EXISTENCE AND REGULARITY OF WEAK SOLUTIONS TO THE PRESCRIBED MEAN CURVATURE EQUATION FOR A NONPARAMETRIC SURFACE

P. AMSTER, M. CASSINELLI, M. C. MARIANI, AND D. F. RIAL

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1. Introduction

The prescribed mean curvature equation with Dirichlet condition for a nonparametric surface $X : \Omega \rightarrow \mathbb{R}^3$, $X(u,v) = (u,v,f(u,v))$ is the quasilinear partial differential equation

$$
(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv} = 2h(u,v,f) \left(1 + |\nabla f|^2\right)^{3/2} \text{ in } \Omega,
$$

$$
f = g \quad \text{in } \partial \Omega.
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$, $h : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $g \in H^1(\Omega)$.

We call $f \in H^1(\Omega)$ a weak solution of (1.1) if $f \in g + H^1_0(\Omega)$ and for every $\varphi \in C^1_0(\Omega)$

$$
\int_\Omega \left( \left(1 + |\nabla f|^2\right)^{-1/2} \nabla f \nabla \varphi + 2h(u,v,f) \varphi \right) du dv = 0.
$$

It is known that for the parametric Plateau’s problem, weak solutions can be obtained as critical points of a functional (see [2, 6, 7, 8, 10, 11]).

The nonparametric case has been studied for $H = H(x,y)$ (and generally $H = H(x_1,\ldots,x_n)$ for hypersurfaces in $\mathbb{R}^{n+1}$) by Gilbarg, Trudinger, Simon, and Serrin, among other authors. It has been proved [5] that there exists a solution for any smooth boundary data if the mean curvature $H'$ of $\partial \Omega$ satisfies

$$
H'(x_1,\ldots,x_n) \geq \frac{n}{n-1} |H(x_1,\ldots,x_n)|
$$

for any $(x_1,\ldots,x_n) \in \partial \Omega$, and $H \in C^1(\overline{\Omega}, \mathbb{R})$ satisfying the inequality

$$
\left| \int_\Omega H \varphi \right| \leq \frac{1-\epsilon}{n} \int_\Omega |D\varphi|
$$

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for any \( \varphi \in C^1_0(\Omega, \mathbb{R}) \) and some \( \epsilon > 0 \). They also proved a non-existence result (see [5, Corollary 14.13]): if \( H'(x_1, \ldots, x_n) < (n/(n-1))|H(x_1, \ldots, x_n)| \) for some \( (x_1, \ldots, x_n) \) and the sign of \( H \) is constant, then for any \( \epsilon > 0 \) there exists \( g \in C^\infty(\overline{\Omega}) \) such that \( \|g\|_\infty \leq \epsilon \) and that Dirichlet’s problem is not solvable.

We remark that the solutions obtained in [5] are classical. In this paper, we find weak solutions of the problem by variational methods.

We prove that for prescribed \( h \) there exists an associated functional to \( h \), and under some conditions on \( h \) and \( g \) we find that this functional has a global minimum in a convex subset of \( H^1(\Omega) \), which provides a weak solution of (1.1). We denote by \( H^1(\Omega) \) the usual Sobolev space, [1].

2. The associated variational problem

Given a function \( f \in C^2(\Omega) \), the generated nonparametric surface associated to this function is the graph of \( f \) in \( \mathbb{R}^3 \), parametrized as \( X(u,v) = (u,v,f(u,v)) \).

The mean curvature of this surface is
\[
h(u,v,f) = \frac{1}{2} \frac{Ef_{vv} - 2Ff_{uv} + Gf_{uu}}{(1 + f_u^2 + f_v^2)^{3/2}}, \tag{2.1}
\]
where \( E, F, \) and \( G \) are the coefficients of the first fundamental form [4, 9].

For prescribed \( h \), weak solutions of (1.1) can be obtained as critical points of a functional.

**Proposition 2.1.** Let \( J_h : H^1(\Omega) \to \mathbb{R} \) be the functional defined by
\[
J_h(f) = \int_{\Omega} \left( ((1 + |\nabla f|^2)^{1/2} + H(u,v,f)) \right) du dv, \tag{2.2}
\]
where \( H(u,v,z) = \int_0^z 2h(u,v,t) dt \). Then (1.1) is the Euler Lagrange equation of (2.2).

