A TOPOLOGICAL LATTICE ON THE SET OF MULTIFUNCTIONS

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ABSTRACT. Let $X$ be a Wilker space and $M(X,Y)$ the set of continuous multifunctions from $X$ to a topological space $Y$ equipped with the compact-open topology. Assuming that $M(X,Y)$ is equipped with the partial order $\preceq$, we prove that $(M(X,Y), \preceq)$ is a topological $V$-semilattice. We also prove that if $X$ is a Wilker normal space and $U(X,Y)$ is the set of point-closed upper semi-continuous multifunctions equipped with the compact-open topology, then $(U(X,Y), \preceq)$ is a topological lattice.

KEY WORDS AND PHRASES. Continuous multifunctions, upper semicontinuous multifunctions, compact-open topology, topological lattice.

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1. INTRODUCTION AND DEFINITIONS.

A mapping $F$ from a set $X$ to a set $Y$ which maps each point of $X$ to a subset of $Y$ is called multifunction. For any subset $A$ of $X$, $F(A) = \bigcup_{x \in A} F(x)$. For any subset $B$ of $Y$, $F^{+}(B) = \{x \in X : F(x) \subseteq B\}$ and $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. Let $X$ and $Y$ be topological spaces.

A multifunction $F$ from $X$ to $Y$ is upper semi-continuous (lower semi-continuous) if and only if $F^{+}(P)$ ($F^{-}(P)$) is open for each open subset $P$ of $Y$ (see Smithson [1]).

A multifunction $F:X \to Y$ is continuous if and only if it is both upper and lower semi-continuous [1].

A multifunction $F:X \to Y$ is point-closed [1] if and only if $F(x)$ is a closed subset of $Y$, for each $x \in X$.

If $F_1$, $F_2$ are two multifunctions from $X$ to $Y$, by $F_1 \vee F_2$, we denote the multifunction from $X$ to $Y$ defined by $(F_1 \vee F_2)(x) = F_1(x) \cup F_2(x)$. Also, by $F_1 \wedge F_2$, we denote the multifunction from $X$ to $Y$ defined by $(F_1 \wedge F_2)(x) = F_1(x) \cap F_2(x)$ in Kuratowski [2].

In the following, by $M(X,Y)$, we denote the set of continuous multifunctions. Also, by $U(X,Y)$, we denote the set of point-closed upper semi-continuous multifunctions.
Let $K$ be a compact subset of $X$ and $P$ an open subset of $Y$. Let $\langle K, P \rangle = \{ F \in M(X, Y) : F(x) \cap P \neq 0 \text{ for all } x \in K \}$ and $[K, P] = \{ F \in M(X, Y) : F(K) \subseteq P \}$. The topology $T_{co}$ on $M(Y, Z)$ generated by the sets of the form $\langle K, P \rangle$ and $[K, P]$, where $K$ is compact in $X$ and $P$ is open in $Y$, is called the compact open topology on $M(X, Y)$ [1].

The topology $T^*_{co}$ on $U(X, Y)$ generated by the sets of the form $[K, P] = \{ F \in U(X, Y) : F(K) \subseteq P \}$, where $K$ is compact in $X$ and $P$ open in $Y$, is called the compact-open topology on $U(X, Y)$.

For simplicity, in what follows, we use the symbols $M(X, Y)$ ($U(X, Y)$) to denote the topological spaces $(M(X, Y), T_{co})$ ($U(X, Y), T^*_{co}$).

We give now the definition of Wilker spaces that we will use in the following: A topological space $X$ satisfies the Wilker's condition (D) for every compact subset $K \subseteq X$ and for every pair of open subsets $A_1, A_2 \subseteq X$ with $K \subseteq A_1 \cup A_2$ there are compact subsets $K_1 \subseteq A_1$ and $K_2 \subseteq A_2$ such that $K \subseteq K_1 \cup K_2$ is called a Wilker space (Wilker [3]). It can be easily proved that the class of Wilker spaces contains properly the class of $T_2$ spaces and also the class of basic locally compact spaces (i.e., those spaces every point of which has a neighborhood basis consisting of compact sets). In [4] basic locally compact spaces are called locally quasi-compact spaces and in Murdehswar [5] they are called spaces which satisfy condition $L_2$.

In this paper we prove that if $X$ is a Wilker space, then the $\vee$-semilattices $(M(X, Y), \vee)$, $(U(X, Y), \vee)$ are topological, i.e., we prove the continuity of the join operation $\vee$. It is also noticed that if $X$ is a normal space, $(U(X, Y), \vee)$ is a semilattice [4, p.4]. Finally, if $X$ is a Wilker normal space, we prove that the meet operation $\wedge$ is continuous, i.e., $(U(X, Y), \wedge)$ is a topological semilattice [4, p.274].

The worth of the above results relies on the fact that the space $U(X, Y)$ ($M(X, Y)$) can be considered as a topological lattice (topological $\vee$-semilattice [4, p.4]).

2. MAIN RESULTS.

**PROPOSITION 2.1.** Let $X$ be a Wilker space. Then the operation $(F_1, F_2) \rightarrow F_1 \vee F_2 : M(X, Y) \times M(X, Y) \rightarrow M(X, Y)$ is continuous. Thus the $\vee$-semilattice $(M(X, Y), \vee)$ is topological.

**PROOF.** Let $(F_1, F_2) \in M(X, Y) \times M(X, Y)$ and $F_1 \vee F_2 \in [K, P]$. Then $(F_1 \vee F_2)(K) \subseteq P$, which implies that $F_1(K) \subseteq P$ and $F_2(K) \subseteq P$. Hence $F_1 \in [K, P]$ and $F_2 \in [K, P]$ and it can be easily proved that $(G_1, G_2) \in [K, P] \times [K, P]$ implies that $G_1 \vee G_2 \in [K, P]$.

Let now $F_1 \vee F_2 \in \langle K, P \rangle$. Then $(F_1 \vee F_2)(x) \cap P \neq 0$ for each $x \in K$. So we have $K \subseteq F_1^{-1}(P) \cup F_2^{-1}(P)$. But since $X$ is a Wilker space there are compact subsets $K_1, K_2$ of $X$ such that $K_i \subseteq F_i^{-1}(P)$, $i = 1, 2$, and $K \subseteq K_1 \cup K_2$. So $F_1 \subseteq \langle K_1, P \rangle$, $F_2 \subseteq \langle K_2, P \rangle$. We prove now that $(G_1, G_2) \subseteq \langle K_1, P \rangle \times \langle K_2, P \rangle$ implies that $G_1 \vee G_2 \subseteq \langle K, P \rangle$.

Let $(G_1, G_2) \subseteq \langle K_1, P \rangle \times \langle K_2, P \rangle$. Then, $K_1 \subseteq G_1^{-1}(P)$, $i = 1, 2$, which implies that $K \subseteq F_1 \cup F_2 \subseteq G_1^{-1}(P) \cup G_2^{-1}(P) = (G_1 \vee G_2)^{-1}(P)$. Therefore $G_1 \vee G_2 \subseteq \langle K, P \rangle$. 


The proof of the following Proposition is the same as that of Proposition 2.1 (first part) and it is omitted.

**PROPOSITION 2.3.** Let $X$ be a Wilker space. Then the operation

$$(F_1, F_2) \mapsto F_1 \vee F_2: U(X,Y) \times U(X,Y) \times U(X,Y)$$

is continuous. Thus, the $\vee$-semilattice

$(U(X,Y), \subseteq)$

is topological.

**LEMMA 2.3.** [2, p.179]. Suppose $X$ is a normal space. Let $F_1: X \to Y$, $F_2: X \to Y$ be two point-closed upper semi-continuous multifunctions and $P$ an open set in $Y$. Then,

$$(F_1 \wedge F_2)^+(P) = \bigcup \{F_1^+(V) \cap F_2^+(W) \}$$

where $V,W$ are open in $Y$, $V \cap W = P$.

**PROPOSITION 2.4.** Consider a Wilker normal space $X$. Let $U(X,Y)$ be the set of point closed upper semi-continuous multifunctions equipped with the compact-open topology. Then $(U(X,Y), \subseteq)$ is a topological lattice.

**PROOF.** It suffices to prove that $(U(X,Y), \subseteq)$ is a topological similattice, i.e., that the meet operation $\wedge$ is continuous. According to the previous lemma, it is obvious that the function $(F_1,F_2) \mapsto F_1 \wedge F_2: U(X,Y) \times U(X,Y) \times U(X,Y)$ is well defined, i.e. that $(U(X,Y), \subseteq)$ is a semilattice.

We prove now that $\wedge$ continuous.

Let an arbitrary $(F_1,F_2) \in U(X,Y) \times U(X,Y)$ and let $F_1 \wedge F_2 \in [K,P]$, where $K$ is compact in $X$ and $P$ is open in $Y$. Then by the previous lemma

$$K \subseteq (F_1 \wedge F_2)^+(P) = \bigcup \{F_1^+(V) \cap F_2^+(W) \},$$

where $V,W$ are open in $Y$, $V \cap W = P$. But since $K$ is compact there are finitely many sets $V_i, W_i, i = 1, \ldots, n$ such that

$$K \subseteq \bigcup_{i=1}^{n} \{F_1^+(V_i) \cap F_2^+(W_i) \}.$$ 

where $V_i, W_i$ are open in $Y$, $V_i \cap W_i = P$, $i = 1, \ldots, n$. Moreover since $X$ is a Wilker space there exist compact subsets of $X$, $K_i, i = 1, \ldots, n$, such that

$$K_i \subseteq F_1^+(V_i) \cap F_2^+(W_i) \text{ and } K \subseteq \bigcup_{i=1}^{n} K_i.$$ 

Thus, $K_i \subseteq F_1^+(V_i), K_i \subseteq F_2^+(W_i), i = 1, \ldots, n$.

Hence $F_1 \in [K_i, V_i], F_2 \in [K_i, W_i], i = 1, \ldots, n$ and finally

$$(F_1,F_2) \in \prod_{i=1}^{n} [K_i, V_i] \times \prod_{i=1}^{n} [K_i, W_i].$$
It remains to prove that for each

\[(G_1, G_2) \in \bigcap_{i=1}^{n} [K_i, V_i] \times \bigcap_{i=1}^{n} [K_i, W_i], \quad G_1 \wedge G_2 \in [K, P].\]

To prove this consider an arbitrary

\[(G_1, G_2) \in \bigcap_{i=1}^{n} [K_i, V_i] \times \bigcap_{i=1}^{n} [K_i, W_i].\]

It must be shown that \(K (G_1 \wedge G_2)^+(P)\). Let an arbitrary \(x \in K\). Then \(x \in K_i\), for some \(i, 1 \leq i \leq n\). Since \(K_i \subseteq G_1(V_i), K_i \subseteq G_2(W_i)\), we have that \(G_1(x) \subseteq V_i, \quad G_2(x) \subseteq W_i\). So \(G_1(x) \cap G_2(x) = (G_1 \wedge G_2)(x) \subseteq V_i \cap W_i \subseteq P\). Thus, \(x \in (G_1 \wedge G_2)^+(P)\), which completes the proof.

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REFERENCES

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<thead>
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<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
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<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
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