varepsilon^{2^*-1} |v_\varepsilon|^{2^*} + C |v_\varepsilon|^2,
\] (3.27)
for $\tau$ sufficiently small.

Recalling that
\[
|v_\varepsilon|_2^2 = \begin{cases} O(\varepsilon) & \text{if } N \geq 5, \\ O(\varepsilon \log \varepsilon) & \text{if } N = 4, \\ O(\varepsilon^{1/2}) & \text{if } N = 3, \end{cases}
\] (3.28)
from (3.27), we obtain
\[
\int_{B_{2R}(x_0)} \frac{a(x)g(t_\varepsilon v_\varepsilon)}{t_\varepsilon} \, dx \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.
\] (3.29)

Using this fact in (3.23), we conclude that
\[
t_\varepsilon \rightarrow S^{1/(2^*-2)}, \quad \text{as } \varepsilon \rightarrow 0.
\] (3.30)

Now, using (3.18)–(3.20) and (3.30), we have
\[
\int_{\mathbb{R}^N} a(x)G(t_\varepsilon v_\varepsilon) \, dx \geq C \int_{B_{2R}(x_0)} G \left( \frac{c\varepsilon^{(N-2)/4}}{\varepsilon + |x-x_0|^2} \right) \left( \frac{\varepsilon^{(N-2)/2}}{(N-2)/2} \right) \, dx,
\] (3.31)
for some positive constants $c$ and $C$. Substituting (3.31) in (3.26), we get
\[
\Psi(t_\varepsilon u_\varepsilon) \leq \frac{1}{N} S^{N/2} + O(\varepsilon^{(N-2)/2}) - C \int_{B_{2R}(x_0)} G \left( \frac{c\varepsilon^{(N-2)/4}}{\varepsilon + |x-x_0|^2} \right) \left( \frac{\varepsilon^{(N-2)/2}}{(N-2)/2} \right) \, dx.
\] (3.32)

But
\[
J_\varepsilon \equiv \frac{1}{\varepsilon^{(N-2)/2}} \int_{B_{2R}(x_0)} G \left( \frac{c\varepsilon^{(N-2)/4}}{\varepsilon + |x-x_0|^2} \right) \left( \frac{\varepsilon^{(N-2)/2}}{(N-2)/2} \right) \, dx \rightarrow \infty, \quad \text{as } \varepsilon \rightarrow 0.
\] (3.33)

In fact, since
\[
J_\varepsilon = \frac{\omega_N}{\varepsilon^{(N-2)/2}} \int_0^R G \left( \frac{c\varepsilon^{(N-2)/4}}{\varepsilon + r^2} \right) r^{N-1} \, dr,
\] (3.34)
\( \omega_N \) is the area of \( S^{N-1} \) making the change of variables \( r = \varepsilon^{1/2} s \) and rescaling \( \varepsilon \), we get

\[
J_\varepsilon = \varepsilon \omega_N \int_0^{R\varepsilon^{-1/2}} G \left( \left[ \frac{\varepsilon^{-1/2}}{1 + s^2} \right]^{(N-2)/2} \right) s^{N-1} ds
\]

(3.35)

for some constant \( R > 0 \).

Then, if \( R \geq 1 \), using (3.35), assertion (3.33) follows directly from hypothesis (1.10). If \( R < 1 \), consider

\[
Z_\varepsilon = \varepsilon \int_{\varepsilon^{-1/2}}^{R\varepsilon^{-1/2}} G \left( \left[ \frac{\varepsilon^{-1/2}}{1 + s^2} \right]^{(N-2)/2} \right) s^{N-1} ds.
\]

(3.36)

Hence, there is \( c > 0 \) such that

\[
|Z_\varepsilon| \leq c\varepsilon G(\varepsilon^{(N-2)/4}) \varepsilon^{-N/2}
\]

(3.37)

which implies, due to the growth of \( g \), that \( |Z_\varepsilon| \) is bounded as \( \varepsilon \to 0 \). Consequently, in the case \( R < 1 \), since

\[
\int_0^{R\varepsilon^{-1/2}} = \int_{\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} - \int_{R\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}
\]

(3.38)

and the last integral is bounded, as \( \varepsilon \to 0 \), it follows that (3.33) is a consequence of (3.37) and, again, of hypothesis (1.10).

Finally, applying (3.33) in (3.32), we see that

\[
\Psi(t_\varepsilon u_\varepsilon) \leq \frac{1}{N} s^{N/2}
\]

(3.39)

for small \( \varepsilon > 0 \), as desired. \( \square \)

Next we prove the following.

**Proposition 3.3.** If \( (u_n) \subset D^{1,2}(\mathbb{R}^N) \) is a sequence such that (3.13) and (3.14) hold, then there exists a subsequence \( u_n \rightharpoonup u_0 \) weakly in \( D^{1,2}(\mathbb{R}^N) \), as \( n \to \infty \), for some \( u_0 \in D^{1,2}(\mathbb{R}^N) \).

**Proof.** The proof finishes if we prove that \( (u_n) \) is bounded. Suppose, on the contrary, that \( (u_n) \) is not bounded in \( D^{1,2}(\mathbb{R}^N) \). We may assume that

\[
\|u_n\| \equiv t_n \longrightarrow +\infty, \quad \text{as } n \longrightarrow \infty.
\]

(3.40)

Define \( v_n = u_n/t_n \). By (3.13) and (3.14), we achieve that

\[
\frac{1}{2} \int_{\mathbb{R}^N} \left| \nabla v_n \right|^2 dx - \int_{\mathbb{R}^N} aG(u_n) t_n dx - \frac{1}{2} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} dx = o_n(1)
\]

(3.41)

and for all \( v \in D^{1,2}(\mathbb{R}^N) \), we get that

\[
\int_{\mathbb{R}^N} \nabla v_n \nabla v dx - \int_{\mathbb{R}^N} aG(u_n) \frac{v}{t_n} dx - \int_{\mathbb{R}^N} \frac{|u_n|^{2^* - 2} u_n v}{t_n} dx = o_n(1) \|v\|_t.
\]

(3.42)
Since \( \|v_n\| = 1 \), by (3.41), we have
\[
\int_{\mathbb{R}^N} \frac{aG(u_n)}{t_n^2} \, dx + \frac{1}{2^*} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} \, dx = \frac{1}{2} + o_n(1),
\]
and taking \( v = v_n \) in (3.42), we infer that
\[
\int_{\mathbb{R}^N} \frac{ag(u_n)u_n}{t_n^2} \, dx + \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} \, dx = 1 + o_n(1).
\]
Observe that, combining (3.43) and (3.44) together with (1.9), we may assume that
\[
\text{supp } a \cap \text{supp } g(u_n) \neq \emptyset, \quad \text{supp } a \cap \text{supp } G(u_n) \neq \emptyset.
\]
From (3.43) and (3.44), we get that
\[
\left( \frac{2}{2^*} - 1 \right) \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} \, dx = \int_{\mathbb{R}^N} \frac{ag(u_n)u_n - 2G(u_n)}{t_n^2} \, dx + o_n(1).
\]
Observe that, by (1.7) and (1.8), we have
\[
|g(s)| \leq C_1 |s| + C_1 |s|^q, \quad s \in \mathbb{R}, \quad 1 < q < 2^* - 1,
\]
and hence, for a given \( \varepsilon \), there exists a \( K > 0 \) such that
\[
|g(t) - 2G(t)| < \varepsilon |t|^{2^*} \quad \text{for } |t| \geq K.
\]
The last integral in (3.46) may be split as
\[
\int_{|u_n| \leq K} \frac{a^+(g(u_n)u_n - 2G(u_n))}{t_n^2} \, dx + \int_{|u_n| \geq K} \frac{a^+(g(u_n)u_n - 2G(u_n))}{t_n^2} \, dx
\]
\[\quad - \int_{|u_n| \geq K} \frac{a^-(g(u_n)u_n - 2G(u_n))}{t_n^2} \, dx.
\]
We bound these integrals. Since (1.14) holds, the first integral is \( o_n(1) \); the second, taking \( K > s_0 \) in (3.9), is nonnegative, and the last one, by (3.48), is bounded as follows:
\[
\int_{|u_n| \geq K} \frac{a^-(g(u_n)u_n - 2G(u_n))}{t_n^2} \, dx \leq \varepsilon \|a^-\|_{\infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2}.
\]
Using these facts in (3.46), we have
\[
\left(\left(\frac{2}{2^*} - 1\right) + \varepsilon \|a^{-}\|_{\infty}\right) \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} \, dx \geq o_n(1). \tag{3.51}
\]

Thereafter, picking a small \(\varepsilon\), we conclude that
\[
\int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} \, dx \longrightarrow 0. \tag{3.52}
\]

We use this limit to contradict the fact that \(\|u_n\| \to \infty\).