**Remark 2.2.** If \( f \in T = g + H_0^1(\Omega) \) is a critical point of \( J_h \), then \( f \) is a weak solution of (1.1).

**Proof.** For \( \varphi \in C^1_0(\Omega) \), integrating by parts we obtain
\[
d J_h(f)(\varphi) = 2 \int_{\Omega} \left( \frac{1}{2} \frac{Ef_{vv} - 2Ff_{uv} + Gf_{uu}}{(1 + f_u^2 + f_v^2)^{3/2}} - h(u,v,f) \right) \varphi du dv. \tag{2.3}
\]

3. Behavior of the functional \( J_h \)

In this section, we study the behavior of the functional \( J_h \) restricted to \( T \). For simplicity we write \( J_h(f) = A(f) + B(f) \), with
\[
A(f) = \int_{\Omega} (1 + |\nabla f|^2)^{1/2} du dv, \quad B(f) = \int_{\Omega} H(u,v,f) du dv. \tag{3.1}
\]

We will assume that \( h \) is bounded.
Lemma 3.1. The functional $A : T \to \mathbb{R}$ is continuous and convex.

Proof. Continuity can be proved by a simple computation. Let $a, b \geq 0$ such that $a + b = 1$. By Cauchy inequality, it follows that
\[
\sqrt{1 + |\nabla (af + bf_0)|^2} \leq a \sqrt{1 + |\nabla f|^2} + b \sqrt{1 + |\nabla f_0|^2}
\] (3.2)
and convexity holds. □

Remark 3.2. As $A$ is continuous and convex, then it is weakly lower semicontinuous in $T$.

Lemma 3.3. The functional $B$ is weakly lower semicontinuous in $T$.

Proof. Since $h$ is bounded, we have
\[
|H(u, v, z)| \leq c|z| + d.
\] (3.3)
From the compact immersion $H^1_0(\Omega) \hookrightarrow L^1(\Omega)$ and the continuity of Nemytskii operator associated to $H$ in $L^1(\Omega)$, we conclude that $B$ is weakly lower semicontinuous in $T$ (see [3, 12]). □

4. Weak solutions as critical points of $J_h$

Let us assume that $g \in W^{1,\infty}$, and consider for each $k > 0$, the following subset of $T$:
\[
\overline{M}_k = \{ f \in T : \| \nabla (f - g) \|_\infty \leq k \}.
\] (4.1)
$\overline{M}_k$ is nonempty, closed, convex, bounded, then it is weakly compact.

Remark 4.1. As $g \in W^{1,\infty}$, taking $p > 2$ we obtain, for any $f \in \overline{M}_k$:
\[
\| f - g \|_p \leq c \| \nabla (f - g) \|_p.
\] (4.2)
Then, by Sobolev imbedding, $\| f - g \|_\infty \leq c_1 \| f - g \|_1, p \leq \tilde{c}k$ for some constant $\tilde{c}$. We deduce that $f \in W^{1,\infty}$ and $f(\Omega) \subset K$ for some fixed compact $K \subset \mathbb{R}$. Thus, the assumption $\| h \|_\infty < \infty$ is not needed.

Let $\rho$ be the slope of $J_h$ in $\overline{M}_k$ defined by
\[
\rho(f_0, \overline{M}_k) = \sup \{ dJ_h(f_0)(f_0 - f) ; f \in \overline{M}_k \}
\] (4.3)
(see [7, 11]), then the following result holds.

Lemma 4.2. If $f_0 \in \overline{M}_k$ verifies
\[
J_h(f_0) = \inf \{ J_h(f) : f \in \overline{M}_k \},
\] (4.4)
then $\rho(f_0, \overline{M}_k) = 0$. 
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Proof.

\[ dJ_h(f_0)(f - f_0) = \lim_{\varepsilon \to 0} \frac{J_h(f_0 + \varepsilon (f - f_0)) - J_h(f_0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{J_h((1 - \varepsilon)f_0 + \varepsilon f) - J_h(f_0)}{\varepsilon}. \] (4.5)

When \(0 < \varepsilon < 1\) we have that \((1 - \varepsilon)f_0 + \varepsilon f \in \overline{M}_k\), and then \(dJ_h(f_0)(f_0 - f) \leq 0\) for all \(f \in \overline{M}_k\). As \(dJ_h(f_0)(f_0 - f) = 0\), we conclude that \(\rho(f_0, \overline{M}_k) = 0\).

Remark 4.3. Let \(J_h\) be weakly semicontinuous and let \(\overline{M}_k\) be a weakly compact subset of \(T\), then \(J_h\) achieves a minimum \(f_0\) in \(\overline{M}_k\). By Lemma 4.2, \(\rho(f_0, \overline{M}_k) = 0\).

As in [7], if \(f_0\) has zero slope, we call it a \(\rho\)-critical point. The following result gives sufficient conditions to assure that if \(f_0\) is a \(\rho\)-critical point, then it is a critical point of \(J_h\).

Theorem 4.4. Let \(f_0 \in \overline{M}_k\) such that \(\rho(f_0, \overline{M}_k) = 0\), and assume that one of the following conditions holds:

(i) \(dJ_h(f_0)(f_0 - g) \geq 0\)
(ii) \(\|\nabla (f_0 - g)\|_{\infty} < k\).

Then \(dJ_h(f_0) = 0\).

Proof. As \(\rho(f_0, \overline{M}_k) = 0\), we have that \(dJ_h(f_0)(f_0 - f) \leq 0\), and then \(dJ_h(f_0)(f_0 - g) \leq dJ_h(f_0)(f - g)\) for any \(f \in \overline{M}_k\).

We will prove that \(dJ_h(f_0)(\varphi) = 0\) for any \(\varphi \in C^1_0\). Let \(\tilde{\varphi} = k\varphi/2\|\nabla \varphi\|_{\infty}\), then \(\pm \tilde{\varphi} + g \in \overline{M}_k\), and then \(dJ_h(f_0)(f_0 - g) \leq \pm dJ_h(f_0)(\tilde{\varphi})\).

Suppose that \(dJ_h(f_0)(\tilde{\varphi}) \neq 0\), then \(dJ_h(f_0)(f_0 - g) < 0\).

If (i) holds, we immediately get a contradiction. On the other hand, if (ii) holds, there exists \(r > 1\) such that \(g + r(f_0 - g) \in \overline{M}_k\). Then \(dJ_h(f_0)(f_0 - g) \leq r dJ_h(f_0)(f_0 - g)\), a contradiction.

□

Examples

Let us assume that \(\int_\Omega ((\nabla (f - g))\nabla g)/\sqrt{1 + |\nabla f|^2} du dv \geq 0\) for any \(f \in \overline{M}_k\). Then condition (i) of Theorem 4.4 is fulfilled for example if

(a) \(|h(u, v, z)| \leq c(z - g(u, v))_+\) for every \((u, v) \in \Omega, z \in \mathbb{R}^2\), for some constant \(c\) small enough.

(b) \(\int_\Omega h(u, v, f)(f - g) du dv \geq 0\) for every \(f \in \overline{M}_k\). As a particular case, we may take \(h(u, v, z) = c(z - g(u, v))\) for any \(c \geq 0\).

(c) \(h(u, v, z) = -c(z - g(u, v))\) for some \(c > 0\) small enough.
Indeed, in all the examples the inequality \( dJ_h(f)(f - g) \geq 0 \) holds for any \( f \in \overline{M}_k \), since

\[
dJ_h(f)(f - g) = \int_{\Omega} \left( \frac{\nabla f \nabla (f - g)}{\sqrt{1 + |\nabla f|^2}} + 2h(u, v, f)(f - g) \right) dudv \\
= \int_{\Omega} \left( \frac{|\nabla (f - g)|^2}{\sqrt{1 + |\nabla f|^2}} + 2h(f - g) \right) dudv + \int_{\Omega} \frac{\nabla (f - g) \nabla g}{\sqrt{1 + |\nabla f|^2}} dudv \\
\geq \int_{\Omega} \left( \frac{|\nabla (f - g)|^2}{\sqrt{1 + |\nabla f|^2}} + 2h(f - g) \right) dudv.
\]

(4.6)

Then the result follows immediately in example (b). In examples (a) and (c), being \( \|\nabla (f - g)\|_{\infty} \leq k \), we can choose \( \hat{k} \) such that \( \sqrt{1 + \|\nabla f\|_\infty^2} \leq \hat{k} \). Then

\[
\int_{\Omega} \left( \frac{|\nabla (f - g)|^2}{\sqrt{1 + |\nabla f|^2}} + 2h(u, v, f)(f - g) \right) dudv \geq \int_{\Omega} \left( \frac{|\nabla (f - g)|^2}{k} - 2c(f - g)^2 \right) dudv \\
\geq \frac{1}{k} \|\nabla (f - g)\|^2_2 - 2c_1^2 \|\nabla (f - g)\|^2_2 \\
= \left( \frac{1}{k} - 2c_1^2 \right) \|\nabla (f - g)\|^2_2.
\]

(4.7)

where \( c_1 \) is the Poincaré’s constant associated to \( \Omega \).

