1. INTRODUCTION. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be analytic in \( |z| < R \). For a non-decreasing sequence of positive numbers \( \{d_n\} \), the Gelfond-Leontev (G-L) derivative of \( f \) is defined as 
\[
Df(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1}.
\]

The \( k \)th iterate \( D^k f \), \( k=1,2,\ldots \), of \( D \) is given by
\[
D^k f(z) = \sum_{n=k}^{\infty} d_n \cdots d_{n-k+1} a_n z^{n-k}
\]
where, \( e_0 = 1 \) and \( e_n = (d_1 d_2 \cdots d_n)^{-1} \), \( n=1,2,\ldots \). If \( d_n \equiv n \), \( Df \) is the ordinary derivative of \( f \); whereas, if \( d_n \equiv 1 \), \( D \) is the shift operator \( L \) which transforms
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ into } Lf(z) = \sum_{n=1}^{\infty} a_n z^{n-1}.
\]

Let,
\[
\psi(z) = \sum_{n=0}^{\infty} e_n z^n
\]
and have radius of convergence \( R_0 \). From the monotonicity of \( \{d_n\} \), we have
\[
R_0 = \lim_{n \to \infty} d_n = \sup \{d_n\}.
\]
Clearly, \( \psi(0) = 1 \) and \( D\psi(z) = \psi(z) \). Thus, \( \psi(z) \) bears the same relationship to the operator \( D \) that the function \( \exp(z) \) bears to the ordinary differentiation.

For an entire function \( f \), Nachbin used the function \( \psi(z) \) as a comparison function for measuring the growth of maximum modulus of \( f \) on \( |z| = r \). Thus, the
growth parameter $\psi$-type of $f$ is defined as the infimum of the positive numbers $\tau$ such that, for sufficiently large $r$,

$$|f(z)| < N\psi(\tau r) \quad (1.4)$$

where, $\psi(z)$ is entire and $N$ is a positive constant. We denote $\psi$-type of $f$ as $\tau_\psi(f)$. It is known [2, p.6] that

$$\tau_\psi(f) = \lim_{n \to \infty} \sup \frac{a_n}{n^{1/n}} \quad (1.5)$$

For $d_n \equiv n$, the $\psi$-type of an entire function $f$ reduces to its classical exponential type and the formula (1.5) gives its well known coefficient characterisation [3, p. 11].

The comparison function $\psi(z)$ can also be used to define a measure of growth analogous to classical order [3, p.8] of an entire function. Thus, for an entire function $f$, let the $\psi$-order $\rho_\psi(f)$ of $f$ be defined as the infimum of positive numbers $\rho$ such that, for sufficiently large $r$,

$$|f(z)| < K\psi(r^\rho) \quad (1.6)$$

where $\psi(z)$ is entire and $K$ is a positive constant.

Shah and Trimble [4,5] showed that if $f$ is entire then, the assumption that the classical derivatives $f^{(n)}_p$ are univalent in $\Delta = \{z: |z| < 1\}$ for a suitable increasing sequence $\{n_p\}_{p=1}^\infty$ of positive integers affects the growth of the maximum modulus of $f$. If instead, we assume that the G-L derivatives $D_n^pf$ of an entire function $f$ are univalent in $\Delta$, then it is natural to enquire in what way the $\psi$-type and $\psi$-order of $f$ are influenced. The present paper is an attempt in this direction. In Theorem 1, we find that if $f$ is entire, $D_n^pf$ are univalent in $\Delta$ and

$$\lim_{p \to \infty} \sup (n_p - n_{p-1}) = \mu, \quad 1 < \mu < \infty,$$

then the $\psi$-type $\tau_\psi(f)$ of $f$ must satisfy

$$\tau_\psi(f) \leq 2(d(\mu+1)\ldots d(2))^{1/\mu}. \quad (1.7)$$

Further, if $\mu = \infty$, then $f$ need not be of finite $\psi$-type. Our Theorem 2 shows that if $f$ is entire, $D_n^pf$ are univalent in $\Delta$ and $n_p \sim n_{p+1}$ as $p \to \infty$, then

$$\rho_\psi(f) \leq \frac{1}{\log d(n_p-n_{p-1})} \left(1 - \lim_{p \to \infty} \sup \frac{\log d(n_p-n_{p-1})}{\log d(n_p)}\right). \quad (1.8)$$