We also may consider that there exists \(v \in D^{1,2}(\mathbb{R}^N)\) such that
\[
v_n \longrightarrow v \quad \text{a.e. in } \mathbb{R}^N, \tag{3.53}
\]
and for all bounded sets \(U \subset \mathbb{R}^N\) and for \(1 \leq t < 2^*\),
\[
v_n \longrightarrow v \quad \text{in } L^t(U),
\]
\[
v_n(x) \longrightarrow v(x), \quad \text{for } x \in U \text{ a.e.,}
\]
\[|v_n(x)| \leq h(x) \quad \text{for } h \in L^t(U), \text{ and a.e. in } U,
\]
as \(n \to \infty\).

In the sequel, we need the following claim which will be proved at the end of this proof.

**Claim.** \(v \equiv 0\).

Proceeding, we take \(\xi \in C_0^\infty(\mathbb{R}^N)\). Inserting \(v = v_n \xi\) in (3.42) and using the claim, we get
\[
\int_{\mathbb{R}^N} |\nabla v_n|^{2^*} \xi \, dx - \int_{\mathbb{R}^N} \frac{a(G(u_n))u_n}{t_n^2} \xi \, dx - \int_{\mathbb{R}^N} |u_n|^{2^* - 2} v_n^2 \xi \, dx = o_n(1). \tag{3.55}
\]

We choose the cutoff function \(\xi \in C_0^\infty(\mathbb{R}^N)\) such that \(\xi \equiv 1\) on \(\Omega^+\) and \(\xi \equiv 0\) on \(\Omega^-\).

Using (3.8) and (3.55), together with (3.45), we obtain
\[
\int_{\mathbb{R}^N} \frac{aG(u_n)}{t_n^2} \xi \, dx = \int_{|u_n| \leq s_0} \frac{aG(u_n)}{t_n^2} \xi \, dx + \int_{|u_n| > s_0} \frac{a(G(u_n))}{t_n^2} \xi \, dx
\]
\[\leq o_n(1) + \frac{1}{\theta} \int_{|u_n| > s_0} \frac{ag(u_n)u_n \xi}{t_n^2} \, dx
\]
\[= o_n(1) + \frac{1}{\theta} \left[ \int_{\mathbb{R}^N} |\nabla v_n|^{2^*} \xi \, dx - \int_{\mathbb{R}^N} |u_n|^{2^* - 2} v_n^2 \xi \, dx \right]
\]
\[\leq o_n(1) + \frac{1}{\theta} - \frac{1}{\theta} \int_{\mathbb{R}^N} \frac{u_n^{2^*} \xi}{t_n^2} \, dx. \tag{3.56}
\]
The above inequality together with (3.8) and (3.42) yields

$$\left(\frac{1}{2} - \frac{1}{\theta}\right) \leq \frac{1}{2^*} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} \, dx + o_n(1),$$

(3.57)

which contradicts (3.52).

**Proof of the claim.** We are going to prove that \(v(x) = 0\) a.e. for \(x \in \Omega^+\), arguing by contradiction. Let \(F = \{x \in \mathbb{R}^N; v(x) \neq 0\}\) and we suppose that there exists \(B_r(x_0)\) such that

$$|F \cap B_{2r}(x_0)| > 0,$$

(3.58)

where \(|\cdot|\) denotes the Lebesgue measure defined in \(\mathbb{R}^N\).

