A NOTE ON THE HOMOMORPHISM THEOREM FOR HEMIRINGS

D. M. OLSON
Department of Mathematics
Cameron University
Lawton, Oklahoma 73501
U.S.A.

(Received April 11, 1978)

ABSTRACT. The fundamental homomorphism theorem for rings is not generally applicable in hemiring theory. In this paper, we show that for the class of N-homomorphism of hemirings the fundamental theorem is valid. In addition, the concept of N-homomorphism is used to prove that every hereditarily semisubtractive hemiring is of type (K).

KEY WORDS AND PHRASES. Hemirings, homomorphism of hemirings, homomorphism Theorem, N-homomorphism, Type (K) ideals, Hereditarily semisubtractive.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 16A78.
1. **INTRODUCTION.**

It is well known that the analogue to the fundamental homomorphism theorem is not necessarily true in general hemiring theory. However, in [1] Allen defined a class of maximal homomorphisms of hemirings for which the exact analogue could be proven. In this article we extend his class to the class of N-homomorphisms for which the homomorphism theorem holds, and examine some properties of N-homomorphisms.

Also, in [2] LaTorre defines the concepts of a hemiring being of type (H) or type (K). He gives results establishing certain classes of hemirings as being all of type (H), but states that no general statement can be made concerning the occurrence of hemiring of type (K). In section 4 we use the concept of an N-homomorphism together with the idea of a semisubtractive hemiring [4] to establish that all hereditarily semisubtractive hemirings are of type (K).

In what follows we use the standard hemiring definitions and terminology which may be found in [1] and [2].

2. **N-HOMOMORPHISMS.**

**DEFINITION 1.** A hemiring homomorphism $\phi$ from $S$ onto $T$ is called a maximal homomorphism if for every $t \in T$, there exists $c_t \in \phi^{-1}(t)$ such that for all $x \in \phi^{-1}(t)$ we have $x + \ker \phi \subseteq c_t + \ker \phi$. [1]

**DEFINITION 2.** A hemiring homomorphism $\phi$ from $S$ onto $T$ is called an N-homomorphism if for every $t \in T$, the collection $\{x + \ker \phi : x \in \phi^{-1}(t)\}$ contains no two sets which are disjoint.

It is easy to see that $\phi : S \to T$ will be an N-homomorphism if and only if whenever $\phi(x) = \phi(y)$ for some $x, y \in S$ we have $k_1, k_2 \in \ker \phi$ such that $x + k_1 = y + k_2$. LaTorre [3] has also characterized maximal homomorphism as follows.
LEMMA 3. A homomorphism $\phi : S \to T$ is maximal if and only if the inverse image of every $t \in T$ is a coset of $\ker \phi$.

With this lemma we can establish the following result.

THEOREM 4. If $\phi : S \to T$ is a maximal homomorphism then $\phi$ is an $N$-homomorphism.

PROOF. Let $\phi : S \to T$ be a maximal homomorphism and let $t \in T$. Now consider $x + \ker \phi$ and $y + \ker \phi$ for $x, y \in \phi^{-1}(t)$. By Lemma 1, $\phi^{-1}(t)$ is a coset of $\ker \phi$, say $\phi^{-1}(t) = c + \ker \phi$. Then $x, y \in c + \ker \phi$.

Let $x = c + k_1$ and $y = c + k_2$, then $x + k_2 = y + k_1$ for $k_1, k_2 \in \ker \phi$ and $x + \ker \phi \cap y + \ker \phi \neq \phi$. Since $t, x$ and $y$ are arbitrary we see that $\phi$ is an $N$-homomorphism.

Since every ring homomorphism is maximal, these are also $N$-homomorphisms. In addition it is easy to verify that every natural map of a hemiring $S$ onto a quotient hemiring $S/I$ is an $N$-homomorphism. The following example shows that the class of $N$-homomorphisms does not coincide with the class of maximal homomorphisms.

EXAMPLE 5. Let $S = \{0, 1, 2, 3, 4\}$ be the hemiring with zero multiplication and addition defined by the following table.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Let $T$ be the subhemiring $\{0, 1\}$ of $S$. Define $\phi : S \to T$ by $\phi : 0, 1 \to 0$
and $2,3,4 \rightarrow 1$ then $\phi$ is a hemiring homomorphism with $\ker \phi = \{0\}$.

Now $\phi^{-1}(1) = \{2,3,4\}$ and no two of $2 + \ker \phi$, $3 + \ker \phi$ and $4 + \ker \phi$ are disjoint. Thus $\phi$ is an N-homomorphism. However, $\phi^{-1}(1)$ is not a coset of $\ker \phi$ so $\phi$ is not a maximal homomorphism.

3. THE FUNDAMENTAL HOMOMORPHISM THEOREM

For the class of N-homomorphism we can establish analogues to certain desirable results from ring theory.

**Lemma 6.** If $\phi$ is an N-homomorphism from $S$ onto $T$ with $\ker \phi = \{0\}$, then $\phi$ is an isomorphism.

**Proof.** Suppose $\phi(x) = \phi(y)$. Then there exist $k_1, k_2 \in \ker \phi$ such that $x + k_1 = y + k_2$. But since $\ker \phi = \{0\}$, $k_1 = k_2 = 0$. Thus $x = y$ and $\phi$ is an isomorphism.

**Theorem 7.** If $\phi$ is an N-homomorphism from $S$ onto $T$ then $S/\ker \phi \cong T$.

**Proof.** Define $\psi: S/\ker \phi \rightarrow T$ by $\psi([s]) = \phi(s)$, where $[s]$ is the equivalence class of $s$ in $S$ determined by the ideal $\ker \phi$ of $S$. Then as usual $\psi$ is a well defined onto homomorphism. If $\psi([s]) = \psi([t])$, then $\phi(s) = \phi(t)$ and, since $\phi$ is an N-homomorphism, there must exist $k_1, k_2 \in \ker \phi$ such that $s + k_1 = t + k_2$. But then by definition $[s] = [t]$ and $\psi$ is an isomorphism.