It is clear that if $0 < \rho_\psi(f) < 1$, then the above inequality gives no relationship between $D_n^pf$ and the $\psi$-order of an entire function $f$. In fact, no such relation of this nature exists. This is illustrated in Theorem 3, wherein for any given
growth of entire functions with univalent derivatives

\[ \rho, 0 < \rho < 1, \text{ and any given increasing sequence } \{n_p\}_{p=1}^\infty \text{ of positive integers, we} \]

construct an entire function \( h \), of \( \psi \)-order \( \rho \), such that \( D P h \) is univalent in \( \Delta \) if and only if \( n = n_p \).

In the sequel, we shall assume throughout that \( d_n \to \infty \) as \( n \to \infty \).

2. \( \psi \)-type and exponents of univalent G-L derivatives.

**Theorem 1.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function and \( \{n_p\}_{p=1}^\infty \) be an increasing sequence of positive integers. Let \( D^n f \) be analytic and univalent in \( \Delta \). Suppose \( \lim_{p \to \infty} \sup_{p} (n_p - n_{p-1}) = \mu, 1 < \mu < \infty \). Then, the \( \psi \)-type \( \tau_{\psi} (f) \) of \( f \) satisfies

\[ \tau_{\psi} (f) < 2 (d(\mu + 1) \ldots d(2))^{1/\mu}. \]  

**Proof.** By the hypothesis,

\[ D^n f(z) = \sum_{k=0}^{\infty} d(n_k + k) a(n_k + k) z^k \]

are univalent in \( \Delta \). Since, for any function \( G(z) = b_0 + b_1 z + b_2 z^2 + \ldots \), univalent in \( \Delta \), it is known \([6]\) that \( |b_n| < n |b_1| \) for \( n = 2, 3, \ldots \), we get

\[ |a(n_k + k)| < k d_k \ldots d_{(k+1)} d(n_{p+1}) \ldots d(2) |a(n_{p+1})| \]

for \( k = 1, 2, \ldots \) and \( p = 2, 3, \ldots \). In particular, putting \( k = n_{p+1} - n_p + 1 \) and inducting upon \( p \), we get, for \( p > 2 \) and \( 2 < k < n_{p+1} - n_p + 1 \),

\[ |a(n_k + k)| < \frac{d_k \ldots d_{(k+1)}}{d_{n_{p+1}} \ldots d_{(2)}} p (n_{p+1} - n_p + 1) d(n_{p+1} - n_p + 1) \ldots d(2) \]

where \( A = d(n_p + 1) \ldots d(2) |a(n_{p+1})| \). Hence, for sufficiently large \( p \),

\[ \frac{a(n_k + k)}{e(n_k + k)} \left( \frac{1}{n_k + k} \right)^{1/\mu} \]

\[ < (1 + o(1)) d_k \ldots d_{(k+1)}^{1/\mu} p (n_{p+1} - n_p + 1) d(n_{p+1} - n_p + 1) \ldots d(2) \]

Since, \( d_k \ldots d_{(k+1)} \) is an increasing function of \( k \), and

\[ (n_{p+1} - n_p) < \mu', \mu' > \mu \], for sufficiently large \( p \),

\[ \frac{1}{n_k + k} \]

\[ < \frac{1}{(n_{p+1} - n_p + 1) \ldots d(1)} \left( \frac{1}{n_{p+1} - n_p + 1} + \frac{1}{n_{p+1}} \right) \]

\[ < \frac{1}{(n_{p+1} - n_p + 1) \ldots d(1)} (1 + o(1)) \]
Further [7], for \( p > 2 \)
\[
\frac{p}{\prod_{i=2}^{p} (n_i - n_{i-1} + 1)} \frac{1}{p(n+2)} < \left(1 + \frac{n_p}{p}\right) \frac{p}{n} < 2 .
\] (2.5)

