ON THE CLASS OF SQUARE PETRIE MATRICES INDUCED BY CYCLIC PERMUTATIONS

BAU-SEN DU

Received 2 September 2003

Let \( n \geq 2 \) be an integer and let \( P = \{1, 2, \ldots, n, n+1\} \). Let \( Z_p \) denote the finite field \( \{0, 1, 2, \ldots, p - 1\} \), where \( p \geq 2 \) is a prime. Then every map \( \sigma \) on \( P \) determines a real \( n \times n \) Petrie matrix \( A_{\sigma} \) which is known to contain information on the dynamical properties such as topological entropy and the Artin-Mazur zeta function of the linearization of \( \sigma \). In this paper, we show that if \( \sigma \) is a cyclic permutation on \( P \), then all such matrices \( A_{\sigma} \) are similar to one another over \( \mathbb{Z}_2 \) (but not over \( \mathbb{Z}_p \) for any prime \( p \geq 3 \)) and their characteristic polynomials over \( \mathbb{Z}_2 \) are all equal to \( \sum_{k=0}^{n} x^k \). As a consequence, we obtain that if \( \sigma \) is a cyclic permutation on \( P \), then the coefficients of the characteristic polynomial of \( A_{\sigma} \) are all odd integers and hence nonzero.

2000 Mathematics Subject Classification: 15A33, 15A36.

1. Introduction. Throughout this paper, let \( n \geq 2 \) be a fixed integer and let \( P = \{1, 2, \ldots, n, n+1\} \). For every integer \( 1 \leq i \leq n \), let \( J_i = [i, i+1] \). Let \( \sigma \) be a map from \( P \) into itself. The linearization of \( \sigma \) on \( P \) is defined as the continuous map \( f_{\sigma} \) from \( [1, n+1] \) into itself such that \( f_{\sigma}(k) = \sigma(k) \) for every integer \( 1 \leq k \leq n+1 \) and \( f_{\sigma} \) is linear on \( J_i \) for every integer \( 1 \leq i \leq n \). Let \( A_{\sigma} = (a_{ij}) \) be the real \( n \times n \) matrix defined by \( a_{ij} = 1 \) if \( f_{\sigma}(J_i) \supset J_j \) and \( a_{ij} = 0 \) otherwise. The definition of \( A_{\sigma} \) may seem opaque. But if we take \( J_i \)'s as the vertices of a directed graph and draw an arrow from the vertex \( J_i \) to the vertex \( J_j \) if \( f_{\sigma}(J_i) \supset J_j \), then \( A_{\sigma} \) will be the adjacency matrix [4, page 17] of the resulting directed graph. For example, the adjacency matrix of the cyclic permutation \( \sigma: 1 \to 2 \to 5 \to 4 \to 3 \to 1 \) is given as

\[
A_{\sigma} = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\] (1.1)

In the theory of discrete dynamical systems on the interval, this adjacency matrix \( A_{\sigma} \) turns out to contain much information on the dynamical properties of the map \( f_{\sigma} \). For example, for some special types (including cyclic permutations) of \( \sigma \), if \( x^n + \sum_{k=0}^{n-1} a_k x^k \) is the characteristic polynomial of \( A_{\sigma} \), then it is shown in [6] that the Artin-Mazur zeta function \( \zeta(z) [2] \) of \( f_{\sigma} \) is \( \zeta(z) = 1/(1 + \sum_{k=1}^{n} a_{n-k} z^k) \). On the other hand, it follows from [1, Theorem 4.4.5, page 222] or [4, Proposition 19, page 204] that the topological entropy of \( f_{\sigma} \) equals \( \max\{0, \log \lambda\} \), where \( \lambda \) is the maximal eigenvalue of \( A_{\sigma} \). Since every cyclic graph defines a communication channel, as defined by Shannon, we can claim that the logarithm of the largest eigenvalue of \( A_{\sigma} \) gives its channel capacity. This motivates further investigation of such matrices \( A_{\sigma} \).
Due to the continuity of $f_\sigma$, it is clear that such matrices $A_\sigma$ have entries either zeros or ones such that the ones in each row occur consecutively. Actually, we have $a_{ij} = 1$ for all $a_i \leq j \leq b_i - 1$, where $a_i = \min\{f_\sigma(i), f_\sigma(i + 1)\}$ and $b_i = \max\{f_\sigma(i), f_\sigma(i + 1)\}$, and $a_{ij} = 0$ elsewhere. For our purpose, we define a Petrie matrix [5] to be a matrix whose entries are either zeros or ones such that the ones in each row occur consecutively. So, the matrix $A_\sigma$ induced by a map $\sigma$ on $P$ is a square Petrie matrix whose determinant is easily seen (by induction) [7] to be either 0 or ±1. For any prime number $p \geq 2$, let $Z_p = \{0, 1, 2, \ldots, p - 1\}$ denote the usual finite field and let $W_{Z_p} = \{\sum_{i=1}^{n} r_i J_i \mid r_i \in Z_p, 1 \leq i \leq n\}$ be the $n$-dimensional vector space over $Z_p$ with $\{f_i \mid 1 \leq i \leq n\}$ as a set of basis. Then the matrix $A_\sigma(\text{mod } 2)$ defines a linear transformation $\psi_\sigma$ on $W_{Z_2}$ such that, for every integer $1 \leq i \leq n$, $\psi_\sigma(J_i) = \sum_{j=1}^{n} a_{ij}J_j$.

If both $\sigma$ and $\rho$ are just permutations on $P$, then it is easy to see that $A_\sigma$ may not be similar to $A_\rho$ over $Z_2$. But if both $\sigma$ and $\rho$ are cyclic permutations on $P$, then we show, in this paper, that $A_\sigma$ is similar to $A_\rho$ over $Z_2$ (but $A_\sigma$ may not be similar to $A_\rho$ over $Z_p$ for any prime $p \geq 3$) and their characteristic polynomials over $Z_2$ are all equal to $\sum_{k=0}^{n} x^k$.

As a consequence, we obtain that if $\sigma$ is a cyclic permutation, then the coefficients of the characteristic polynomial of $A_\sigma$ are all odd integers and hence nonzero (not true in general if $\sigma$ is not cyclic) with constant term ±1.

2. On the Petrie matrix $A_\sigma$ over $Z_2$ with any map $\sigma$ on $P$. In the following, we let $[x : y]$ denote the closed interval on the real line with $x$ and $y$ as endpoints. For integers $1 \leq k < j \leq n + 1$, we let $[k, j]$ denote the element $\sum_{i=k}^{j-1} J_i$ of $W_{Z_2}$ and call $k$ and $j$ the endpoints (this terminology will be used in the proof of Theorem 3.2 in Section 3) of the element $\sum_{i=k}^{j-1} J_i$. Part (2) of the following lemma is proved in [4, pages 22-23], which will be needed in Section 3. Here, we present a different proof (see also [3]).

**Lemma 2.1.** Let $n, P, J_i$'s, $\sigma, f_\sigma, W_{Z_2}, \psi_\sigma, A_\sigma$ be defined as in Section 1. Let $\rho$ be a map from $P$ into itself and let $\psi_\rho$ and $A_\rho$ be defined similarly. Then the following hold.

1. Let $1 \leq k < j \leq n + 1$ be any integers. Then for any element $[k, j] = \sum_{i=k}^{j-1} J_i$ in $W_{Z_2}$, $\psi_\sigma([k, j]) = [f_\sigma(k) : f_\sigma(j)]$.

2. $\psi_\rho \circ \psi_\sigma = \psi_{\rho \circ \sigma}$ and $(A_\sigma)(A_\rho) = A_{\rho \circ \sigma}(\text{mod } 2)$. Consequently, if $\sigma$ is a permutation on $P$, then $\psi_\sigma$ is invertible with inverse $\psi_{\sigma^{-1}}$ and $A_\sigma$ is nonsingular with determinant ±1.

**Proof.** It follows from the definition of $\psi_\sigma$ in Section 1 that $\psi_\sigma(J_i) = [f_\sigma(i) : f_\sigma(i + 1)]$ for every integer $1 \leq i \leq n$. Thus, we obtain that $\psi_\sigma([k, j]) = \psi_\sigma(\sum_{i=k}^{j-1} J_i) = \sum_{i=k}^{j-1} \psi_\sigma(J_i) = \sum_{i=k}^{j-1} [f_\sigma(i) : f_\sigma(i + 1)] = [f_\sigma(k) : f_\sigma(j)]$ since $1 + 1 = 0$ in $Z_2$. This proves part (1).

By part (1), $\psi_\sigma([k, j]) = [f_\sigma(k) : f_\sigma(j)]$. Similarly, $\psi_\rho([k, j]) = [f_\rho(k) : f_\rho(j)]$. So, $(\psi_\rho \circ \psi_\sigma)(J_i) = \psi_\rho([f_\sigma(i) : f_\sigma(i + 1)]) = [f_\rho(f_\sigma(i)) : f_\rho(f_\sigma(i + 1))] = [(\rho \circ \sigma)(i) : (\rho \circ \sigma)(i + 1)] = \psi_{\rho \circ \sigma}(J_i)$ since, on the finite set $P$, $f_\sigma = \sigma$ and $f_\rho = \rho$. This shows that $\psi_\rho \circ \psi_\sigma = \psi_{\rho \circ \sigma}$ on $W_{Z_2}$. Thus, if $\sigma$ is a permutation on $P$, then $\psi_{\sigma^{-1}} \circ \psi_\sigma = \psi_{\sigma^{-1} \circ \sigma}$ is the identity map on $W_{Z_2}$, and so $\psi_\sigma$ is an invertible linear transformation on $W_{Z_2}$ with inverse $\psi_{\sigma^{-1}}$. The rest of part (2) can be easily proved and is omitted. This proves part (2) and completes the proof of Lemma 2.1.
3. On the Petrie matrix $A_\sigma$ with any cyclic permutation $\sigma$ on $P$. We will need the following elementary result. We include its proof for completeness.

**Lemma 3.1.** Let $1 \leq j \leq n$ be any fixed integer and let $b$ denote the greatest common divisor of $j$ and $n + 1$. Let $s = (n + 1)/b$. For every integer $1 \leq k \leq s - 1$, let $1 \leq m_k \leq n$ be the unique integer such that $kj \equiv m_k \pmod{n + 1}$. Then the $m_k$’s are all distinct and \( \{m_k \mid 1 \leq k \leq s - 1\} = \{kb \mid 1 \leq k \leq s - 1\} \).

**Proof.** Let $B = \{m_k \mid 1 \leq k \leq s - 1\}$ and $C = \{kb \mid 1 \leq k \leq s - 1\}$. For every integer $1 \leq k \leq s - 1$, since $j/b$ and $(n + 1)/b$ are relatively prime, the congruence equation $(j/b)x \equiv k \pmod{(n + 1)/b}$ has a solution in $1 \leq x \leq (n + 1)/b$. Consequently, for every integer $1 \leq k \leq s - 1$, the congruence equation $jx \equiv kb \pmod{n + 1}$ has a solution in $1 \leq x \leq s - 1$. Since $1 \leq kb \leq n$ for every $1 \leq k \leq s - 1$, we obtain that $C \subseteq B$. Since both $B$ and $C$ contain exactly $s - 1$ elements, we have $B = C$. That is, \( \{m_k \mid 1 \leq k \leq s - 1\} = \{kb \mid 1 \leq k \leq s - 1\} \). This completes the proof.

**Theorem 3.2.** Let $n$, $P$, $J_i$’s, $\sigma$, $f_{\sigma}$, $W_{Z_2}$, $\psi_\sigma$, $A_\sigma$ be defined as in Section 1. Assume that $\sigma$ is also a cyclic permutation on $P$. Then the following hold.

1. For every integer $1 \leq i \leq n$, $\sum_{k=0}^{n} \psi_\sigma^k(J_i) = 0$. Consequently, $\sum_{k=0}^{n} \psi_\sigma^k(w) = 0$ for all $w \in W_{Z_2}$.

2. Let $1 \leq i \leq n$ and $1 \leq j \leq n$ be two fixed integers such that $1 \leq i < f_{\sigma}^j(i) \leq n$ and let $J = [i, f_{\sigma}^j(i)] = \sum_{k=i}^{f_{\sigma}^j(i)-1} j_k$. Assume that $j$ and $n + 1$ are relatively prime. Then the set $\{\psi_\sigma^k(J) \mid 0 \leq k \leq n - 1\}$ is a basis for $W_{Z_2}$.

3. For any cyclic permutations $\sigma$ and $\rho$ on $P$, $\psi_\sigma$ and $\psi_\rho$ are similar on $W_{Z_2}$. Consequently, the Petrie matrices over $Z_2$ of all cyclic permutations on $P$ are similar to one another and have the same characteristic polynomial $\sum_{k=0}^{n} x^k$.

4. The coefficients of the characteristic polynomial of $A_{\sigma}$ are all odd integers (and hence nonzero) with constant term $\pm 1$.

**Remark 3.3.** Part (3) of the above theorem does not hold if the Petrie matrices of cyclic permutations are over the finite field $Z_p$ for any prime $p \geq 3$. For example, if $P = \{1, 2, 3, 4, 5\}$, $\sigma$ denotes the cyclic permutation $1 \rightarrow 2 \rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 1$, and $\rho$ denotes the cyclic permutation $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$, then $A_\sigma$ and $A_\rho$ are not similar over $Z_p$ for any prime $p \geq 3$ because the characteristic polynomials of $A_\sigma$ and $A_\rho$ are $x^4 - x^3 - 3x^2 - 3x - 1$ and $x^4 - x^3 - x^2 - x - 1$, respectively, which are distinct over $Z_p$ for any prime $p \geq 3$.

**Proof.** For any fixed integer $1 \leq i \leq n$, let $1 \leq j \leq n$ be the unique integer such that $f_{\sigma}^j(i) = i + 1$, and so $J_i = [i, i + 1] = [i, f_{\sigma}^j(i)]$. Let $b$ denote the greatest common divisor of $j$ and $n + 1$ and let $s = (n + 1)/b$. For every integer $1 \leq k \leq s - 1$, let $1 \leq m_k \leq n$ be the unique integer such that $kj \equiv m_k \pmod{n + 1}$. Then, by Lemma 3.1, we obtain that \( \{m_k \mid 1 \leq k \leq s - 1\} = \{kb \mid 1 \leq k \leq s - 1\} \). Let $m_0 = 0$. Then $\{m_k \mid 0 \leq k \leq s - 1\} = \{kb \mid 0 \leq k \leq s - 1\}$. Hence, the set $\{0, 1, 2, 3, \ldots, n - 1, n\}$ is the disjoint union of the sets $\{m_k + m \mid 0 \leq k \leq s - 1\}$, $0 \leq m \leq b - 1$. Therefore, $\sum_{k=0}^{n} \psi_\sigma^k(J_i) = \sum_{k=0}^{b-1} \psi_\sigma^{kj}(J_i)$ (since $kj \equiv m_k \pmod{n + 1}$) $= [i : f_{\sigma}^1(i)] + [f_{\sigma}^1(i) : f_{\sigma}^2(i)] + [f_{\sigma}^2(i) : f_{\sigma}^3(i)] + \cdots + [f_{\sigma}^{(s-2)}j(i) : f_{\sigma}^{(s-1)}j(i)] + [f_{\sigma}^{(s-1)}j(i) : i] = 0$. So, $\sum_{k=0}^{n} \psi_\sigma^k(J_i) = \sum_{m=0}^{b-1} \psi_\sigma^{m} (\sum_{k=0}^{s-1} \psi_\sigma^{m_k}(J_i)) = 0$. This proves part (1).
For the proof of part (2), we first show that if $E$ is a nonempty subset of $\{1, 2, 3, \ldots, n-1, n\}$ such that $J + \sum_{k \in E} \psi_{\sigma}^k(J) = 0$, then $E = \{1, 2, 3, \ldots, n-1, n\}$. Indeed, for every integer $1 \leq k \leq n$, let $1 \leq m_k \leq n$ be the unique integer such that $kj \equiv mk (\text{mod } n + 1)$. Assume that $m_1 = j \notin E$. Then, for any $m \in E$, $m \neq 0, j$. Since $\psi_{\sigma}^m(J) = \psi_{\sigma}^m([i, f_j^i(i)]) = [f_j^m(i) : f_j^{m+1}(i)]$, the endpoints of $\psi_{\sigma}^m(J)$ do not contain the point $f_j^1(i)$. Thus, in the expression of $\psi_{\sigma}^m(J)$ as a sum of the basis elements $J_k$’s, it contains either both the basis elements $J_{f_j^1(i)-1}$ and $J_{f_j^1(i)}$ or none of them. But, since $J = [i, f_j^1(i)] = J_1 + J_{i+1} + \cdots + J_{f_j^1(i)-1}$ contains the element $J_{f_j^1(i)-1}$ but not the element $J_{f_j^1(i)}$, in its expression as a sum of the basis elements $J_k$’s, we obtain that in the expression of $J + \sum_{m \in E} \psi_{\sigma}^m(J)$ as a sum of the basis elements $J_k$’s, the coefficient of $J_{f_j^1(i)-1}$ is different from that of $J_{f_j^1(i)}$ by 1. This implies that $J + \sum_{m \in E} \psi_{\sigma}^m(J) \neq 0$, which is a contradiction. Therefore, $m_1 = j \in E$.

