ON SEQUENCES OF CONTRACTIVE MAPPINGS AND THEIR FIXED POINTS

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ABSTRACT. By using a condition of Reich, we establish two fixed point theorems concerning sequences of contractive mappings and their fixed points. A suitable example is also given.

KEY WORDS AND PHRASES: Complete metric space, Fixed point, Sequence of mappings.

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1. INTRODUCTIONS.

Throughout this paper, \((X,d)\) denotes a complete metric space and \(T\) stands for a mapping of \(X\) into itself. It is well known that each of the following conditions ensure the existence and uniqueness of a fixed point of \(T\):

(A) (Banach). There exists a number \(k, 0 \leq k < 1\), such that for each \(x,y\) in \(X\),

\[
d(Tx, Ty) \leq k \cdot d(x,y)
\]

(B) (Rakotch [1]). There exists a monotonically decreasing function \(g: (0,\infty) \to [0,1]\) such that for each \(x,y\) in \(X\),

\[
d(Tx, Ty) \leq g(d(x,y)) \cdot d(x,y).
\]

(C) (Reich [2]). There exist nonnegative numbers \(a,b\) such that for each \(x,y\) in \(X\),

\[
d(Tx, Ty) \leq a \cdot d(x,y) + b \cdot [d(x,Tx)+d(y,Ty)].
\]
(D) (Reich [3]). There exist monotonically decreasing functions $a, b: (0, \infty) \to [0, 1)$ such that for each $x, y$ in $X$, $x \neq y$,
\[ d(Tx, Ty) \leq a(d(x, y)) \cdot d(x, y) + b(d(x, y)) \cdot [d(x, Tx) + d(y, Ty)], \]
(1.1)
where for any $t > 0$,
\[ a(t) + 2b(t) \leq 1. \]
(1.2)

(E) (Reich [4]). There exist functions $a, b: (0, \infty) \to [0, 1)$ such that (1.1) holds for each $x, y$ in $X$, $x \neq y$, satisfying (1.2) and
\[ \limsup_{r \to t^+} [a(r) + 2b(r)] < 1. \]

(F) (Hardy and Rogers [5]). There exist monotonically decreasing functions $a, b, c: (0, \infty) \to [0, 1)$ such that for each $x, y$ in $X$, $x \neq y$,
\[ d(Tx, Ty) \leq a(d(x, y)) \cdot d(x, y) + b(d(x, y)) \cdot [d(x, Tx) + d(y, Ty)] + c(d(x, y)) \cdot [d(x, Ty) + d(y, Tx)], \]
(1.3)
where for any $t > 0$,
\[ a(t) + 2b(t) + 2c(t) \leq 1. \]
(1.4)

It is not hard to show that, adopting the same proof of [4], that $T$ has a unique fixed point if

(G) There exist functions $a, b, c: (0, \infty) \to [0, 1)$ such that (1.3) holds for each $x, y$ in $X$, $x \neq y$, satisfying (1.4) and
\[ \limsup_{r \to t^+} [a(r) + 2b(r) + 2c(r)] < 1. \]

Evidently (A) implies (B) and (C), (B) and (C) imply (D), (D) imply (E) and (F), (E) and (F) imply (G). Suitable examples can be found in Rhoades [6] to illustrate some of the above implications. In the sequel, $N$ stands for the set of natural numbers.

The following result was established in [5] and [6].

**Theorem 1.** Let $T_n$, $n \in N$, be mappings of $(X, d)$ into itself satisfying condition (F) with the same functions $a, b, c$ and with fixed points $z_n$. Suppose that a mapping $T$ of $X$ into itself can be defined pointwise by $T(x) = \lim_{n \to \infty} T_n(x)$ for any $x$ in $X$. Then $T$ has a unique fixed point $z$ and $z = \lim_{n \to \infty} z_n$.

Theorem 1 generalizes an analogous result of Bonsall [7], Theorem 6 of [2] and Theorem 4 of [3] established for mappings $T_n$ satisfying conditions (A), (C) and (D), respectively. Results due to Chatterjea [8] and Singh [9], Kannan type mappings, are also included in Theorem 1.

The proof of Theorem 1 consists essentially in the fact that the sequence $\{z_n\}$ is regular, i.e. it possesses a limit $z$(say). It appears that a result corresponding to Theorem 1 for mappings satisfying condition (G) does not exist in print in the
2. RESULTS.

We first state the following result more general than Theorem 1.

**THEOREM 2.** Let $T_n, n \in \mathbb{N}$, be mappings of $(X,d)$ into itself satisfying condition $(G)$ with the same functions $a, b, c$ and with fixed points $z_n$. Suppose that a mapping $T$ of $X$ into itself can be defined pointwise by $T(x) = \lim_{n \to \infty} T_n(x)$ for any $x$ in $X$ and the sequence $\{z_n\}$ is regular. Then $T$ has a unique fixed point $z$ and $z = \lim_{n \to \infty} z_n$.

**PROOF.** Since the metric $d: X \times X \to [0,\infty)$ is continuous, the limit mapping $T$ satisfies the inequality (1.3). By condition $(G)$, $T$ has a unique fixed point $z$. We claim that

$$
\ell = \inf_{n \in \mathbb{N}} d(z_n, z) = 0,
$$

otherwise assume $\ell > 0$. By observing that $d(z_n, z) > \ell > 0$ and hence $z_n \neq z$ for any $n \in \mathbb{N}$, we deduce using (1.3) and the triangular property of the metric $d$,

$$
d(z_n, z) = d(T_n z_n, Tz) \leq d(T_n z_n, Tz) + d(T_n z, Tz) \\
\leq a \cdot d(z_n, z) + b \cdot d(z, Tz) \\
+ c \cdot [d(z_n, z) + d(z, Tz) + d(z, T z_n)] + d(T_n z, Tz) \\
= (a + 2c) \cdot d(z_n, z) + (1 + b + c) \cdot d(T_n z, Tz)
$$

for any $n \in \mathbb{N}$, where $a = a(d(z_n, z))$ and similarly for $b$ and $c$.

Thus

$$
d(z_n, z) \leq \frac{2d(T_n z, z)}{1 - (a + 2c)}
$$

for any $n \in \mathbb{N}$. If we denote with $\{z_{k(n)}\}$ a subsequence of $\{z_n\}$ such that

$$
\ell \leq d(z_{k(n)}, z) < \ell + 1/n,
$$

we obtain that

$$
\lim_{n \to \infty} d(z_{k(n)}, z) = \ell > 0.
$$

Following Reich [4], we observe that the assumptions about the functions $a, b, c$ of condition $(G)$ imply the existence of two functions $h, k: (0, \infty) \to (0, \infty)$ for which, given $t > 0$, there exists an $h(t) > 0$ such that $0 \leq t < h(t)$ implies

$$
a(t) + 2b(t) + 2c(t)k(t) < 1.
$$

By (2.2), let $p \in \mathbb{N}$ such that

$$
0 \leq d(z_{k(n)}, z) - \ell < h(\ell)
$$
for any \( n > p \). From (2.3), it follows that
\[
a (d(Z_{k(n)}, z)) + 2c (d(Z_{k(n)}, z)) \leq k(\ell) < 1
\]
for any \( n > p \). On the other hand, since the sequence \( \{T_{k(n)}(z)\} \) converges to \( z = T(z) \), we can find an integer \( q \in \mathbb{N} \) such that
\[
d(Tz, T_{k(n)}(z)) < \ell (1 - k(\ell))/2
\]
for any \( n > q \). By (2.1), then we have for any \( n \geq \max\{p, q\} \),
\[
2d(Tz, T_{k(n)}(z)) \leq d(Z_{k(n)}, z) < 1 - [a (d(Z_{k(n)}, z)) + 2c (d(Z_{k(n)}, z))]
\]
a contradiction. Thus \( \ell = 0 \) and therefore the sequence \( \{z_{k(n)}\} \) converges to \( z \). Since \( \{z_n\} \) is regular, it has limit \( z \) and this concludes the proof.