Using (2.5) and the preceding inequality in (2.4), we get for sufficiently large \( p \),
\[
\frac{1}{p} \frac{a(n+k)}{e(n+k)} \frac{1}{(np+2)(d(n_1 - n_{1-1} + 1)\ldots d(2))} < 2(1+\alpha(1)) \frac{1}{(np+k)} .
\] (2.6)

Now, if \( a_j > 0, t_j > 0, \Sigma t_j > 0 \) and \( \max_{1 \leq j < N-1} \frac{a_j}{N} < \frac{a_N}{N} \) then clearly,
\[
\sum_{j=1}^{N} a_j t_j < \frac{a_N}{N} .
\] (2.7)

Further, \( \log(d(j+1)\ldots d(2))/j \) is an increasing function of \( j \) for \( 1 < j < \mu, \mu = 1,2,\ldots \). Thus, if \( 1 < j < \mu \),
\[
\frac{\log(d(j+1)\ldots d(2))}{j} < \log(d(\mu+1)\ldots d(2))
\] (2.8)

Let \( p > p_0, 1 < \gamma < \mu \). Suppose \( t_\gamma \) is the number of \( j_i \)'s in \([p_0,p]\) such that
\[
n_j + n_j = \gamma \text{ for } j = j_i .\text{ Then, by (2.7) and (2.8),}
\]
\[
\sum_{\gamma=1}^{\mu} t_\gamma \log(d(\gamma+1)\ldots d(2)) = \frac{p}{p_0+1} \frac{1}{(n_j - n_{j-1})} \sum_{\gamma=1}^{\mu} t_\gamma (\log(d(\gamma+1)\ldots d(2))
\] (2.9)

The above inequality implies that
\[
\prod_{i=2}^{p} (d(n_i - n_{i-1} + 1)\ldots d(2)) \frac{1}{p} < \exp \left\{ \frac{p}{p_0+1} \frac{1}{(n_j - n_{j-1})} \sum_{\gamma=1}^{\mu} t_\gamma (\log(d(\gamma+1)\ldots d(2))
\] (2.10)

Using the estimate (2.9) in (2.6) and proceeding to limits
\[
\lim_{k+\infty} \frac{a_k}{e_k} = \lim_{k+\infty} \left\{ \frac{a(n+k)}{e(n+k)} \frac{1}{p} : 2 < k < n_{p+1} - n_{p+1}, \ p > 2 \right\} < 2(d(\mu+1)\ldots d(2))^{1/\mu}.
\]

This completes the proof of the theorem.
REMARK 1. In Theorem 1, it is sufficient to take the function \( f \) to be analytic in \( |z| < R \), for some \( R, 0 < R < \infty \), if the sequence \( \{d_n\}_{n=1}^{\infty} \) in the definition of G-L derivative of \( f \) satisfies the condition \( \lim_{m \to \infty} \left( \frac{1}{m} \log d(i)/m \right) = 0 \). In fact, for an analytic function \( f \) in \( |z| < R \), if \( D^p f \) are univalent in \( \Delta \),

\[
\lim_{p \to \infty} \sup_{n \geq p} \left( \frac{n - n_{p-1}}{p} \right) = \mu, \quad 1 < \mu < \infty, \text{ and }
\lim_{m \to \infty} \frac{1}{m} \sum_{i=2}^{m} \log d(i) = 0
\]

holds, then \( f \) is necessarily entire. To see this, we use (2.5) and

\[
(d_k \ldots d_1)^{1/(n+k)} \leq 1 + o(1)
\]

for sufficiently large \( p \) in (2.3) to get

\[
|a(n+k)|^{1/(n+k)} \leq 2(1+o(1)) \exp \left( \frac{1}{n_p} \sum_{i=2}^{n_p} \log(d(n_i - n_{i-1} + 1) \ldots d(2)) \right)
\]

for sufficiently large \( p \). But since, for sufficiently large \( p \), \( (n_p - n_{p-1}) < \mu' \), \( \mu' > \mu \),

\[
\sum_{i=2}^{n_p} \log(d(n_i - n_{i-1} + 1) \ldots d(2)) \to 0 \text{ as } p \to \infty.
\]

Thus, by (2.10) and the condition \( \lim_{m \to \infty} \left( \frac{1}{m} \log d(i)/m \right) = 0 \),

\[
\lim_{k \to \infty} \sup_{k \leq n} \left( \frac{1}{k} \right) = \lim_{k \to \infty} \sup_{k \leq n} \left( \frac{1}{k} \right) = 0.
\]

REMARK 2. The inequality (2.1) can be improved by imposing suitable additional restrictions on the sequence \( \{d_n\}_{n=1}^{\infty} \). For example, let the sequence \( \{d_n\}_{n=1}^{\infty} \) be such that

\[
\frac{(d(n+2))^n}{d(n+1) \ldots d(2)} > \frac{2}{3(n+1)}, \ n=1,2,3, \ldots.
\]  

Note that (2.11) is satisfied for \( d_n = n^\alpha, \alpha > 1 \).

Because of (2.11), the function \( s(j) \) defined by

\[
s(j) = \frac{\log(d(j+1) \ldots d(2)) + \log(j+1)}{j}
\]

is an increasing function of \( j \) and so for \( j=1,2,3, \ldots; \mu=1,2, \ldots \).
Let $t_{\gamma}$ be the same as in the proof of Theorem 1. Using (2.7) and (2.12), we get

$$
\frac{\log(d(j+1)\ldots d(2))+\log(j+1)}{\mu} < \frac{\log(d(\gamma+1)\ldots d(2))+\log(\gamma+1)}{\mu}.
$$

(2.12)

Again, we have

$$
\frac{\log(d(n_{i-1}-n_{i-2}+1)d(n_{i-1}-n_{i-2}+1)\ldots d(2))}{\mu} < \exp \left\{ o(1) + \frac{1}{\mu} \log(d(n_{i-1}-n_{i-2}+1)\ldots d(2)) \right\}.
$$

The above inequality, when employed in (2.4), gives

$$
\frac{\log(d(n_{i-1}-n_{i-2}+1)d(n_{i-1}-n_{i-2}+1)\ldots d(2))}{\mu} < \exp \left\{ o(1) + \frac{1}{\mu} \log(d(n_{i-1}-n_{i-2}+1)\ldots d(2)) \right\}.
$$

Now, on proceeding to limits, we get

$$
\tau_{\psi}(f) < \frac{1}{\mu}(d(\mu+1)\ldots d(2))^{1/\mu}.
$$

(2.13)

It is clear that the bound on $\tau_{\psi}(f)$ in (2.13) is better than that in (2.1).

REMARK 3. By taking $\mu=1$, Theorem 1 gives $\tau_{\psi}(f) = 2d(2)$, a result recently proved in [8].

Theorem 1 shows that if $(n_{p+1}-n_{p}) = O(1)$, then $f$ is of finite $\psi$-type.

We now give an example to show that if $\lim_{p \to \infty} \sup_{p} (n_{p+1}-n_{p}) = \infty$, then $f$ need not be of finite $\psi$-type.

EXAMPLE. Let $\{n_{p}\}_{p=1}^{\infty}$ be an increasing sequence of positive integers such that $n_{p+1} - n_{p} > 2$ for all $p$. Further, assume that the sequence $\{d_{n}\}_{n=1}^{\infty}$ is such that

(i) $d_{1} = 1$ and $\log d(n) \sim \log n$ as $n \to \infty$.

(ii) $n_{p} = o(n_{p})$.

(iii) $n_{p} = o(n \log d(n))$.
where, \( n_p = \sum_{i=2}^{p} \log(d(n_i-n_{i-1}+1)...d(2)) \).

Let \( \psi \) be a non-decreasing step function such that \( \psi(n_1) = \psi(n_2) \),

\[
\psi(n_p) = \frac{\exp(n_p)}{2^{p-1}}, \quad p > 2
\]

and

\[
\psi(x) = \psi(n_p) \quad n < x < n_{p+1}.
\]

Let

\[
g_{j+1}(n) = \begin{cases} \frac{\psi(j)}{d(j+1)...d(2)} (j-n_p+p+1) & \text{if } j = n_p \text{ for some } p \\ 0 & \text{otherwise.} \end{cases}
\]

Define

\[
g(z) = \sum_{j=0}^{\infty} g_j z^j
\]

We first show that \( g \) is an entire function. We have

\[
\lim_{k \to \infty} \sup_{1/k} |g_k|^{1/k} = \lim_{p \to \infty} \frac{\psi(n_p)}{d(n_p+1)...d(2)}^{1/n_p+1}
\]

\[
< \limsup_{p \to \infty} \frac{\exp(n_p/n_p)}{(d(n_p+1)...d(2))^{1/n_p+1}}
\]

\[
= \lim_{p \to \infty} \exp(\frac{n_p}{n_p} - \frac{1}{n_p+1} \sum_{i=2}^{n_p} \log d(i))
\]

Since \( \log d(n) \sim \log n \) as \( n \to \infty \), using the condition (iii), we get from the above inequality that

\[
\lim_{k \to \infty} \sup_{1/k} |g_k|^{1/k} = 0.
\]

Hence \( g \) is entire. It is easily seen that \( g \) is of order 1. But, by the condition (ii),

\[
\lim_{k \to \infty} \sup_{1/k} |g_k|^{1/k} = \lim_{p \to \infty} \frac{\psi(n_p)}{\exp(n_p/n_p)} \left( \frac{1}{d(n_p+1)...d(2)}^{1/n_p+1} \right)
\]

Thus, \( f \) is not of finite \( \psi \)-type. It remains to see that

\[
D^n g(z) = \sum_{k=1}^{\infty} d(n_p+k+1)...d(n_p+k-n_p+2) z^{n_{p+k}-n_p+1}
\]

are univalent in \( \Delta \). To this end, it is enough to prove that

\[
\left| \sum_{k=1}^{\infty} \frac{d(n_p+k+1)...d(2)}{d(n_p+k-n_p+1)...d(2)} |a(n_p+k+1)| \right|
\]
or, equivalently to show that
\[
\psi(n^p_{p+k}) \leq \frac{1}{k!} \frac{\exp(n^p_{p+k}-n^p_{p})}{d(n^p_{p+k}-n^p_{p}+1)\ldots d(2)} < \psi(n^p_{p})
\]

Using the definition of \( \psi \), the last inequality reads as
\[
\psi(n^p_{p+k}) \leq \frac{1}{k!} \frac{\exp(n^p_{p+k}-n^p_{p})}{d(n^p_{p+k}-n^p_{p}+1)\ldots d(2)} < 1.
\]

Now, an induction on \( k \), gives, for \( k=1,2,3,\ldots \)
\[
\exp(n^p_{p+k}-n^p_{p}) \leq \prod_{i=1}^{p+k} d(n^p_{p+k}-n^p_{p}+1)\ldots d(2) < d(n^p_{p+k}-n^p_{p}+1)\ldots d(2)
\]

Hence, (2.14) is clearly satisfied.

3. \( \psi \)-ORDER AND EXPONENTS OF UNIVALENT G-L DERIVATIVES.

A function \( S(x) \), continuous on \([1,\infty)\), is said to be Slowly Oscillating (S.O.) if for every positive number \( c > 0 \),
\[
\lim_{x \to \infty} \frac{S(cx)}{S(x)} = 1.
\]

A function \( H(n) \) is said to be the restriction of a Slowly Oscillating function \( S(x) \) if \( S(n) = H(n) \) for every positive integer \( n \). It is known \([9]\) that, as \( k \to \infty \)
\[
L \sum_{i=1}^{k} H(i) \sim kH(k).
\]

THEOREM 2. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function of \( \psi \)-order \( \rho_\psi \) and \( \{n_p\}_{p=1}^{\infty} \) be a strictly increasing sequence of positive integers. Let \( D^{n_p}_p f \) be analytic and univalent in \( \Delta \), such that \( n_p \sim n_{p+1} \) as \( p \to \infty \). If \( \log d(n) \) is the restriction of a slowly oscillating function on integers, then
\[
\rho_\psi(f) \leq \frac{1}{\limsup_{p \to \infty} [\log d(n^p_{p+p-1})]}.
\]

We need the following lemmas.

**Lemma 1.** Let \( \gamma \) be defined by (1.3). Let \( \gamma_n = \min_{x > 0} \psi(x^a) x^{-n}, a > 0 \).

Then,
\[
\gamma_n \leq e^{-\frac{n(1 - \frac{1}{a})}{a} (\frac{a(n+a)}{a})}
\]

**Proof.** Since \( \{d^{n-p}_{n}\}_{n=1}^{\infty} \) is increasing, we note that for any pair of integers \( k \) and \( n \), \( e_k \leq d^{n-k}_{n} \). Thus,
Let $0 < w < 1$. Setting $x = w^{1/a}$, we get

$$
\psi(x^n) = e^{\frac{w}{a} x} \leq e^{\frac{1}{a} x} \leq e^{\frac{1}{a} w^{1/a}}.
$$

Choosing $w = (n/n+a)^{1/a}$ to minimize the right-hand side of the above inequality, we have

$$
\psi(x^n) \leq e^{\frac{1}{a} (n/a)} (e(n+a)).
$$

**Lemma 2.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of $\psi$-order $\rho_\psi$, where the sequence $\{d(n)\}$ in $Df$ is such that $\log d(n)$ is the restriction of a slowly oscillating function on positive integers.

Then,

$$
\rho_\psi(f) = \limsup_{n \to \infty} \frac{n \log d(n)}{\log |a_n|}.
$$

**Proof.** By Cauchy's inequality, we get

$$
|a_n| \leq M(r) r^{-n}, \quad M(r) = \max_{|z| < r} |f(z)|.
$$

Since $f$ is of $\psi$-order $\rho_\psi(f) = \rho$, for any $\varepsilon > 0$, $|f(z)| < M(r^{\rho+\varepsilon})$.

So that

$$
|a_n| < M(r^{\rho+\varepsilon}) r^{-n}.
$$

Using Lemma 1, we have

$$
|a_n| < M e^{\frac{n(1-a)}{a}} (e(n+a)).
$$

But, since $\log d(n)$ is the restriction of a S.O. function, by (3.1),

$$
\sum_{n=0}^{\infty} \log d(n) = n \log d(n) \text{ as } n \to \infty. \text{ Thus, it follows from (3.5)}
$$

$$
\limsup_{n \to \infty} \frac{n \log d(n)}{-\log |a_n|} < \rho.
$$

To prove that equality holds in (3.4), suppose that

$$
\limsup_{n \to \infty} \frac{n \log d(n)}{-\log |a_n|} < \rho.
$$

Then, there exist $\rho_1 < \rho$ such that $|a_n|^{1/\rho_1} \leq \varepsilon_n$ for $n > n_0$. It now follows that, for $|z| = r$,

$$
|f(z)| < \sum_{n=0}^{n_0} |a_n| r^n + \sum_{n=n_0+1}^{\infty} |a_n| r^n
$$

$$
< 0(1) + \sum_{n=n_0+1}^{\infty} \varepsilon_n^{1/\rho_1} r^n.
$$

\[\_]
Choose

\[ N(r) = \frac{\log \psi(r)}{\log r} \]

It is easily seen that \( N(r) \to \infty \) as \( r \to \infty \). Since for all values of \( k \) and \( n \),
\[ e_n < e_k d_k^{k-n} \]
we have
\[ \sum_{n=0}^{\infty} e_n^{1/\rho_1} r^n < \sum_{n=0}^{\infty} e_k d_k^{k-n/\rho_1} r^n \]
\[ = \frac{1/\rho_1}{d_k} e_k d_k^{k-n/\rho_1} \left( -\frac{r}{1/\rho_1} \right)^n. \]