Thus,

$$0 = J + \sum_{m \in E} \psi_{\sigma}^m(J)$$

$$= J + \psi_{\sigma}^j(J) + \sum_{m \in E \setminus \{1\}} \psi_{\sigma}^m(J)$$

$$= [i, f_j^1(i)] + [f_j^1(i) : f_j^2(i)] + \sum_{m \in E \setminus \{1\}} \psi_{\sigma}^m(J)$$

$$(3.1)$$

$$= [i : f_j^2(i)] + \sum_{m \in E \setminus \{1\}} \psi_{\sigma}^m(J).$$

Proceeding in this manner finitely many times, we obtain that $\{m_1, m_2, \ldots, m_{n-1}\} \subset E$ and

$$0 = J + \sum_{m \in E} \psi_{\sigma}^m(J)$$

$$= [i : f_j^2(i)] + \sum_{m \in E \setminus \{1, m_2\}} \psi_{\sigma}^m(J)$$

$$= [i : f_j^3(i)] + \sum_{m \in E \setminus \{1, m_2, m_3\}} \psi_{\sigma}^m(J)$$

$$(3.2)$$

$$= \cdots = [i : f_j^{n_j}(i)]$$

$$+ \sum_{m \in E \setminus \{1, m_2, \ldots, m_{n-1}\}} \psi_{\sigma}^m(J).$$

In particular, $0 = [i : f_j^{n_j}(i)] + \sum_{m \in E \setminus \{1, m_2, \ldots, m_{n-1}\}} \psi_{\sigma}^m(J)$. If $m \in E$ and $m \neq m_n$, then, as above, since $m \neq 0$ and $m \neq m_n \equiv nj (\text{mod } n + 1)$, the endpoints of $\psi_{\sigma}^m(J)$ do not contain the point $f_j^{n_j}(i)$. Hence, in the expression of $\psi_{\sigma}^{m_n}(J)$ as a sum of the basis elements $J_k$’s, it contains either both the basis elements $J_{f_j^{n_j}(i)-1}$ and $J_{f_j^{n_j}(i)}$ or none of them. But, since $[i, f_j^{n_j}(i)] = J_1 + J_{i+1} + \cdots + J_{f_j^{n_j}(i)-1}$ contains the element $J_{f_j^{n_j}(i)-1}$, not the element $J_{f_j^{n_j}(i)}$, in its expression as a sum of the basis elements $J_k$’s, we obtain that in the expression of $[i : f_j^{n_j}(i)] + \sum_{m \in E \setminus \{1, m_2, \ldots, m_{n-1}\}} \psi_{\sigma}^m(J)$ as a sum of the basis elements $J_k$’s, the coefficient of $J_{f_j^{n_j}(i)-1}$ is different from that
of $J_{f_{\sigma}^n(i)}$ by 1. This implies that $\{i : f_{\sigma}^n(i) + \sum_{m \in E \setminus \{m_1, m_2, \ldots, m_{n-1}\}} z^m(J) \neq 0\}$, which is a contradiction. Thus, $m_n = nj \in E$. Since, by assumption, $j$ and $n + 1$ are relatively prime, we see that, by Lemma 3.1, $\{m_1, m_2, \ldots, m_n\} = \{1, 2, \ldots, n - 1, n\}$. Since $\{m_1, m_2, \ldots, m_n\} \subset E \subset \{1, 2, \ldots, n - 1, n\}$, we obtain that $E = \{1, 2, \ldots, n - 1, n\}$. This proves our assertion.