**Theorem 3.** Let \( T_n \) be mappings of \((X, d)\) into itself with at least one fixed point \( z_n \). If \( \{T_n\} \) converges uniformly to a mapping \( T \) of \( X \) into itself satisfying condition (G) and if the sequence \( \{z_n\} \) is regular, then \( z = \lim_{n \to \infty} z_n \), where \( z \) is the unique fixed point of \( T \).

**Proof.** We have that
\[
d(z_n, z) = d(z_n, Tz) \leq d(z_n, Tz_n) + d(Tz_n, Tz)
\]
\[
\leq d(z_n, Tz_n) + a \cdot d(z_n, z) + b \cdot d(z_n, Tz_n) + c \cdot [d(z_n, z) + d(z_n, Tz_n)]
\]
\[
= (a + 2c) \cdot d(z_n, z) + (1 + b + c) \cdot d(z_n, Tz_n).
\]
for any \( n \in \mathbb{N} \), where \( a = a (d(z_n, z)) \) and similarly for \( b \) and \( c \). Thus
\[
d(z_n, z) \leq \frac{2d(z_n, Tz_n)}{1 - (a + 2c)}
\]
for any \( n \in \mathbb{N} \) and proceeding as in the proof of Theorem 2, we get the thesis.

**Remark 1.** It is evident that there certainly exists a subsequence of \( \{z_n\} \) converging to \( z \), even if \( \{z_n\} \) is not regular. It is not yet known if the regularity of \( \{z_n\} \) is a necessary condition in Theorems 2 and 3.

**Remark 2.** Theorem 3 generalizes Theorem 5 of Ray [13], which in turn extends Theorem 9 of Reich [3].

**Remark 3.** Following Ray [13] and Fraser and Nadler [14], one can establish a result analogous to Theorem 10 of Reich [3] by using condition (G).
3. AN EXAMPLE.

In order to illustrate the degree of generality of Theorem 2 over Theorem 1, we furnish an example which shows that there exist mappings $T_n$ of $X$ into itself satisfying condition (G) but no condition (F).

**EXAMPLE.** Let $X=\{0,1\}$ be equipped with metric $d$ defined as follows,

$$d(x,y) = \begin{cases} |x-y| & \text{if } x,y \in [0,1], \\ x+y & \text{if one of } x,y \in \mathbb{N} - \{1\} \end{cases}$$

Then $(X,d)$ is a complete metric space because it is isometric to a closed subspace of the space of absolutely summable sequences. For further details, see Boyd and Wong [15].

Now we define $T_n: x \mapsto x$, for $n \geq 3$,

$$T_n(x) = \begin{cases} x-(n-1)x^2/(2n-3) & \text{if } x \in [0,1], \\ x-1 & \text{if } x \in \mathbb{N} - \{1\} \end{cases}$$

Further, $T_n$ does not satisfy condition (F) for $n \geq 3$, otherwise we should have for $y=0$ and $x=t \in (0,1]$, 

$$t - \frac{(n-1)t^2}{2n-3} \leq a(t) \cdot t + b(t) \cdot \frac{(n-1)t^2}{2n-3} + c(t) \cdot \frac{(n-1)t^2}{2n-3} + t$$

$$\leq \frac{(n-1)t^2}{2n-3} \cdot [a(t)+2c(t)] \cdot t + [b(t)-c(t)] \cdot \frac{(n-1)t^2}{2n-3}$$

$$< 1 - 2b(t) \cdot t + b(t) \cdot \frac{(n-1)t^2}{2n-3}.$$ 

Since $b(t) < 1/2$ for any $t > 0$, we obtain

$$1 - \frac{(n-1)t}{2n-3} < 1 - 2b(t) + \frac{1}{2} \cdot \frac{(n-1)t}{2n-3}$$

for any $t \in (0,1]$. This implies, for $n \geq 3$, that

$$1 - t \leq 1 - \frac{3}{2} \cdot \frac{(n-1)t}{2n-3} < 1 - 2b(t),$$

i.e. $b(t) \leq 2b(t) < t$ for any $t \in (0,1]$. Then, since $b$ is monotonically decreasing in $(0,\infty)$, we should have $b(t)=0$ for any $t>0$. 

Therefore, for each \( x, y \) in \( X \), \( x \neq y \), and \( n \geq 3 \), the condition \((F)\) reduces to

\[
d(\mathbf{T}_n X, \mathbf{T}_n Y) \leq a(d(x, y)) \cdot d(x, y) + c(d(x, y)) \cdot [d(x, T_n y) + d(y, T_n x)].
\]

Now for \( x = 0 \) and \( y \in q \mathbb{N} \setminus \{1\} \), we deduce that

\[
q - 1 \leq a(q) \cdot q + c(q) \cdot [q - 1 + q] \leq a(q) \cdot q + 2c(q) \cdot q.
\]

This implies, since \( a \) and \( c \) are monotonically decreasing functions, that

\[
\frac{q - 1}{q} \leq a(1) + 2c(1).
\]

As \( q \to \infty \), we obtain \( 1 \leq a(1) + 2c(1) < 1 \), a contradiction which shows that the condition \((F)\) is not satisfied by \( \mathbf{T}_n \) for \( n \geq 3 \).

On the other hand, for any \( n \in \mathbb{N} \) the condition \((G)\) holds if we choose \( b(t) = c(t) = 0 \) for any \( t > 0 \) and \( a(t) = 1 - t/2 \) if \( 0 < t \leq 1 \), \( a(t) = 1 - t^2 \) if \( t > 1 \). The condition \((G)\) is obviously satisfied by \( \mathbf{T}_1 \) and \( \mathbf{T}_2 \). For \( n \geq 3 \) and \( x, y \) in \([0,1]\), \( x \neq y \), we get

\[
d(\mathbf{T}_n X, \mathbf{T}_n Y) = |x - y| \cdot \left[ 1 - \frac{n - 1}{2n - 3} \cdot (x + y) \right]
\]

\[
< |x - y| \cdot \left[ 1 - \frac{1}{2} (x + y) \right] \leq |x - y| \cdot \left[ 1 - \frac{1}{2} |x - y| \right]
\]

\[
= c(d(x, y)) \cdot d(x, y).
\]

Furthermore, if one of \( x, y \) lies in \( \mathbb{N} \setminus \{1\} \) with \( x \neq y \) and \( n \geq 3 \), then we have

\[
d(\mathbf{T}_n X, \mathbf{T}_n Y) = \mathbf{T}_n x + \mathbf{T}_n y \leq x + y - 1
\]

\[
= (x + y) \cdot \left[ 1 - 1/(x + y) \right]
\]

\[
+ c(d(x, y)) \cdot d(x, y).
\]

We now define \( \mathbf{T}(x) = x - x^2/2 \) if \( 0 \leq x \leq 1 \), \( \mathbf{T}(x) = x - 1 \) if \( x \) is in \( \mathbb{N} \setminus \{1\} \). Of course, \( z_n = 0 \) are the unique fixed points of \( \mathbf{T} \) and \( \mathbf{T}_n \), respectively and we have

\[
\lim_{n \to \infty} \mathbf{T}_n (x) = \mathbf{T}(x)
\]

for any \( x \) in \( X \). Thus the conclusion of Theorem 2 holds good since the sequence \( \{z_n\} \) converges to \( z \).

The idea of this example appears in [15].

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REFERENCES

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