Pick \(\xi \in C_0^\infty(B_{2r}(x_0))\) such that \(\xi(x) = 1\) if \(x \in B_r(x_0)\). Replacing \(v = v_n \xi\) in (3.42), we have

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 \xi \, dx + \int_{\mathbb{R}^N} v_n \nabla v_n \nabla \xi \, dx - \int_{\mathbb{R}^N} |u_n|^{2^* - 2} v_n^2 \xi \, dx$$

$$= t_n^{p - 2} \left[ \int_{\mathbb{R}^N} a \left| \frac{|v_n|}{|t_n v_n|^p} \right|^p t_n v_n \xi \, dx \right] + o_n(1).$$

(3.59)

But, since \(|t_n v_n(x)| \to \infty\), for \(x \in F\), as \(n \to \infty\), using (1.8), the growth conditions of \(g\), (3.54), and the Lebesgue dominated convergence theorem, we get

$$\int_{\mathbb{R}^N} a|v_n|^p g(t_n v_n) \xi \, dx \to 0, \text{ as } n \to \infty,$$

(4.1)

$$\int_{\mathbb{R}^N} aG(u_n) \, dx \to 0, \text{ as } n \to \infty.$$
By a Brézis-Lieb result (see [9]), we have that
\[
\int_{\mathbb{R}^N} |u_n|^{2^* - 2} u_n v \, dx \longrightarrow \int_{\mathbb{R}^N} |u_0|^{2^* - 2} u_0 v \, dx, \quad \text{as } n \longrightarrow \infty.
\] (4.4)

Hence, by (4.1), passing to the limit in (3.14), we achieve that
\[
\langle \Psi'(u_0), v \rangle = 0, \quad \forall v \in E,
\] (4.5)

that is, \( u_0 \) is a weak solution for (1.5).

To see that \( u_0 \neq 0 \), suppose on the contrary, that \( u_0 \equiv 0 \). If
\[
\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \longrightarrow l, \quad \text{as } n \longrightarrow \infty,
\] (4.6)

then by (4.2) and (3.14) with \( v = u_n \), we get
\[
\int_{\mathbb{R}^N} |u_n|^{2^*} \, dx \longrightarrow l, \quad \text{as } n \longrightarrow \infty.
\] (4.7)

Using (4.3), (4.6), (4.7) and passing to the limit in (3.13) yields
\[
l = Nc > 0
\] (4.8)

with the choice that
\[
c < \frac{1}{N} S^{N/2},
\] (4.9)

since (3.16) holds.

Passing to the limit in definition (3.15) with \( u_n \), and regarding (4.6) and (4.7), we get that \( l \geq S^{N/2} \). But this inequality contradicts (4.9).

**Proof of (4.1).** Using (1.7) and (1.8), we see that for a given \( \varepsilon > 0 \),
\[
|g(s)| \leq \varepsilon |s| + C |s|^{p-1}, \quad \forall s \in \mathbb{R},
\] (4.10)

for some \( C > 0 \). Hence, combining (4.10), (1.11), and a similar reasoning for \( u_n \) such as that made in (3.54), there exists \( h \in L^t(U) \), \( U \subset \mathbb{R}^N \), \( 1 \leq t < 2^* \), such that
\[
|ag(u)| \leq C \left( \frac{|h|}{|x|^a} + \frac{|h|^{p-1}}{|x|^a} \right) \in L^1(B_R(0)), \quad \text{for some } R, C > 0.
\] (4.11)

Thus, applying the Lebesgue dominated convergence theorem yields
\[
\int_{|x| \leq R} ag(u_n) v \, dx \longrightarrow \int_{|x| \leq R} ag(u) v \, dx, \quad \text{as } n \longrightarrow \infty.
\] (4.12)
The proof finishes if we prove that
\[
\lim_{R \to \infty} \int_{|x| > R} |ag(u_n)v| \, dx = 0, \quad \text{uniformly in } n.
\] (4.13)

By (1.12), (2.2), (4.10), and Hölder’s inequality, we have
\[
\int_{|x| > R} |ag(u_n)v| \, dx \leq \epsilon \int_{|x| > R} |a||u_n||v| \, dx + \int_{|x| > R} |a||u_n|^{p-1}|v| \, dx
\leq \epsilon C\|u_n\|\|v\| + C\left(\int_{|x| > R} |a| \, dx\right)^{1/r} \|u_n\|^{p-1}\|v\|^{2^*},
\] (4.14)
where \( r = 2^*/(2^* - p) \). Since the sequence \((u_n)\) is bounded in \(E\) norm, if \(R > 0\) is chosen in the above inequality, such that
\[
\left(\int_{|x| > R} |a| \, dx\right)^{1/r} < \epsilon,
\] (4.15)
we assure that (4.13) holds. \(\square\)

**Proof of (4.2) and (4.3).** The proof is made using similar reasoning as those made in the previous proof. \(\square\)

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