It is clear that the class of N-homomorphism is the largest class for which the mapping $\psi$, as defined in the proof of Theorem 7, will be an isomorphism.

**Theorem 8.** If $\phi$ is an N-homomorphism from $S$ onto $T$ and $K$ is an ideal of $T$, then $s/\phi^{-1}(K) \cong T/K$.

**Proof.** Define $\psi: S \rightarrow T/K$ by $\psi(s) = [\phi(s)]$. Then one can quickly check
to see that $\psi$ is a homomorphism from $S$ onto $T/K$. Now if $\psi(s) = \psi(t)$ then $[\psi(s)] = [\psi(t)]$ which implies that $\phi(s) + k_1 = \phi(t) + k_2$ for some $k_1, k_2 \in k$. Choose $\ell_1$ and $\ell_2$ from $\phi^{-1}(k_1)$ and $\phi^{-1}(k_2)$ respectively. Then $\phi(s + \ell_1) = \phi(t + \ell_2)$. Since $\phi$ is an $N$-homomorphism, there exist $z_1, z_2 \in \ker \phi \subseteq \phi^{-1}(K)$ such that $s + (\ell_1 + z_1) = t + (\ell_2 + z_2)$. But since $\ell_1 + z_1, \ell_2 + z_2 \in \phi^{-1}(K)$, they are both in $\ker \psi$ and thus $\psi$ is an $N$-homomorphism.

Finally $\phi^{-1}(K)^* \subseteq \ker \psi = \{s \in S : \psi(s) = [\psi(s)] = 0\} = \{s \in S : \phi(s) \in K^*\}$. But if $\phi(s) \in K^*$ then $\phi(s) + \phi(k) \in K$ for some $k \in \phi^{-1}(K)$. Thus $s + k \in \phi^{-1}(K)$ so $s \in \phi^{-1}(K)^*$. This gives us that $\ker \psi = \phi^{-1}(K)^*$ so by the preceding theorem $S/\phi^{-1}(K) \cong S/\phi^{-1}(K)^* \cong T/K$.

4. HEMIRINGS OF TYPE K

DEFINITION 9. A hemiring $S$ is of type $(K)$ provided that if $I$ is a $k$-ideal of $S$ and $n : S \rightarrow S/I$ is the natural homomorphism, then $n$ preserves $k$-ideals. [2]

DEFINITION 10. A hemiring $S$ is said to be semisubtractive if for every pair of elements $a$ and $b$ in $S$ at least one of the equations $a + x = b$ or $b + x = a$ is solvable in $S$. [4]

DEFINITION 11. A hemiring $S$ is hereditarily semisubtractive if each ideal of $S$ is semisubtractive as a hemiring.

Clearly every ring is hereditarily semisubtractive and also the hemiring of non-negative integers under the operations of $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$. In view of the following lemma any semisubtractive hemiring whose ideals are all $k$-ideals is also hereditarily semisubtractive.

LEMMA 12. If $S$ is a semisubtractive hemiring and $K$ is a $k$-ideal of
S then K is semisubtractive.

PROOF. Let a,b ∈ K. Since a,b ∈ S, there exists an element s ∈ S for which either a + s = b or b + s = a. For the sake of argument say a + s = b. But then a + s ∈ K with a ∈ K and since K is a k-ideal this requires s to be K. Thus K is indeed semisubtractive.

THEOREM 13. If S is hereditarily semisubtractive and φ is an N-homomorphism from S onto T then φ preserves k-ideals.

PROOF. Let K be a k-ideal of S and K = φ(K). We shall show that K ⊆ K and thus that K is a k-ideal. We use the notation s for the image of s under φ whenever it is convenient. If x ∈ K then x + k_1 k_2 for some k_1,k_2 ∈ K. Then φ(x + k_1) = φ(k_2) so there exist z_1,z_2 ∈ ker φ such that x + k_1 + z_1 = k_2 + z_2. Since K + ker φ is semisubtractive as an ideal of S we have either there exists t ∈ K + ker φ such that k_1 + t + z_2 or there exists t ∈ K + ker φ such that k_1 = z_2 + t.

In the first case we see that x + k_1 + t + z_1 = k_2 + z_2 + t which implies that x + z_2 + z_1 = k_2 + z_2 + t. Then x = φ(x) = φ(x + z_1 + z_2) = φ(k_1 + z_2 + t) ∈ φ(K) = K.

In the second case we get x + k_1 + z_1 + t = k_2 + z_2 + t so x + k_1 + z_1 + t = k_2 + k_1. Now t ∈ K + ker φ so t = k_3 + z_3 and as a result we have x + k_1 + k_3 + (z_1 + z_3) = k_2 + k_1. Since K is a k-ideal of S we have x + z_1 + z_3 ∈ K. Then x = φ(x + z_1 + z_3) ∈ φ(K) = K. In any case we get K ⊆ K and so φ(K) = K is a k-ideal of T as desired.

COROLLARY 14. If S is an hereditarily semisubtractive hemiring then S is of type (K).

PROOF. If I is a k-ideal of S the natural map η : S → S/I is an N-homomorphism. By Theorem 13 η preserves k-ideals which makes S a hemiring
of type (K).

REFERENCES


Special Issue on
Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today’s economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems).

This special issue will include (but not be limited to) the following topics:

- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

**Lean Yu,** Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

**Shouyang Wang,** Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

**K. K. Lai,** Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk