SINGULAR NONLINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^N$

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Abstract. This paper deals with existence, uniqueness and regularity of positive generalized solutions of singular nonlinear equations of the form $-\Delta u + a(x)u = h(x)u^{-\gamma}$ in $\mathbb{R}^N$ where $a, h$ are given, not necessarily continuous functions, and $\gamma$ is a positive number. We explore both situations where $a, h$ are radial functions, with $a$ being eventually identically zero, and cases where no symmetry is required from either $a$ or $h$. Schauder’s fixed point theorem, combined with penalty arguments, is exploited.

1. Introduction

This paper addresses existence, uniqueness and regularity questions on generalized solutions of the singular nonlinear elliptic problem

(*) \[
\begin{cases}
-\Delta u + a(x)u = h(x)u^{-\gamma} \quad \text{in } \mathbb{R}^N \\
u > 0 \quad \text{in } \mathbb{R}^N
\end{cases}
\]

where $a, h$ are nonnegative $L^\infty_{\text{loc}}$ functions, $h \not\equiv 0$, (eventually we consider $a \equiv 0$), $\gamma > 0$ and $N \geq 3$. We point out that the search of positive solutions of the Dirichlet problem for the equation

$-\Delta u + a(x)u = h(x)u^{-\gamma} \quad \text{in } \Omega$

where $\Omega$ is a smooth bounded domain has deserved the attention of many authors. Nowosad [1] studied a related Hammerstein equation, namely

$u(x) = \int_0^1 K(x, y)(u(x))^{-\gamma}dy,$

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where $\gamma = 1$, $\int_0^1 K(x,y)dy \geq \delta > 0$ and $K(x,y)$ is positive semidefinite. Nowosad’s work was extended by Karlin and Nirenberg [2] where more general Hammerstein equations were considered including the case $\gamma > 0$ in the equation above. Crandall-Rabinowitz and Tartar [3] studied the Dirichlet problem

$$Lu = f(x,u) \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega$$

where $L$ is a linear second order elliptic operator and $f: \Omega \times (0,+,\infty) \to \mathbb{R}$ is singular in the sense that $f(x,r) \to \infty$ as $r \to 0^+$. Examples such as $f(x,r) = r^{\gamma-1}$ with $\gamma > 1$, $\gamma < 1$ or $\gamma = 1$ were covered.

There is by now an extensive literature on singular elliptic problems. With respect to the case of bounded domains $\Omega \subset \mathbb{R}^N$ we would like to further mention Gomes [4], Lazer and McKenna [5], Cac and Hernandez [8], Chen [9], Lair and Shaker [10], Shangbin [13] while for the case $\Omega = \mathbb{R}^N$ we recall Kuzano and and Swanson [11], Lair and Shaker [12,14]. This reference list is far from complete. In the earlier papers concerning $\Omega = \mathbb{R}^N$, $h(x)$ is assumed at least continuous and several techniques are developed such as the method of lower and upper solutions. In this paper we assume $h(x)$ only integrable and use the Schauder fixed point theorem and elliptic estimates. Singular equations appear in the theory of heat conduction in electrically conducting materials, (Fulks and Maybee [6]), in binary communications by signals (Nowosad [1]) and in the theory of pseudoplastic fluids (Nachman and Callegari [7]).

The following condition on $a$ will be required in the first one of our main results stated below:

$$(a)_R \quad a(x) \geq a_0 \text{ for } |x| \geq R \text{ for some } a_0, R > 0.$$ 

In what follows we take $\gamma, \alpha \in (0,1)$ and $h \in L^\theta \cap L^2$ where $\theta \equiv \frac{2}{2-(1-\gamma)}$.

**Theorem 1.** Assume $(a)_R$. Then $(\ast)$ has a unique solution $u \in D^{1,2} \cap W^{2,p}_{loc}$ where $1 < p < \infty$ with $\int a(x)u^2 < \infty$. If $a, h$ are radial functions the solution is radial, as well, and in fact, $u(x) \to 0$ as $|x| \to \infty$. Moreover if $a, h \in C^{\alpha}_{loc}$ then $u \in C^{2,\alpha}_{loc}$.