Let \( k \) be chosen such that \( (r/d_k^{1/\rho_1}) < 1 \). Then,
\[ \sum_{n=0}^{\infty} e_n^{1/\rho_1} r^n < \frac{d_k^{k+1/\rho_1}}{1/\rho_1} e_k^{1/\rho_1} \left( \frac{r}{1/\rho_1} - r \right) \]

Since the left hand side of (3.7) is independent of \( k \), letting \( k \to \infty \), we get
\[ \sum_{n=0}^{\infty} e_n^{1/\rho_1} r^n < 1. \]

Thus
\[ \sum_{n=N(r)}^{\infty} e_n^{1/\rho_1} r^{n-o(1)} = \psi(r^{\rho_1}) \]

Since, \( r^N(r) = \exp(N(r)\log r) = \psi(r^{\rho_1}) \), it now follows from (3.6)
\[ |f(z)| < 0(1) + \sum_{n=0}^{N(r)} e_n^{1/\rho_1} r^n + o(1) \]
\[ < 0(1) \psi(r^{\rho_1}). \]

Since \( \rho_1 < \rho \) and \( \rho \) is the \( \psi \)-order of \( f \), the above inequality contradicts the definition of \( \psi \)-order. Thus, equality must hold in (3.4). This proves the lemma.

**PROOF OF THEOREM 2.** Since \( D \) are univalent in \( \Delta \), from (2.2), we get for sufficiently large \( p \) and \( 2k<n+1-p \).
\[ |a(n+k)|^{1/(n+k)} \leq (1+o(1)) \left( \frac{d_k \ldots d_1}{d_{k+n} \ldots d_1} \right)^{1/(n+k)} \sum_{l=2}^{\infty} \left( (n_1-n_1-1)d(n_1-n_1-1)+d(2) \right)^{1/(n+k)} \]

Further, we have
\[
\frac{1}{(n+k)} \leq (\frac{d_{p+1}}{d_1})^{p} < (\frac{d_{n+p}}{d_1})^{p+1}
\]

and

\[
-\frac{1}{(n+k)} \leq (\frac{d_{p+1}}{d_1})^{p} < (\frac{d_{n+p}}{d_1})^{p+2}.
\]

Using these inequalities, (2.5) and (3.8), it follows that, for sufficiently large \( p \),

\[
|a(n+k)|^{\frac{1}{(n+k)}} \leq \frac{2(1+\omega(1))}{1/n} \left( \log \frac{d_{n+p}}{d_1}\right)^{1/n} \frac{n_{n+1}}{n_{n+1}^{p+1}}
\]

Let,

\[ M_p = \max \{ \log d_{n_i-n_{i-1}+1}: 2 \leq i \leq p \}. \]

Since \( \log d(n) \) is the restriction of a slowly oscillating function on integers, by (3.1)

\[
\frac{1}{n} \left[ \sum_{i=2}^{n_p} (n_i-n_{i-1}) \log d_{n_i-n_{i-1}+1} - \sum_{i=1}^{n_p-1} \log d_i \right]
\]

\[
\leq \frac{n_{p+1}}{n_p} \frac{M_p}{p+1} \log d_{n_p}.
\]

Consequently, for sufficiently large \( p \),

\[
\frac{(n+k) \log d_{n+k}}{\log d_{n+p+1}} < \frac{\log d_{n+p+1}}{-\log |a(n+k)|} \log d_{n_p} - \frac{n_{p+1}}{n_p} \frac{M_p}{p+1} \log 2.
\]

Again, from the definition of S.O. function \( \log d_{n_p} \sim \log d_{n_p+1} \) as \( p \to \infty \).

Hence,

\[
\rho \psi < \frac{1}{1 - \lim \sup_{p \to \infty} \frac{M_p}{\log d_{n_p}}}.
\]

If \( M_p \) is bounded, there is nothing to prove. So, let \( M_p \to \infty \) as \( p \to \infty \).