Thus, the result holds for \( c \leq 1/2\hat{k}c_1^2 \).

Remark 4.5. As in the preceding examples, it can be proved that if \( dJ_h(f)(f - g) \geq 0 \) for any \( f \in \overline{M}_k \), then \( g \) is a weak solution of (1.1). Indeed, if \( dJ_h(g) \neq 0 \), from Theorem 4.4 it follows that \( \rho(g, \overline{M}_k) > 0 \). As \( J_h \) achieves a minimum in every \( \overline{M}_k \), we may take \( k \geq k_n \to 0 \), and \( f_n \) such that \( \rho(f_n, \overline{M}_k) = 0 \). As \( \overline{M}_k \subset \overline{M}_k \), condition (i) in Theorem 4.4 holds, and then \( dJ_h(f_n) = 0 \). It is immediate that \( f_n \to g \) in \( W^{1, \infty} \), and then it follows easily that \( dJ_h(g) = 0 \).

Furthermore, for constant \( g \) we can see that if \( dJ_h(f)(f - g) \geq 0 \) for any \( f \in \overline{M}_k \), then \( g \) is a global minimum of \( J_h \) in \( \overline{M}_k \): let us define \( \varphi(t) = J_h(tf + (1-t)g) \), then \( \varphi'(t) = dJ_h(tf + (1-t)g)(f - g) \). As \( 0 \leq dJ_h(tf + (1-t)g)(tf + (1-t)g - g) = tdJ_h(tf + (1-t)g)(f - g) \) it follows that \( J_h(f) - J_h(g) = \varphi(1) - \varphi(0) = \varphi'(c) \geq 0 \).

5. Multiple solutions

In this section, we study the multiplicity of weak solutions of (1.1). Consider

\[
\mathcal{N}_k = \left\{ f \in \overline{M}_k \cap H^2 : \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_2 \leq k \right\},
\]

(5.1)
Theorem 5.1. Let \( f_0 \in \overline{N}_k \) be a local minimum of \( J_h \) and assume that \( J_h(f_1) < J_h(f_0) \) for some \( f_1 \in \overline{N}_k \). Let

\[
c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_h(\gamma(t)),
\]

where \( \Gamma = \{ \gamma \in C([0,1], \overline{N}_k) : \gamma(0) = f_0, \gamma(1) = f_1 \} \). Then there exists \( f \in \overline{N}_k \) such that \( J_h(f) = c \) and \( \rho(f, \overline{N}_k) = 0 \).

We remark that \( f \) is not a local minimum of \( J_h \). This kind of \( f \) is called an unstable critical point.

The proof of Theorem 5.1 follows from Theorem 3 in [7] and Lemmas 5.2, 5.3, and 5.4 below.

Lemma 5.2. The functional \( J_h \) is \( C^1(\overline{N}_k) \).

Proof. Let \( f, f_0 \in \overline{N}_k \). Then

\[
\left| dJ_h(f)(\varphi) - dJ_h(f_0)(\varphi) \right| \leq \| \varphi \|_{H^1_0} \left( \left\| \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} - \frac{\nabla f_0}{\sqrt{1 + |\nabla f_0|^2}} \right\|_2 + \left\| N_h(f_0) - N_h(f) \right\|_2 \right),
\]

where \( N_h \) is the Nemytskii operator associated to \( h \). Let

\[
\left\| \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} - \frac{\nabla f_0}{\sqrt{1 + |\nabla f_0|^2}} \right\|_2 \leq \left\| 1 + |\nabla f|^2 \nabla f - \sqrt{1 + |\nabla f|^2} \nabla f_0 \right\|_2 \leq \kappa \| f_0 - f \|_{H^1_0}
\]

and \( N_h : L^2 \rightarrow L^2 \) continuous, the result holds. \( \square \)

Lemma 5.3. The slope \( \rho \) is \( H^1 \)-continuous.

Proof. Let \( f_n \in \overline{N}_k \) such that \( f_n \rightarrow f_0 \) in \( H^1_0 \). For \( \epsilon > 0 \) we take \( g_n \in \overline{N}_k \) such that \( \rho(f_n, \overline{N}_k) - \epsilon/2 < dJ_h(f_n)(f_n - g_n) \). Then

\[
\rho(f_n, \overline{N}_k) - \rho(f_0, \overline{N}_k) \leq dJ_h(f_n)(f_n - g_n) + \epsilon - dJ_h(f_0)(f_0 - g_n) \\
\leq \| dJ_h(f_n) \|_{(H^1_0)^*} \| (f_n - f_0) \|_{H^1_0} \\
+ \| dJ_h(f_n) - dJ_h(f_0) \|_{(H^1_0)^*} \| (f_0 - g_n) \|_{H^1_0} + \epsilon/2 < \epsilon
\]

for \( n \geq n_0 \). Operating in the same way with \( \rho(f_0, \overline{N}_k) - \rho(f_n, \overline{N}_k) \), we conclude that \( \rho(f_n, \overline{N}_k) \rightarrow \rho(f_0, \overline{N}_k) \). \( \square \)
Lemma 5.4 (Palais Smale condition). Let \((f_n)_{n \in \mathbb{N}} \subset \overline{N}_k\) such that \(\lim_{n \to \infty} \rho(f_n, \overline{N}_k) = 0\). Then \((f_n)_{n \in \mathbb{N}}\) has a convergent subsequence in \(H^1_0(\Omega)\).

Proof. As \(f_n \in \overline{N}_k\), we may suppose that \(f_n \to f\) weakly. Let \(\Psi_n = f_n - f\). We will see that \(\Psi_n \to 0\). Indeed,

\[
d J_h(f_n)(\Psi_n) = \int_{\Omega} \left( \frac{\nabla f_n}{\sqrt{1 + |\nabla f_n|^2}} \nabla \Psi_n + 2h(u, v, f_n) \Psi_n \right) du dv
\]

\[
= \int_{\Omega} \frac{1}{\sqrt{1 + |\nabla f_n|^2}} |\nabla \Psi_n|^2 du dv + \int_{\Omega} \frac{\nabla \Psi_n}{\sqrt{1 + |\nabla f_n|^2}} \nabla f du dv
\]

\[
+ \int_{\Omega} 2h(u, v, f_n) \Psi_n du dv. \tag{5.6}
\]

Then for some constant \(c\)

\[
c \|\nabla \Psi_n\|_2 \leq \rho(f_n, \overline{N}_k) - \int_{\Omega} \frac{\nabla \Psi_n}{\sqrt{1 + |\nabla f_n|^2}} \nabla f du dv - \int_{\Omega} 2h(u, v, f_n) \Psi_n du dv. \tag{5.7}
\]

By Rellich-Kondrachov theorem \(\Psi_n \to 0\) in \(L^2(\Omega)\), and then

\[
\left| \int_{\Omega} 2h(u, v, f_n) \Psi_n du dv \right| \leq 2\|h\|_{\infty} |\Omega|^{1/2}\|\Psi_n\|_2 \to 0, \tag{5.8}
\]

\[
\left| \int_{\Omega} \frac{\nabla \Psi_n}{\sqrt{1 + |\nabla f_n|^2}} \nabla f du dv \right|
\]

\[
= \left| - \int_{\Omega} \frac{\Delta f}{\sqrt{1 + |\nabla f_n|^2}} \Psi_n du dv - \int_{\Omega} \Psi_n \nabla (1 + |\nabla f_n|^2)^{-1/2} \nabla f du dv \right| \tag{5.9}
\]

\[
\leq \|\Delta f\|_2 \|\Psi_n\|_2 + \|\nabla f_n\|_{\infty} \|\nabla f\|_{\infty} \|D^2 f_n\|_2 \|\Psi_n\|_2 \to 0. \tag{5.9}
\]

\(\square\)

Example 5.5. Now we will show with an example that problem (1.1) may have at least three \(\rho\)-critical points in \(N_k\).

Let \(g = g_0\) be a constant, and \(h(u, v, z) = -c(z - g_0)\) for some constant \(c > 0\). Then, \(g_0\) is a minimum of \(J_h\) in \(M_{k_1}\) for \(k_1\) small enough, and a local minimum in \(M_k\) for any \(k \geq k_1\).

Moreover, taking \(\Omega = B_R\), \(f(u, v) = g_0 + R^2 - (u^2 + v^2)\), it follows that

\[
J_h(f) - J_h(g_0) = 2\pi \left( o(R^3) - \frac{c}{6}R^6 \right), \tag{5.10}
\]

and taking \(k = 2\sqrt{\pi}R\) it holds that \(f \in \overline{N}_k\). Hence, if \(R\) is big enough, it follows that \(g_0\) is not a global minimum in \(\overline{N}_k\). Furthermore, we see that the proof of Lemma 4.2 may be repeated in \(\overline{N}_k\), and then the minimum of \(J_h\) in \(\overline{N}_k\) is a \(\rho\)-critical point. From Theorem 5.1 there is a third \(\rho\)-critical point which is not a local minimum of \(J_h\).
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6. Regularity

As we proved, problem (1.1) admits (for an appropriate $k > 0$) a weak solution in a subset $M(k) = \{ f \in T / \| \nabla (f - g) \|_{\infty} \leq k \}$. Consider $p > 2$, and $f_0 \in W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ a weak solution of (1.1). Then $L_{f_0}f_0 = 2h(u, v, f_0)(1 + \nabla f_0^2)^{3/2}$ in $\Omega$ where for any $f \in C^1(\overline{\Omega})$ $L_f : W^{2,p} \rightarrow L^p$ is the strictly elliptic operator given by

\[ L_f \phi = (1 + f^2_v)\phi_{uu} + (1 + f^2_u)\phi_{vv} - 2f_uf_v\phi_{uv}. \] (6.1)

In order to prove the regularity of $f_0$, we study equation (6.2)

\[ L_{f_0}\phi = 2h(u, v, f_0)(1 + \nabla f_0^2)^{3/2} \text{ in } \Omega, \phi = g \text{ in } \partial \Omega. \] (6.2)

**Proposition 6.1.** Let us assume that $\partial \Omega \in C^{2, \alpha}$, $g \in C^{2, \alpha}$, and $h \in C^{\alpha}$ for some $0 < \alpha \leq 1 - 2/p$. Then, if $\phi \in W^{2,p}$ is a strong solution of (6.2), $\phi \in C^{2, \alpha}(\overline{\Omega})$.

**Proof.** By Sobolev imbedding $\phi \in C^{1, \alpha}(\overline{\Omega})$. Then $L_{f_0}\phi \in C^{\alpha}(\overline{\Omega})$ and the coefficients of the operator $L_{f_0}$ belong to $C^{\alpha}$. By Theorem 6.14 in [5], the equation $Lw = L_{f_0}\phi$ in $\Omega$, $w = g$ in $\partial \Omega$ is uniquely solvable in $C^{2, \alpha}(\overline{\Omega})$, and the result follows from the uniqueness in Theorem 9.15 in [5]. □

**Remark 6.2.** As a simple consequence, we obtain that $f_0 \in C^{2, \alpha}(\overline{\Omega})$, by the uniqueness in $W^{2,p}$ given by [5, Theorem 9.15].

**Corollary 6.3.** Let us assume that $\partial \Omega \in C^{k, \alpha}$, $g \in C^{k+2, \alpha}$, and $h \in C^{k, \alpha}$ for some $0 < \alpha \leq 1 - 2/p$. Then $f_0 \in C^{k+2, \alpha}(\overline{\Omega})$.

**Proof.** It is immediate from Proposition 2.1 and Theorem 6.19 in [5]. □

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**References**


P. AMSTER: DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UBA. PAB I, CIUDAD UNIVERSITARIA, 1428. BUENOS AIRES, ARGENTINA

E-mail address: pamster@dm.uba.ar

M. CASSINELLI: DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UBA. PAB I, CIUDAD UNIVERSITARIA, 1428. BUENOS AIRES, ARGENTINA

E-mail address: mcmarian@dm.uba.ar

D. F. RIAL: DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UBA. PAB I, CIUDAD UNIVERSITARIA, 1428. BUENOS AIRES, ARGENTINA
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Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

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- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

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Lead Guest Editor

Gerrit K. Janssens, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; Gerrit.Janssens@uhasselt.be

Guest Editor

Cathy Macharis, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; Cathy.Macharis@vub.ac.be