In our second result we take $a \equiv 0$ and $h$ radially symmetric that is, we study the problem

$$\ast_0 \quad \begin{cases} -\Delta u = h(|x|)u^{-\gamma} \text{ in } \mathbb{R}^N \\ u > 0 \text{ in } \mathbb{R}^N. \end{cases}$$

This problem shall be treated by first perturbing the equation by a radially symmetric term, then using the earlier result in the case $a, h$ are radial functions and finally taking limits.

**Theorem 2.** Let $a \equiv 0$ and let $h$ be radially symmetric. Then $(\ast)_0$ has a unique radially symmetric solution $u \in D^{1,2} \cap W^{2,p}_{loc}$, $1 < p < \infty$ and $u(x) \to 0$ as $|x| \to \infty$. Moreover, if $h \in C^{\alpha}_{loc}$ then $u \in C^{2,\alpha}_{loc}$. 
2. Preliminaries

The main goal in this section is to prove theorem 1. For that purpose let $\epsilon > 0$ and consider the problem

\[
\begin{cases}
-\Delta u + a(x)u = \frac{h(x)}{(u+\epsilon)^\gamma} & \text{in } \mathbb{R}^N \\
u > 0 & \text{in } \mathbb{R}^N.
\end{cases}
\]

We are going to show by applying the Schauder fixed point theorem that (2.1) has a solution $u \in W^{2,p}_{\text{loc}},$ $1 < p < \infty,$ and then by passing to the limit as $\epsilon \to 0$ we arrive at a solution of (*).

In order to deal with a first step namely, existence of a solution of (2.1), let $f \in L^2$ and consider the linear equation

\[
-\Delta u + a(x)u = f(x) \text{ in } \mathbb{R}^N.
\]

Recalling that the Hilbert space $\mathcal{D}^{1,2}$ is defined as the closure of $C_0^\infty$ with respect to the gradient norm $\|\varphi\|_2^2 = \int |\nabla \varphi|^2$ we introduce the space

\[
E \equiv \left\{ u \in \mathcal{D}^{1,2} \mid \int au^2 < \infty \right\}
\]

which endowed with the inner product and norm given respectively by

\[
\langle u, v \rangle = \int (\nabla u \cdot \nabla v + auv) \quad \text{and} \quad \|u\|^2 = \langle u, u \rangle
\]

is itself a Hilbert space. Under condition $(a)_R$ it follows that $u \in E$ iff $u \in W^{1,2}(\mathbb{R}^N)$.

Yet if $f \in L^2$ it follows by minimizing over $E$ the energy functional associated with (2.2),

\[
I(u) = \frac{1}{2}\|u\|^2 - \int f u
\]

that (2.2) has a weak solution $u \in E$, that is,

\[
\int (\nabla u \nabla \varphi + au\varphi) = \int f(x)\varphi, \quad \varphi \in E.
\]

The solution $u$ is, in fact, unique. Letting $S : L^2 \to E$ be the solution operator associated to (2.2) that is $Sf = u$ for $f \in L^2$ it follows that $S$ is linear and moreover

\[
\|Sf\| \leq C|f|_2, \quad f \in L^2
\]

for some $C > 0$. In addition, splitting $u$ into $u^+ - u^-$ where $u^\pm$ are respectively the positive and negative parts of $u$, taking $\varphi = -u^-$ above and noticing that $u^- \in E$ we infer that

\[
Sf \geq 0 \text{ whenever } f \geq 0.
\]

Now let $u \in L^2$ with $u \geq 0$. Since

\[
0 \leq \frac{h(x)}{(u+\epsilon)^\gamma} \leq \frac{h(x)}{\epsilon^\gamma}
\]

(2.3)
and \( \frac{h(x)}{\epsilon} \in L^2 \) the operator

\[
Tu \equiv S \left[ \frac{h(x)}{(u+\epsilon)^\gamma} \right]
\]
is continuous in \( L^2 \), and as a matter of fact, letting \( w \equiv T(0) \) we have

\[
w = S \left[ \frac{h(x)}{\epsilon} \right].
\]

Considering

\[
K \equiv \{ v \in L^2 \mid 0 \leq v \leq w \text{ a.e. in } \mathbb{R}^N \}
\]

we shall prove that the following result holds true.

**Lemma 3.** The set \( K \subset L^2 \) is closed, convex and bounded and moreover \( T(K) \subset K \) and \( T(K) \) is a compact subset of \( L^2 \).

Using lemma 3 and the Schauder fixed point theorem there is some \( u_\epsilon \in K \) satisfying

\[
u_\epsilon = S \left[ \frac{h(x)}{(u_\epsilon+\epsilon)^\gamma} \right]
\]

that is

\[
\begin{cases}
  f(\nabla u_\epsilon \nabla \varphi + a u_\epsilon \varphi) = f(\frac{h(x)\varphi}{(u_\epsilon+\epsilon)^\gamma}), & \varphi \in E \\
u_\epsilon \geq 0 \text{ a.e. in } \mathbb{R}^N, & u_\epsilon \in E.
\end{cases}
\]

Now since by (2.3)

\[
\frac{h(x)}{(u_\epsilon+\epsilon)^\gamma} \in L^\infty_{\text{loc}}
\]

it follows by the elliptic regularity theory that \( u_\epsilon \in W^{2,p}_{\text{loc}}, \ 1 < p < \infty \), and further if \( B \subset \mathbb{R}^N \) is a ball, then

\[-\Delta u_\epsilon + a(x)u_\epsilon = \frac{h(x)}{(u_\epsilon+\epsilon)^\gamma} \text{ a.e. in } B.
\]

In fact, it follows by the maximum principle that \( u_\epsilon > 0 \) in \( B \) and so

\[
\begin{cases}
  -\Delta u_\epsilon + a(x)u_\epsilon = \frac{h(x)}{(u_\epsilon+\epsilon)^\gamma} \text{ a.e. in } \mathbb{R}^N \\
u_\epsilon > 0 \text{ in } \mathbb{R}^N.
\end{cases}
\]

On the other hand, if \( f \in L^2_{\text{rad}} \) we get by minimizing the functional \( I \) above over the space

\[
E_{\text{rad}} \equiv \left\{ u \in W^{1,2}_{\text{rad}} \mid \int a(r)u^2 < \infty \right\}
\]

which endowed with the inner product and norm given above is also a Hilbert space, a weak solution \( u \in E_{\text{rad}} \) of (2.2) that is

\[
\int (\nabla u \nabla \varphi + au\varphi) = \int f(x)\varphi, \ \varphi \in E_{\text{rad}}.
\]

The solution is also unique and as before the solution operator associated to (2.2), namely \( S : L^2_{\text{rad}} \rightarrow E_{\text{rad}} \) satisfies

\[
\|Sf\| \leq C\|f\|_2
\]
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for \( f \in L^2_{rad} \) and further
\[ Sf \geq 0 \] whenever \( f \geq 0 \).

Letting
\[ K \equiv \{ v \in L^2_{rad} \mid 0 \leq v \leq w \text{ a.e. in } \mathbb{R}^N \} \]
we have a corresponding symmetric variant of lemma 3 and so there is some \( u_\epsilon \in E_{rad} \) with
\[ \int (\nabla u_\epsilon \nabla \varphi + a(r)u_\epsilon \varphi) = \int \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} \varphi, \quad \varphi \in E_{rad}. \]

Proof of Lemma 3.

It is easy to show that \( K \) is convex, closed and bounded. So we will only show that \( T(K) \subset K \) and \( \overline{T(K)} \) is compact in \( L^2 \). If \( v \in K \) then
\[ T(0) - T(v) = S \left[ \frac{1}{\epsilon^\gamma} - \frac{1}{(v+\epsilon)^\gamma} \right] \geq 0 \]
that is \( T(v) \leq w \) and hence \( T(K) \subset K \).

In order to show that \( \overline{T(K)} \subset L^2 \) is compact let \( v_n \) be a sequence in \( T(K) \) say \( v_n = T(u_n) \) for some \( u_n \in K \). By (2.3)
\[ \frac{h(x)}{(u_n + \epsilon)^\gamma} \text{ is bounded in } L^2 \]
so that
\[ T(u_n) = S \left[ \frac{h(x)}{(u_n + \epsilon)^\gamma} \right] \text{ is bounded in } E. \]
Thus, passing to subsequences,
\[ T(u_n) \to v \text{ for some } v \in E \]
and
\[ T(u_n) \to v \text{ a.e. in } \mathbb{R}^N. \]
On the other hand, since \( 0 \leq T(u_n) \leq w \) it follows by Lebesgue’s theorem that
\[ T(u_n) \to v \text{ in } L^2. \]
showing that \( \overline{T(K)} \) is compact in \( L^2 \), ending the proof of lemma 3. The radial case is handled similarly. 

The next result states that the family \( u_\epsilon \) increases when \( \epsilon \) decreases.

Lemma 4. If \( 0 < \epsilon < \epsilon' \) then \( u_{\epsilon'} \leq u_\epsilon \) in \( \mathbb{R}^N \).

Proof of Lemma 4.

Letting \( \omega \equiv u_{\epsilon'} - u_\epsilon \) we get
\[ -\Delta \omega + a(x)\omega = h(x) \left[ \frac{1}{(u_{\epsilon'} + \epsilon')^\gamma} - \frac{1}{(u_\epsilon + \epsilon)^\gamma} \right] \text{ a.e. in } \mathbb{R}^N. \]
which gives
\[ \int |\nabla \omega^+|^2 + a(x) \omega^+ = \int h(x) \left[ \frac{1}{(u_\epsilon^+ + \epsilon)v} - \frac{1}{(u_\epsilon^+ + \epsilon)} \right] \omega^+ \leq 0 \]
showing that \( \omega^+ = 0 \) and thus \( u_\epsilon^+ \leq u_\epsilon \) a.e. in \( \mathbb{R}^N \), finishing the proof of lemma 4.

3. Proofs of Main Results

**Proof of Theorem 1.**

**Step 1** (the non-symmetric case).

Let \( \epsilon_n > 0 \) be a decreasing sequence converging to 0 and set \( u_n = u_{\epsilon_n} \). We claim that
\[
\|u_n\| \text{ is bounded.}
\]
Indeed,
\[
(3.1) \int (|\nabla u_n|^2 + a|u_n|^2) = \int \frac{h(x)u_n}{(u_n + \epsilon_n)^\gamma} \leq \int h(x)u_n^{1-\gamma} \leq C|\theta||u_n|^{1-\gamma}
\]
for some \( C > 0 \), showing that \( u_n \) is bounded in \( E \). Hence, passing to subsequences, we have
\[
u_n \rightharpoonup u \text{ in } E, \text{ and } u_n \to u \text{ a.e. in } \mathbb{R}^N.
\]
Moreover since by lemma 4 \( 0 < u_1 \leq u_n \) in \( \mathbb{R}^N \) we infer that if \( \varphi \in E \) has compact support then \( \text{supp}(\varphi) \subset B \) for some ball \( B \subset \mathbb{R}^N \) and
\[
\frac{|h(x)\varphi|}{(u_n + \epsilon_n)^\gamma} \leq H(x) \text{ for some } H \in L^1
\]
which gives, by applying Lebesgue’s theorem to
\[
\int (\nabla u_n \nabla \varphi + au_n \varphi) = \int \frac{h(x)\varphi}{(u_n + \epsilon_n)^\gamma}
\]
that
\[
\left\{ \begin{array}{l}
\int (\nabla u \nabla \varphi + au \varphi) = \int \frac{h(x)\varphi}{u^{\gamma}} \\
u \geq u_1 > 0 \text{ in } \mathbb{R}^N.
\end{array} \right.
\]
Using the regularity theory again we arrive at
\[
\left\{ \begin{array}{l}
-\Delta u + a(x)u = h(x)u^{-\gamma} \text{ a.e. in } \mathbb{R}^N \\
u \in W^{2,p}_{\text{loc}}, \ 1 < p < \infty \\
u > 0 \text{ in } \mathbb{R}^N.
\end{array} \right.
\]
In order to prove uniqueness let \( M \in C_0^\infty \) such that
\[
M(x) = 1 \text{ if } |x| \leq 1, \ M(x) = 0 \text{ if } |x| \geq 2 \text{ and } 0 \leq M \leq 1.
\]
Given \( \varphi \in E \), an integer \( j \geq 1 \) and letting
\[
\varphi^j(x) \equiv M(\frac{x}{j})\varphi(x), \ x \in \mathbb{R}^N
\]

it follows that \( \varphi^j \in E \) and \( \text{supp}(\varphi^j) \) is compact. Moreover as we will show in the Appendix
\[ \tag{3.2} \varphi^j \to \varphi \text{ in } E. \]
Now assume \( u, v \) are two solutions of (*) and let \( w_j \equiv u^j - v^j \). Then
\[
\langle u - v, u^j - v^j \rangle = \int (\nabla (u - v) \nabla w_j + a(x)(u - v)w_j) = \int h(x) \left( \frac{1}{u^j} - \frac{1}{v^j} \right) w_j.
\]
Assuming, by contradiction, that \( u \neq v \) and once
\[
\langle u - v, u^j - v^j \rangle \to \|u - v\|^2
\]
we have
\[
\int h(x) \left( \frac{1}{u^j} - \frac{1}{v^j} \right) w_j > 0
\]
for large values of \( j \). On the other hand,
\[
\int h(x) \left( \frac{1}{u^j} - \frac{1}{v^j} \right) w_j \leq \int_{\Omega_j} h(x) u^{1-\gamma} + \int_{\Omega_j} h(x) v^{1-\gamma}
\]
where \( \Omega_j \equiv B_{2j} \setminus B_j \). Therefore, passing to the limit as \( j \to \infty \) and noticing that the two integrals in the right hand side tend to zero we get a contradiction, that is \( u = v \).

Assume now, \( h \in C^{\alpha}_{loc} \). Then by the elliptic regularity theory more precisely, interior elliptic estimates, we get \( u \in C^{2,\alpha}_{loc} \). This proves theorem 1 (in the case of Step 1).

**Step 2** (the symmetric case: \( a, h \) are radial).

From section 2 we have found by Schauder’s Theorem some radial function \( u_\epsilon \in K \), \( u_\epsilon \neq 0 \) satisfying \( u_\epsilon = T u_\epsilon \), which means
\[ \tag{3.3} \int (\nabla u_\epsilon \nabla v + a(r) u_\epsilon v) = \int \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} v, \ v \in E_{\text{rad}}. \]
We will show next that \( u_\epsilon \in W^{2,p}_{loc}(\mathbb{R}^N \setminus \{0\}) \) for \( 1 < p < \infty \), and
\[
-\Delta u_\epsilon + a(r) u_\epsilon = \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} \text{ a.e. in } \mathbb{R}^N \setminus \{0\}.
\]
Indeed, changing variables we get from (3.3)
\[
\int_{S} \int_{0}^{\infty} (u'_\epsilon v' + a(r) u_\epsilon v) r^{N-1} drdS = \int_{S} \int_{0}^{\infty} \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} vr^{N-1} drdS
\]
where \( S \subset \mathbb{R}^N \) is the unit sphere. Making
\[
v \equiv r^{-(N-1)} \psi, \ r > 0, \ \psi \in C^\infty_0(0, \infty)
\]
we have
\[
\int_0^\infty \left( \left(r^{(N-1)}u'_\epsilon\right) \left(r^{-(N-1)}\psi\right)' + au_\epsilon \psi \right) dr = \int_0^\infty \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} \psi(r) dr,
\]
for \(\psi \in C_0^\infty(0, \infty)\), and labelling
\[
\frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} - a(r)u_\epsilon \equiv \hat{H}(r), \quad r > 0
\]
we get
\[
-\frac{1}{r^{N-1}} (r^{(N-1)}u'_\epsilon)' = \hat{H}(r) \text{ in } (0, \infty)
\]
in the distribution sense. But since \(a, h, u_\epsilon \in L^p_{\text{loc}}(0, \infty)\), \(1 < p < \infty\) it follows that \(\hat{H} \in L^p_{\text{loc}}(0, \infty)\) and using the regularity theory we infer that \(u_\epsilon \in W^{2,p}_{\text{loc}}(0, \infty)\) and
\[
-\frac{1}{r^{N-1}} (r^{(N-1)}u'_\epsilon)' = \hat{H}(r) \text{ a.e. in } (0, \infty).
\]
By the maximum principle,
\[
u_\epsilon > 0 \text{ in } (0, \infty).
\]
Since \(u_\epsilon \in W^{2,p}_{\text{loc}}(\mathbb{R}^N \setminus \{0\})\) and
\[
-\Delta u_\epsilon = \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} \text{ a.e. in } \mathbb{R}^N \setminus \{0\},
\]
we also have
\[
-\Delta u_\epsilon + a(r)u_\epsilon = \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} \text{ a.e. in } \mathbb{R}^N \setminus \{0\}.
\]
Now, let \(\epsilon_n > 0\) such that \(\epsilon_n \to 0\) and label \(u_{\epsilon_n} \equiv u_n\). Following the proof of lemma 4 we have \(u_n \geq u_1 > 0\). On the other hand we claim that
\[
\|u_n\| \text{ is bounded.}
\]
Indeed, as in (3.1) we have
\[
\int \left( |\nabla u_n|^2 + a |u_n|^2 \right) \leq C \|h\| \|u_n\|^{1-\gamma}
\]
so that \(u_n\) is bounded in \(E_{\text{rad}}\). Passing to subsequences we have
\[
u_n \to u \text{ in } E_{\text{rad}}, \text{ and } u_n \to u \text{ a.e. in } \mathbb{R}^N.
\]
On the other hand, if \(v \in E_{\text{rad}}\) has compact support then, as in section 1, applying Lebesgue’s Theorem to
\[
\int (\nabla u_n \nabla v + a(r)u_nv) = \int \frac{h(r)}{(u_n + \epsilon_n)^\gamma} v,
\]
gives
\[
\int (\nabla u \nabla v + a(r)uv) = \int \frac{h(r)}{u^\gamma} v.
\]
Now changing variables, making again \( v \equiv r^{-(N-1)} \psi \) where \( r > 0 \) and \( \psi \in C_0^\infty(0, \infty) \) and arguing as above we obtain \( u \in W_{loc}^{2,p}(\mathbb{R}^N \setminus \{0\}) \) and