For \( p > 2 \), let,

\[
A_p = \frac{\log d_{n_p-n_{p-1}+1}}{\log d_{n_p}}
\]

and

\[
B_p = \frac{M_p}{\log d_{n_p}}.
\]

But as \( M_p = \max \{ \log d_{n_i-n_{i-1}+1}: 2 \leq i \leq p \} \), for each \( p > 2 \), there is some
q_p^p, q_p < p such that \( M_p = \log d(n_p - n_{p-1}) \). Hence

\[ B < A. \]

Taking \( q_p \rightarrow \infty \),

\[ \limsup_{p \to \infty} B < \limsup_{p \to \infty} A. \]

Now (3.2) follows from (3.10).

**COROLLARY.** Suppose the conditions of Theorem 2 are satisfied. If as \( p \to \infty \),

\[ \log d(n_p - n_{p-1}) = o(\log d(n_p)) \]

then,

\[ \rho_{\psi}(f) < 1. \]

**THEOREM 3.** Let \( 0 < \rho < 1 \). Let \( \{n_p\}_{p=1}^{\infty} \) be a strictly increasing sequence of non-negative integers. Then, there is an entire function \( h \) of \( \psi \)-order \( \rho \) such that \( D^n h \) is univalent in \( \Delta \) if and only if \( n = n_p \) for some \( p \).

**PROOF.** Suppose \( \rho > 0 \) and \( \{d_n\}_{n=1}^{\infty} \) is an increasing sequence of positive numbers such that \( \log d(n) \) is the restriction of a slowly oscillating function on integers and \( d_1 = 1 \). Let,

\[ h_{j+1} = \begin{cases} \frac{1}{2^p d(n_p + 1) \cdots d(2) (j-n_p + 1)} & \text{if } j = n_p \text{ for some } p \\ 0 & \text{otherwise.} \end{cases} \]

Define, \( h(z) = \sum_{j=0}^{\infty} h_j z^j \). Then, \( h(z) \) is an entire function and

\[ \rho_{\psi}(h) = \limsup_{k \to \infty} \frac{k \log d(k)}{-\log |h_k|} = \limsup_{p \to \infty} \frac{(n_p + 1) \log d(n_p + 1)}{p \log 2 + \frac{1}{\rho} \log(d(n_p) \cdots d(2))} = \rho, \]

To show that \( D^n h \) given by

\[ D^n h(z) = \sum_{k=0}^{\infty} (n_p + k - n_p + 1) \frac{d(n_p + k + 1) \cdots d(2)}{d(n_p + k - n_p + 1) \cdots d(2)} h(n_p + k + 1) \frac{n_p + k - n_p + 1}{z} \]

is univalent in \( \Delta \), it is enough to prove that

\[ \sum_{k=1}^{n_p} (n_p + k - n_p + 1) \frac{d(n_p + k + 1) \cdots d(2)}{d(n_p + k - n_p + 1) \cdots d(2)} |h(n_p + k + 1)| < d(n_p + 1) \cdots d(2) |h(n_p + 1)|. \]
Since $p < 1$,
\[ \sum_{k=1}^{m} \frac{d(n_{p+k}+1) \ldots d(2)}{d(n_{p+k} - n_{p}+1) \ldots d(2)} \left| h(n_{p+k}+1) \right| \]
\[ \leq \frac{1}{2^p} \sum_{k=1}^{m} \frac{1 - \frac{1}{p}}{d(n_{p+k} - n_{p}+1) \ldots d(2)} \]
\[ \leq \frac{1}{2^p} \left( d(n_{p}+1) \ldots d(2) \right) \sum_{k=1}^{m} \frac{1}{2^k} \]
\[ = d(n_{p}+1) \ldots d(2) \left| h(n_{p}+1) \right|. \]

As $D^{n+1}h(0) = 0$ unless $n=n_p$ for some $p$, only $D^n h_p$ are univalent in $\Delta$.

If $p=0$, then take $h_{j+1}^{*}$ defined by
\[ h_{j+1}^{*} = \begin{cases} \frac{1}{2^p d(n_{p}+1) \ldots d(2) (j-n_{p}+1)} & \text{if } j=n_p \text{ for some } p, \\ 0 & \text{otherwise}. \end{cases} \]

in place of $h_{j+1}$ in the Taylor series of the function $h(z)$.

REFERENCES


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