EIGENFUNCTION EXPANSION FOR A REGULAR FOURTH ORDER EIGENVALUE PROBLEM WITH EIGENVALUE PARAMETER IN THE BOUNDARY CONDITIONS

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INTRODUCTION.

The regular right-definite eigenvalue problems for second order differential equations with eigenvalue parameter in the boundary conditions, have been studied in Walter [1], Fulton [2] and Hinton [3].

The object of this paper is to prove the expansion theorem for the following regular fourth order eigenvalue problem:

\[ u: \begin{align*}
   \begin{cases}
   (Ku'')'' - (Pu')' + qu = \lambda u, & x \in [a, b] \\
   u(a) = (Pu')(a) = (Ku'')(a) = 0 \\
   (Ku'')(b) = (Pu')(b) = -\lambda u(b)
   \end{cases}
\end{align*} \tag{1.1} \]

where P, q and K are continuous real-valued functions on [a, b]. We assume that P(x) > 0, q(x) > 0, and K(x) > 0 while \( \lambda \) is a complex number.

Recently, Zayed [4] has studied the special case of the problem (1.1) wherein \( K(x) = \alpha^2 \), \( \alpha^2 \) is a constant and \( q(x) = 0 \).

Further, problem (1.1), in general, describes the transverse motion of a rotating beam with tip mass, such as a helicopter blade (Ahn [5]) or a bob pendulum suspended from a wire (Ahn [6]).
Ahn [7] has shown that the set of eigenvalues of problem (1.1) is not empty, has no finite accumulation points and is bounded from below. He used an integral-equation approach.

In this paper, our approach is to give a Hilbert space formulation to the problem (1.1) and self-adjoint operator defined in it such that (1.1) can be considered as the eigenvalue problem of this operator.

2. HILBERT SPACE FORMULATION.

We define a Hilbert space $H$ of two-component vectors by

$$H = L^2(a,b) \oplus \mathbb{C};$$

with inner product

$$\langle f, g \rangle = \int_a^b \overline{f_1} g_1 \, dx + \overline{f_2} g_2, \quad f, g \in H$$

and norm

$$\| f \|_H^2 = \int_a^b |f_1|^2 \, dx + |f_2|^2$$

where

$$f = (f_1, f_2) = (f_1(x), f_1(b)) \in H$$

and

$$g = (g_1, g_2) = (g_1(x), g_1(b)) \in H.$$

We can define a linear operator $A: D(A) \rightarrow H$ by

$$Af = (\tau f_1, -(Kf_1''(b)) + (Pf_1')(b)) \quad \forall f = (f_1, f_2) \in D(A)$$

where the domain $D(A)$ of $A$ is a set of all $f = (f_1, f_2) \in H$ which satisfy the following:

(i) $f_1$, $f_1'$, $f_1''$ and $f_1'''$ are absolutely continuous with

$$\tau f_1 \in L^2(a,b) \text{ and } \int_a^b (K|f_1''|^2 + P|f_1'|^2 + q|f_1|^2) \, dx < \infty.$$

(ii) $f_1(a) = (Pf_1')(a) = (Kf_1')(a) = 0$

(iii) $f_2 = f_1(b)$.

REMARK 2.1. The parameter $\lambda$ is an eigenvalue of (1.1) and $f_1$ is a corresponding eigenfunction of (1.1) if and only if

$$f = (f_1, f_1(b)) \in D(A) \quad \text{and} \quad Af = \lambda f$$

Therefore, the eigenvalues and the eigenfunctions of problem (1.1) are equivalent to the eigenvalues and the eigenfunctions of operator $A$. 


We consider the following assumptions:

1. \[ \lim_{x \to b} [K'(x)f_1(x) - K(x)f_1'(x)] = 0, \tag{2.5} \]
2. \[ \lim_{x \to b} [K'(x)g_1(x) - K(x)g_1'(x)] = 0. \]

**Lemma 2.1.** The linear operator \( A \) in \( H \) is symmetric.

**Proof.** On using the boundary conditions of (1.1) we get,

\[
\langle Af, g \rangle = \int_a^b (tf_1)g_1 \, dx + [-(Kf_1')'](b) + (Pf_1')(b)g_1(b) \\
= \int_a^b (Kf_1'')g_1 \, dx - \int_a^b (Pf_1')g_1 \, dx + \int_a^b qf_1g_1 \, dx - (Kf_1')(b)g_1(b) \\
+ (Pf_1')(b)g_1(b) \tag{2.6} \]

Integrating the first term of (2.6) by parts four times and integrating the second term of (2.6) by parts twice, we get

\[
\langle Af, g \rangle = \int_a^b f_1[(\overline{Kg_1}'')] - (\overline{Pg_1}')' + qg_1 \, dx + f_1(b)[-(\overline{Kg_1}'')(b) + (\overline{Pg_1})'(b)] \\
+ f_1''(b) [K'(b)\overline{g_1}'(b) - K(b)\overline{g_1}'(b)] - \overline{g_1}'(b) [K'(b)f_1(b) - K(b)f_1'(b)] \]

Applying the conditions (2.5) and using the boundary conditions of (1.1), we obtain

\[
\langle Af, g \rangle = \int_a^b f_1(\overline{g_1}) \, dx + f_1(b) [-\overline{(Kg_1'')} + (\overline{Pg_1})'] = \langle f, Ag \rangle. \]

**Remark 2.2.** For all \( f, f' \) in \( D(A) \) and \( f_2 = f_1(b) \neq 0 \), the domain \( D(A) \) is dense in \( H \).

Since the operator \( A \) in \( H \) is symmetric and dense in \( H \), \( A \) is self-adjoint.

3. The Boundedness.

We shall show that the self-adjoint operator \( A \) is unbounded from above and bounded from below. We also show that \( A \) is strictly positive.

**Lemma 3.1.**

(i) If \( f, f' \) are absolutely continuous with \( f(a) = 0 \) and \( P(x) > 0 \) in \([a, b]\), then we have \( P(x) > c_1 \) for some constant \( c_1 > 0 \) such that

\[
\int_a^b P(x)|f'(x)|^2 \, dx > c_1|f(b)|^2. 
\]

(ii) For \( f \in C^2[a, b] \), there exists a positive constant \( c_2 \) such that

\[
\int_a^b |f(x)|^2 \, dx < c_2 \int_a^b |f''(x)|^2 \, dx. 
\]
PROOF.

(i) Since $P(x) > 0$ on $[a,b]$, we have $P(x) > c_1$ for some $c_1 > 0$.

Consequently, on using Schwartz's inequality, we get

$$\int_a^b P(x)|f'(x)|^2 \, dx > c_1 \int_a^b |f'(x)|^2 \, dx > c_1 \left( \int_a^b |f'(x)| \, dx \right)^2 > c_1 |f(b)|^2$$

where $\int_a^b f'(x) \, dx = f(b) - f(a) = f(b)$, since $f(a) = 0$.

(ii) By using Theorem 2 in [8, p.67], we have for $f(x) \in C^1[a,b]$,

$$\int_a^b |f(x)|^2 \, dx < 4(b-a)^2 \int_a^b |f'(x)|^2 \, dx$$

Since $\left| \frac{d|f(x)|}{dx} \right| \leq 4 \frac{df(x)}{dx}$,

then

$$\int_a^b |f(x)|^2 \, dx < 4(b-a)^2 \int_a^b \left| \frac{d|f(x)|}{dx} \right|^2 \, dx < 16(b-a)^2 \int_a^b |f'(x)|^2 \, dx \quad (3.1)$$

Applying (3.1) again for $|f'(x)|$, we get

$$\int_a^b |f'(x)|^2 \, dx < 16(b-a)^2 \int_a^b |f''(x)|^2 \, dx \quad (3.2)$$

from (3.1) and (3.2) we get

$$\int_a^b |f(x)|^2 \, dx < c_2 \int_a^b |f''(x)|^2 \, dx$$

where the constant $c_2 = 256(b-a)^4$.

LEMMA 3.2. The linear operator $A$ is bounded from below.

PROOF. On using the boundary conditions of (1.1) we get

$$\langle Af, f \rangle = \int_a^b (\gamma f_1 \overline{f_1}) \, dx + \left[ -(Kf''_1)(b) + (Pf'_1)(b) \right] f_1(b)$$

$$= \int_a^b (Kf''_1) \overline{f_1} \, dx - \int_a^b (Pf'_1) \overline{f_1} \, dx + \int_a^b qf_1 \overline{f_1} \, dx - (Kf''_1)(b)f_1(b)$$

$$+ (Pf'_1)(b)f_1(b). \quad (3.3)$$

Integrating (3.3) by parts twice and using the boundary conditions of (1.1), we obtain

$$\langle Af, f \rangle = f''_1(b) [K'(b)f_1(b) - K(b)f''_1(b)] + \int_a^b Kf''_1 \, dx$$

$$+ \int_a^b Pf_1^2 \, dx + \int_a^b qf_1 \, dx.$$

On using (2.5) (ii) and lemma (3.1), we get

$$\langle Af, f \rangle > \int_a^b \frac{K(x)}{c_2} |f_1(x)|^2 \, dx + c_1 |f_1(b)|^2 + \int_a^b q(x)|f_1(x)|^2 \, dx$$

$$= \int_a^b \frac{K(x)}{c_2} + q(x) |f_1(x)|^2 \, dx + c_1 |f_2|^2.$$
where
\[ c_3 = \inf_{x \in [a, b]} \frac{K(x)}{c_2} + q(x) \]

Therefore
\[ \langle Af, f \rangle > c \quad \|f\|^2 \quad (3.4) \]

where the constant \( c = \min (c_3, c_1) \).

It follows, from (3.4), that the operator \( A \) is bounded from below.

Since \( c_1 > 0, K(x) > 0, q(x) > 0, c_2 > 0 \) and \( c = \min (c_3, c_1) \) then the constant \( c \) is positive \( (c > 0) \) and hence \( A \) is strictly positive.

REMARK 3.1.
(i) Since \( A \) is a symmetric operator (from lemma 2.1) then \( A \) has only real eigenvalues.
(ii) By Lemma 3.2, we deduce that the set of all eigenvalues of \( A \) is also bounded from below.
(iii) Since \( A \) is strictly positive, then the zero is not an eigenvalue of \( A \).

By using theorem 3 in [8, p.60] we can state that:

Since \( A \) is symmetric and bounded from below, then for every eigenvalue \( \lambda_1 \) of \( A \) in \( H \), \( \lambda_1 > c \) where the constant \( c \) is the same as in (3.4). This means that \( 0 < c < \lambda_1 < \lambda_2 < \ldots < \lambda_1 \) according to the size and \( \lambda_1 \to +\infty \) as \( i \to \infty \).

This implies that the set of all eigenvalues of \( A \) is unbounded from above.

REMARK 3.2. Since the operator \( A \) is self-adjoint, then \( A \) has only real eigenvalues and the eigenfunctions of \( A \) are orthonormal. By using theorem 3 in [8, p.30], the density of the domain \( D(A) \) in \( H \) gives us the completeness of the orthonormal system of eigenfunctions \( \phi_1, \phi_2, \phi_3, \ldots \) of \( A \).

4. THE EIGENFUNCTIONS OF THE OPERATOR \( A \).

We suppose \( \phi_\lambda(x), \psi_\lambda(x), \chi_\lambda(x) \) and \( \gamma_\lambda(x) \), where \( \lambda \in \mathbb{C} \) is not an eigenvalue of \( A \), are the fundamental set of solutions of the fourth order differential equation of (1.1) with the initial conditions:

\[
\begin{align*}
\phi_\lambda(a) &= 0, & (P\phi_\lambda')(a) &= 0, & \phi''(a) &= 1, & (K\phi''')(a) &= 0 & (4.1) \\
\psi_\lambda(a) &= 0, & (P\psi_\lambda')(a) &= 0, & \psi''(a) &= 0, & (K\psi''')(a) &= 1 & (4.2) \\
\chi_\lambda(b) &= 0, & (P\chi_\lambda')(b) &= 1, & \chi''(b) &= 0, & (K\chi''')(b) &= 1 & (4.3) \\
\gamma_\lambda(b) &= 1, & (P\gamma_\lambda')(b) &= 1+\lambda, & \gamma''(b) &= 0, & (K\gamma''')(b) &= 1 & (4.4) 
\end{align*}
\]

Therefore the Wronskian is
\[
W = \lim_{x \to b} [ \chi_\lambda(x)(P\chi_\lambda')(x) - (P\chi_\lambda')(x)\gamma_\lambda(x) ] = -1 \neq 0
\]

Thus the solutions $C_k(x), \psi_k(x), \chi_k(x)$ and $\gamma_k(x)$ are linearly independent of $\tau u = \lambda u$. Putting $x = b$, we obtain the Wronskian in the form:

$$W = \psi''(b) [\lambda \phi'(b) - (p\phi')(b) + (K\phi'')(b)]$$

$$- \phi''(b) [\lambda \psi'(b) - (p\psi')(b) + (K\psi'')(b)] \neq 0 \quad (4.5)$$

Now, for $f = (f_1, f_2) \in H$, we define $\Phi = (\Phi_1, \Phi_2) \in D(A)$ as the unique solution of $(\lambda I - A)\Phi = f$.

Application of variation of parameter method yields the unique solution $\Phi \in D(A)$ of $(\lambda I - A)\Phi = f$, $f \in H$ with:

$$(\lambda I - \tau) \Phi_1 = f_1 \quad (4.6)$$

Therefore

$$\Phi_1(x) = \int_a^b \frac{[\lambda \phi'(c) + \psi'(c) + \gamma(c)]}{W} f_1(c) dc$$

where

$$a_1(c) = \frac{-P(c)}{K(c)} \begin{vmatrix} \psi_\lambda(c) & \chi_\lambda(c) & \gamma_\lambda(c) \\ \phi_\lambda(c) & \chi_\lambda(c) & \gamma_\lambda(c) \\ \psi_\lambda''(c) & \chi_\lambda''(c) & \gamma_\lambda''(c) \end{vmatrix}$$

$$a_2(c) = \frac{P(c)}{K(c)} \begin{vmatrix} \phi_\lambda(c) & \chi_\lambda(c) & \gamma_\lambda(c) \\ \phi_\lambda'(c) & \chi_\lambda'(c) & \gamma_\lambda'(c) \\ \phi_\lambda''(c) & \chi_\lambda''(c) & \gamma_\lambda''(c) \end{vmatrix}$$

$$a_3(c) = \frac{-P(c)}{K(c)} \begin{vmatrix} \phi_\lambda(c) & \psi_\lambda(c) & \gamma_\lambda(c) \\ \phi_\lambda'(c) & \psi_\lambda'(c) & \gamma_\lambda'(c) \\ \phi_\lambda''(c) & \psi_\lambda''(c) & \gamma_\lambda''(c) \end{vmatrix}$$

and

$$a_4(c) = \frac{P(c)}{K(c)} \begin{vmatrix} \phi_\lambda(c) & \psi_\lambda(c) & \chi_\lambda(c) \\ \phi_\lambda'(c) & \psi_\lambda'(c) & \chi_\lambda'(c) \\ \phi_\lambda''(c) & \psi_\lambda''(c) & \chi_\lambda''(c) \end{vmatrix}$$

while $d_1, d_2, d_3$ and $d_4$ are constants.

Calculation of $\Phi_1(b), \Phi_1'(b)$ and $\Phi_1''(b)$ from (4.7) and substitution into (4.6) with the initial conditions (4.3) and (4.4), we can get the constants $d_1, d_2, d_3$ and $d_4$ as follows:
EIGENFUNCTION EXPANSION FOR A FOURTH ORDER EIGENVALUE PROBLEM

The form of equations (4.8) and (4.9) shows that the inverse operator \((\lambda I - A)^{-1}\) is actually compact; for details of argument of theorem 5 in [8, p.120] can be used.

5. EXPANSION THEOREM.

We now arrive at the problem of expanding an arbitrary function \(f(x) \in H\) for \(x \in [a,b]\) in terms of the eigenfunctions of (1.1). The results of our investigations are summarized in the following theorem:

**THEOREM 5.1.** The operator \(A\) in \(H\) has unbounded set of real eigenvalues of finite multiplicity, (they have at most multiplicity four), without accumulation points in \((-\infty, \infty)\) and they can be ordered according to the size, \(0 < \lambda_1 < \lambda_2 < \ldots < \lambda_i\) with \(\lambda_i \rightarrow \pm \infty\) as \(i \rightarrow \infty\). If the corresponding eigenfunctions \(\phi_1, \phi_2, \phi_3, \ldots\) form a complete orthonormal system, then for any function \(f(x) \in H\), we have the expansion:

\[
f(x) = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle \phi_i \quad (4.10)
\]

which is a uniformly convergent series.

The above theorem has some interesting corollaries for particular choices of \(f\).

**COROLLARY 4.1.** If \(f_1 \in L^2(a,b)\) and \(f = (f_1, 0) \in H\), then we have

\[
(i) \quad f_1 = \sum_{i=1}^{\infty} \left( \int_a^b f_1 \phi_i dx \right) \phi_i(x)
\]

\[
(ii) \quad 0 = \sum_{i=1}^{\infty} \left( \int_a^b f_1 \phi_i dx \right) \phi_{i2}
\]
COROLLARY 4.2. If \( \phi = (\phi_1(x), \phi_1) \in D(A) \) and \( f \in (0,1) \in H \), we have:

\[
(1) \quad 0 = \sum_{i=1}^{\infty} \phi_{12}(b) \phi_{11}(x) = \sum_{i=1}^{\infty} \phi_{11}(b) \phi_{11}(x).
\]

\[
(1') \quad 1 = \sum_{i=1}^{\infty} [\phi_{12}(b)]^2 = \sum_{i=1}^{\infty} [\phi_{11}(b)]^2.
\]

REFERENCES


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