\[
-\frac{1}{r^{N-1}}(r^{(N-1)} u')' + a(r)u = \frac{h(r)}{u^\gamma} \quad \text{a.e. in } (0, \infty)
\]

and in addition,

\[
-\Delta u + a(r)u = \frac{h(r)}{u^\gamma} \quad \text{a.e. in } \mathbb{R}^N \setminus \{0\}.
\]

So, if \( \varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}) \) then

\[
\int (\nabla u \nabla \varphi + a(r)u \varphi) = \int \frac{h(r)}{u^\gamma} \varphi
\]

that is

\[
-\Delta u + a(r)u = \frac{h(r)}{u^\gamma} \quad \text{in } \mathbb{R}^N \setminus \{0\}
\]

in the distribution sense. Next we show that \( u \in W_{loc}^{2,p}(\mathbb{R}^N) \) and

\[
\int (\nabla u \nabla \varphi + a(r)u \varphi) = \int \frac{h(r)}{u^\gamma} \varphi, \quad \varphi \in C_0^\infty(\mathbb{R}^N).
\]

Indeed, let \( \eta \in C^\infty(\mathbb{R}^N) \) such that

\[
\eta(x) = 0 \text{ for } |x| \leq 1, \quad \text{and } \eta(x) = 1 \text{ for } |x| \geq 2
\]

and let

\[
\psi_\varepsilon(x) \equiv \eta\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0.
\]

If \( \varphi \in C_0^\infty(\mathbb{R}^N) \) then \( \psi_\varepsilon \varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}) \) and from above

\[
\int (\nabla u \nabla (\psi_\varepsilon \varphi) + a(r)u(\psi_\varepsilon \varphi)) = \int \frac{h(r)}{u^\gamma} (\psi_\varepsilon \varphi)
\]

so that

\[
\int (\psi_\varepsilon \nabla u \nabla \varphi + \varphi \nabla u \nabla \psi_\varepsilon + a(r)u \psi_\varepsilon \varphi) = \int \frac{h(r)}{u^\gamma} \psi_\varepsilon \varphi.
\]

Making \( \varepsilon \to 0 \) and using Lebesgue’s Theorem we infer that

\[
\int \psi_\varepsilon \nabla u \nabla \varphi \to \int \nabla u \nabla \varphi,
\]

\[
\int a(r)u \psi_\varepsilon \varphi \to \int a(r)u \varphi
\]

and

\[
\int \frac{h(r)}{u^\gamma} \psi_\varepsilon \varphi \to \int \frac{h(r)}{u^\gamma} \varphi.
\]

**Claim.** \( \int \varphi \nabla u \nabla \psi_\varepsilon \to 0 \).

Assuming the Claim has been proved we have

\[
\int (\nabla u \nabla \varphi + a(r)u \varphi) = \int \frac{h(r)}{u^\gamma} \varphi
\]
and since \( a, h \in L^\infty_{\text{loc}} \) we get by the regularity theory that \( u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N) \) for 
\[
1 < p < \infty
\]
and if in addition \( a, h \in C^\alpha_{\text{loc}} \) then \( u \in C^{2,\alpha}_{\text{loc}} \) by the interior Schauder estimates.

**Verification of the Claim.**
Using Schwarz inequality we have
\[
|\int \varphi \nabla u \nabla \psi| \leq |\varphi|_{\infty} \left( \int_{|x| \leq 2\epsilon} |\nabla u|^2 \right)^{\frac{1}{2}} \left( \int_{|x| \leq 2\epsilon} |\nabla \psi|^2 \right)^{\frac{1}{2}} 
\leq |\varphi|_{\infty} |\nabla \eta|_2 \left( \int_{|x| \leq 2\epsilon} |\nabla u|^2 \right)^{\frac{1}{2}} \epsilon^{\frac{N-2}{2}}
\]
where \( N \geq 3 \). Letting \( \epsilon \to 0 \) shows the Claim.

As for the uniqueness the argument in the proof of theorem 1 (Step 1) applies ending the proof of theorem 1 (in case of Step 2). The proof of theorem 1 is finished.

**Proof of Theorem 2.**
In order to solve \((*)_0\) we consider the family of problems
\[
\begin{cases}
-\Delta u + \frac{1}{k} u = h(|x|) u^{-\gamma} & \text{in } \mathbb{R}^N \\
u > 0 & \text{in } \mathbb{R}^N.
\end{cases}
\tag{3.4}
\]
where \( k \geq 1 \) is an integer. Making \( a(x) \equiv \frac{1}{k} \) in theorem 1 (radial case), it follows that (3.4) has a solution \( u_k \in H^{1}_{rad} \cap W^{2,p}_{\text{loc}}, 1 < p < \infty \) satisfying
\[
\int |\nabla u_k|^2 + \frac{1}{k} u_k^2 = \int h(r) u_k^{-\gamma}.
\]
Using both Hölder’s inequality and the continuous embedding \( D^{1,2} \to L^{2^*} \) in the equality above we infer that
\[
\int |\nabla u_k|^2 \leq C_1 \text{ for some } C_1 > 0.
\tag{3.5}
\]
By a well known property of radial functions \( u \in D^{1,2} \), namely
\[
|u(x)| \leq \frac{C_2}{|x|^\frac{N-2}{2}} \| u \|_{D^{1,2}}, \ x \neq 0 \text{ for some } C_2 > 0
\]
we get
\[
0 \leq u_k(x) \leq \frac{C}{|x|^\frac{N-2}{2}}, \ x \neq 0 \text{ for some } C > 0.
\tag{3.6}
\]
We shall need the following result which says that the sequence \( u_k \) increases with \( k \).

**Lemma 5.** If \( k < k' \) then \( u_k \leq u_{k'} \), a.e. in \( \mathbb{R}^N \).
By the boundedness of $u_k$ in $D^{1,2}$ and lemma 5 there is some radial function $u \in D^{1,2}$ such that

$$u_k \to u \text{ in } D^{1,2}, \quad u_k \to u \text{ a.e. in } \mathbb{R}^N$$

and

$$u_1 \leq u_2 \leq \ldots \leq u_k \leq \ldots \leq u \text{ a.e. in } \mathbb{R}^N.$$ 

Now if $\varphi \in C_0^\infty(\mathbb{R}^N)$ then

(3.7) $$\int (\nabla u_k \nabla \varphi + \frac{1}{k} u_k \varphi) = \int h u_k^{-\gamma} \varphi.$$ 

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that $\text{supp}(\varphi) \subset \Omega$. Then

$$|hu_k^{-\gamma} \varphi| \leq hu_1^{-\gamma} |\varphi| \in L^p(\Omega), \quad 1 \leq p < \infty$$

and

$$\int hu_k^{-\gamma} \varphi \to \int hu^{-\gamma} \varphi.$$ 

On the other hand, using (3.6) we get

$$\frac{1}{k} \int u_k \varphi \to 0.$$ 

Passing to the limit in (3.7) gives

$$\int \nabla u \nabla \varphi = \int hu^{-\gamma} \varphi.$$ 

Since $0 < u_1 \leq u$ and $u_1 \in W^{2,p}_{loc}(\mathbb{R}^N)$ it follows that $hu^{-\gamma} \in L^p_{loc}(\mathbb{R}^N)$ and by the regularity theory $u \in W^{2,p}_{loc}(\mathbb{R}^N)$. In addition $u \in C^{2,\alpha}_{loc}$ when $h \in C^{\alpha}_{loc}$. This proves Theorem 2. \hspace{1cm} \blacksquare

**Proof of Lemma 5.**

Letting $\omega = u_k - u_{k'}$ we have

$$\int |\nabla \omega^+|^2 + \frac{1}{k'} (\omega^+)^2 \leq \int \nabla \omega \nabla \omega^+ + \frac{1}{k'} \omega \omega^+ \leq \int h \left( \frac{1}{u_k} - \frac{1}{u_{k'}} \right) \omega^+$$

showing that $\omega^+ = 0$ and so $\omega \leq 0$, ending the proof of lemma 5. \hspace{1cm} \blacksquare

**Verification of (3.2).**

Indeed,

$$a|\varphi^j - \varphi|^2 \leq 4a \varphi^2 \in L^1 \text{ and } a|\varphi^j - \varphi|^2 \to 0 \text{ a.e. in } \mathbb{R}^N$$

so that by Lebesgue’s theorem

$$\int a|\varphi^j - \varphi|^2 \to 0.$$
Now
\[ \frac{\partial \varphi^j}{\partial x_i} = \frac{1}{j} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi + M \left( \frac{x}{j} \right) \frac{\partial \varphi}{\partial x_i}. \]

Hence
\[ \int | \frac{\partial \varphi^j}{\partial x_i} - \frac{\partial \varphi}{\partial x_i} |^2 = \int \left| \frac{1}{j^2} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi + M \left( \frac{x}{j} \right) \frac{\partial \varphi}{\partial x_i} \right|^2 \leq C \int \left| \frac{1}{j^2} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 + \int \left| M \left( \frac{x}{j} \right) \frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_i} \right|^2. \]

Arguing as above we infer that
\[ M \left( \frac{x}{j} \right) \frac{\partial \varphi}{\partial x_i} \to \frac{\partial \varphi}{\partial x_i} \text{ in } L^2. \]

It remains to show that
\[ \int \left| \frac{1}{j^2} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 \to 0. \]

At first we remark that
\[ \int \left| \frac{1}{j^2} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 = \int_{B_{2j} \setminus B_j} \left| \frac{1}{j^2} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 \leq C \int_{B_{2j} \setminus B_j} \left| \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2. \]

Now using Hölder inequality with exponents \( \frac{N}{N-2} \) and \( \frac{N}{2} \) in the last integral we obtain
\[ \int \left| \frac{1}{j^2} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 \leq C \left( \int_{B_{2j} \setminus B_j} 1 dx \right)^{\frac{2}{N}} \left( \int_{B_{2j} \setminus B_j} | \varphi |^2 dx \right)^{\frac{N}{2}} \leq C \left( \int_{B_{2j}} 1 dx \right)^{\frac{2}{N}} \left( \int_{B_{2j}} | \varphi |^2 dx \right)^{\frac{N}{2}} \leq C \omega_N \frac{2j}{j^2} \left( \int_{B_{2j}} | \varphi |^2 dx \right)^{\frac{N}{2}} \]

where \( \omega_N \) denotes the volume of the unit sphere of \( \mathbb{R}^N \).

Next passing to the limit we get
\[ \int \left| \frac{1}{j^2} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 \to 0. \]

This shows that \( \varphi^j \to \varphi \) in \( E \) proving (3.2).

REFERENCES


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