Now, assume that $\sum_{k=0}^{n-1} \alpha(k) z^k(J) = 0$, where $\alpha(k) = 0$ or 1 in $Z_2$, $0 \leq k \leq n - 1$. If $\alpha(0) = 0$ and $\alpha(\ell) \neq 0$ for some integer $1 \leq \ell < n - 1$, let $\ell$ be the smallest such integer; then, $\psi_\sigma$ is invertible by Lemma 2.1(2), we obtain that $J + \sum_{k=1}^{n-\ell} \alpha(k) z^k(J) = 0$. So, without loss of generality, we may assume that $\alpha(0) \neq 0$. That is, we assume that $J + \sum_{k=1}^{n-1} \alpha(k) z^k(J) = 0$. Let $E = \{k \mid 1 \leq k \leq n - 1, \alpha(k) \neq 0\}$. Then, we have $J + \sum_{k \in E} \phi^k(J) = 0$. But then it follows from what we have just proved above that $E = \{1, 2, \ldots, n - 1, n\}$. This contradicts the assumption that $E \subset \{1, 2, \ldots, n - 1, n\}$. So, the set $\{\psi^k(J) \mid 0 \leq k \leq n - 1\}$ is linearly independent and hence, by [8], is a basis for $W_{Z_2}$. This proves part 2.

Let $\theta$ denote the cyclic permutation $1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow i \rightarrow i + 1 \rightarrow \cdots \rightarrow n \rightarrow n + 1 \rightarrow 1$ on $P$ and let $\sigma$ be any cyclic permutation on $P$. Choose any fixed integer $1 \leq j \leq n$ such that $j$ and $n + 1$ are relatively prime and let $J = [1, f_{\sigma}^j(1)]$. Then, by part (2), the set $\{\psi^k(J) \mid 0 \leq k \leq n - 1\}$ is a basis for $W_{Z_2}$. Let $\phi$ be the linear transformation on $W_{Z_2}$ defined by $\psi^k(J) = \psi^k(J) = 1 \leq k \leq n$. Then $\phi$ is an isomorphism on $W_{Z_2}$. Furthermore, $(\phi \circ \psi_\theta)(J_n) = \psi^k(\phi(J_n)) = \sum_{k=1}^n \psi^{k-1}(J) = \psi^k(J)$ (by part (1)) = $\psi_\sigma(\psi^{n-1}(J)) = \psi_\sigma(\phi(J_n)) = (\psi_\sigma \circ \phi)(J_n)$ and, for every integer $1 \leq k \leq n - 1$, $(\phi \circ \psi_\theta)(J_k) = \phi(\psi_\theta(J_k)) = \psi_\theta(\phi(J_k)) = \psi_\theta(\phi(J_k)) = (\psi_\sigma \circ \phi)(J_k)$. Thus, $\psi_\sigma$ is similar to $\psi_\theta$ through $\phi$. Since the property of similarity is obviously transitive, we obtain that if $\rho$ is any cyclic permutation on $P$, then $\psi_\sigma$ and $\psi_\theta$ are similar on $W_{Z_2}$. Consequently, by [8], the Petrie matrices (over $Z_2$) of all cyclic permutations on $P$ are similar to one another and so have the same characteristic polynomial $\sum_{k=0}^n x^k$ since $\sum_{k=0}^n x^k$ is easily verified to be the characteristic polynomial of the Petrie matrix $A_\theta$ over $Z_2$. This proves part 3.

Finally, let $\sigma$ be a cyclic permutation on $P$. Since $A_\sigma$ is a real $n \times n$ matrix with entries either zeros or ones, the coefficients of the characteristic polynomial of $A_\sigma$ are all integers. By taking every entry in $A_\sigma$ modulo 2 and applying part 3 and the fact that the determinants of Petrie matrices are either 0 or ±1, we obtain that the characteristic polynomial of $A_\sigma \pmod{2}$ is equal to $\sum_{k=0}^n x^k$. Consequently, the coefficients of the characteristic polynomial of $A_\sigma$ are all odd integers with constant term ±1. This proves part 4 and completes the proof of Theorem 3.2. □

Acknowledgment. The author would like to thank Professor Peter Jau-Shyong Shiue for his interest and many helpful suggestions which led to the improvement of this paper.

References


Bau-Sen Du: Institute of Mathematics, Academia Sinica, Taipei 11529, Taiwan

*E-mail address: mabsdu@sinica.edu.tw*
Special Issue on
Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>December 1, 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob’evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru