THE RADIUS OF STARLIKENESS FOR CONVEX FUNCTIONS OF COMPLEX ORDER

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We will give the relation between the class of Janowski starlike functions of complex order and the class of Janowski convex functions of complex order. As a corollary of this relation, we obtain the radius of starlikeness for the class of Janowski convex functions of complex order.

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1. Introduction. Let $F$ be the class of analytic functions in $D = \{z \mid |z| < 1\}$, and let $S$ denote those functions in $F$ that are univalent and normalized by $f(0) = 0, f'(0) = 1$. Furthermore, let $\Omega$ be the family of functions $\omega(z)$ regular in $D$ and satisfying $\omega(0) = 0, |\omega(z)| < 1$ for $z \in D$.

For arbitrary fixed numbers $-1 \leq B < A \leq 1$, denoted by $P(A,B)$, the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots,$$

(1.1)

which is regular in $D$ on the condition such that

$$p(z) = \frac{1 + A \omega(z)}{1 + B \omega(z)}$$

(1.2)

for some functions $\omega(z) \in \Omega$ and every $z \in D$. This class was introduced by Janowski [7].

Moreover, let $S^*(A,B,b)$ be denoted by the family of functions $f(z) \in S$ such that $f(z)$ is in $S^*(A,B,b)$ if and only if $f(z)/z \neq 0$,

$$1 + \frac{1}{b} \left( z \cdot \frac{f'(z)}{f(z)} - 1 \right) = p(z), \quad (b \neq 0, \text{ complex})$$

(1.3)

for some functions $p(z) \in P(A,B)$ and every $z \in D$.

Finally, let $C(A,B,b)$ denote the family of functions which are regular:

$$1 + \frac{1}{b} \cdot z \cdot \frac{f''(z)}{f'(z)} = p(z), \quad (b \neq 0, \text{ complex})$$

(1.4)

for some functions $p(z) \in P(A,B)$ and every $z \in D$. 


We note that $P(-1,1)$ is the class of Carathéodory functions, and therefore the class $C(A,B,b)$ contains the following classes. $b = 1$, $C(1,-1,1) = C$ is the well-known class of convex functions [2], and $C(1,-1,b) = C(b)$ is the class of convex functions of complex order [7, 8]. $C(1,1,1 - \beta)$, $0 \leq \beta < 1$ is the class of convex functions of order $\beta$ [9]. For $A = 1$, $B = -1$, $b = e^{-i\lambda} \cdot \cos \lambda$, $|\lambda| < \pi/2$ is the class of functions for which $zf'(z)$ is $\lambda$-spirallike [3, 6, 11, 12, 13, 14]. For $A = 1$, $B = -1$, $b = (1 - \beta)e^{-i\lambda} \cdot \cos \lambda$, $0 \leq \beta < 1$, $|\lambda| < \pi/2$ is the class of functions for which $zf'(z)$ is $\lambda$-spirallike of order $\beta$ [3, 6, 11, 12, 13, 14].

2. Representation theorem for the class $S^*(A,B,b)$. The following lemma, well known as Jack’s lemma, is required in our investigation.

**Lemma 2.1** [4, 5]. Let $w(z)$ be a nonconstant and analytic function in the unit disc $D$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at the point $z_0$, then $z_0 w'(z_0) = kw(z_0)$ and $k \geq 1$.

**Lemma 2.2.** $e^{-i\alpha} f(e^{i\alpha}z)$, $\alpha \in [0,2\pi)$ is in $C(A,B,b)$ whenever $f(z)$ is in $C(A,B,b)$.

**Proof.** If $f(z) \in C(A,B,b)$, then

$$g(z) = e^{-i\alpha} f(e^{i\alpha}z) \Rightarrow 1 + \frac{1}{b} z \frac{g'(z)}{g(z)} = 1 + \frac{1}{b} (e^{i\alpha}z) \frac{f'(e^{i\alpha}z)}{f(e^{i\alpha}z)}. \quad (2.1)$$

We note that similarly the class $S^*(A,B,b)$ is invariant under the rotation so that $e^{-i\alpha} f(e^{i\alpha}z)$, $\alpha \in [0,2\pi)$ is in $S^*(A,B,b)$ whenever $f(z)$ is in $S^*(A,B,b)$.

**Lemma 2.3.** If $g(z) \in S^*(A,B,b)$, then

$$g(z) = \begin{cases} z(1 + Bw(z))^{b(A-B)/B}, & B \neq 0, \ k = 1, \\ ze^{bAw(z)}, & B = 0, \ k = 1, \end{cases} \quad (2.2)$$

for some $w(z) \in \Omega$ and for all $z$ in $D$, and conversely.

**Proof.** The proof of this lemma is completed in four steps, and we have used Nicola Tuneski’s technique for the special case of $k = 1$ [15].

**First Step.** If $B \neq 0$ and

$$g(z) = z(1 + Bw(z))^{b(A-B)/B}, \quad (2.3)$$

then by taking logarithmic derivative of (2.3) followed by a brief computation using Jack’s lemma and the definition of subordination, we obtain

$$1 + \frac{1}{b} \left( z \frac{g'(z)}{g(z)} - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad \text{for } k = 1, \quad (2.4)$$

and so from the definition of $S^*(A,B,b)$ it follows that $g(z) \in S^*(A,B,b)$. (See [10].)
SECOND STEP. If $B = 0$, then we have $g(z) = ze^{bAw(z)}$. Similarly, we get

$$1 + \frac{1}{b} \left( zg'(z) \cdot g'(z) - 1 \right) = 1 + Aw(z), \quad \text{for } k = 1. \quad (2.5)$$

The equality shows that $g(z) \in S^*(A,B,b)$.

THIRD STEP. Conversely, if $g(z) \in S^*(A,B,b)$ and $B \neq 0$, then we have

$$1 + \frac{1}{b} \left( zg'(z) \cdot g'(z) - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}. \quad (2.6)$$

Equation (2.6) can be written in the form

$$\frac{g'(z)}{g(z)} = \frac{b(A-B)(w(z)/z)}{1 + Bw(z)} + \frac{1}{z}. \quad (2.7)$$

If we use Jack’s lemma in (2.7) for $k = 1$, we obtain

$$\frac{g'(z)}{g(z)} = \frac{b(A-B)w'(z)}{1 + Bw(z)} + \frac{1}{z}. \quad (2.8)$$

Integrating both sides of equality (2.8), we get (2.3).

FOURTH STEP. Again, conversely, if $g(z) \in S^*(A,B,b)$ and $B = 0$, then in the same way we obtain $g(z) = ze^{bAw(z)}$ which completes the proof.

LEMMA 2.4. Let $f(z)$ be regular and analytic in $D$, and normalized so that $f(0) = 0$, $f'(0) = 1$. A necessary and sufficient condition for $f(z) \in C(A,B,b)$ is that for each member $g(z) = z + b_1 z + b_2 z^2 + \cdots$ of $S^*(A,B,b)$ the following equation holds:

$$g(z,\zeta) = z \left( \frac{f(z) - f(\zeta)}{z - \zeta} \right)^2, \quad \zeta, z \in D, \zeta \neq z, \zeta = nz, |n| \leq 1. \quad (2.9)$$

PROOF. If $f(z) \in C(A,B,b)$, then this function is analytic, regular, and continuous in the unit disc. Therefore, equality (2.9) can be written in the form

$$g(z) = z(f'(z))^2. \quad (2.10)$$

If we take the logarithmic derivative of equality (2.10) followed by simple calculations, we get

$$1 + \frac{1}{2b} \left( zg'(z) \cdot g'(z) - 1 \right) = 1 + \frac{1}{b} z^2 f''(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \quad (2.11)$$

On the other hand, $b$ is a complex number and $b \neq 0$. Therefore, $b_1 = 2b$ is a complex number and $2b \neq 0$, thus (2.11) can be written in the form

$$1 + \frac{1}{b_1} \left( zg'(z) \cdot g'(z) - 1 \right) = 1 + \frac{1}{b_1} z^2 f''(z). \quad (2.12)$$
Considering equality (2.12), the definition of \( C(A,B,b) \), and the definition of \( S^*(A,B,b) \), we obtain \( g(z) \in S^*(A,B,2b) \).

Conversely, if \( g(z) \in S^*(A,B,b) \), and \( g(z) = z\left((f(z) - f(\zeta))/(z - \zeta)\right) \) holds, then from Lemma 2.3 we get

\[
g(z) = z\left(\frac{f(z) - f(\zeta)}{z - \zeta}\right)^2 = \begin{cases} z\left(1 + Bw(z)\right)^{b(A-B)/B}, & B \neq 0, \\ ze^{bAw(z)}, & B = 0. \end{cases} \tag{2.13}
\]

If we take the logarithmic derivative with respect to \( z \) of (2.13) followed by simple calculations, we get

\[
1 + \frac{1}{b}\left(z\frac{g'(z)}{g(z)} - 1\right) = \frac{1}{b}\left[\frac{2zf'(z)}{f(z) - f(\zeta)} - \frac{z + \zeta}{z - \zeta}\right] + 1 - \frac{1}{b} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad B \neq 0,
\]

\[
1 + \frac{1}{b}\left(z\frac{g'(z)}{g(z)} - 1\right) = \frac{1}{b}\left[\frac{2zf''(z)}{f(z) - f(\zeta)} - \frac{z + \zeta}{z - \zeta}\right] + 1 - \frac{1}{b} = 1 + Aw(z), \quad B = 0.
\tag{2.14}
\]

Furthermore, if we write \( F(z,\zeta) = (1/b)[2zf'(z)/(f(z) - f(\zeta)) - (z + \zeta)/(z - \zeta)] + 1 - 1/b \), then we have

\[
\lim_{\zeta \to z} F(z,\zeta) = 1 + \frac{1}{b}z\frac{f''(z)}{f'(z)}. \tag{2.15}
\]

Considering relations (2.14) and (2.15) together, we obtain \( f(z) \in C(A,B,b) \).

**COROLLARY 2.5.** If \( f(z) \in C(A,B,b) \), then

\[
2\left[1 + \frac{1}{b}\left(z\frac{f'(z)}{f(z)} - 1\right)\right] - 1 = p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.16}
\]

**PROOF.** If we take \( \zeta = 0 \) in \( F(z,\zeta) \), we obtain the desired result of this corollary. \( \Box \)

3. The radius of starlikeness for the class \( C(A,B,b) \)

**LEMMA 3.1.** If \( f(z) \in C(A,B,b) \), then

\[
\left|z\frac{f'(z)}{f(z)} - 2 - \frac{B^2 - b(2B^2 - AB)r^2}{(1-B^2)r^2}\right| \leq \frac{|b|(A-B)r}{2(1-B^2)r^2}. \tag{3.1}
\]

**PROOF.** If \( p(z) \in P(A,B) \), then

\[
\left|p(z) - \frac{1 - ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}. \tag{3.2}
\]
The inequality (3.2) was proved by Janowski [7]. Considering Corollary 2.5 and inequality (3.1), then we get

\[
\left| 2 \left[ 1 + \frac{1}{b} \left( z f'(z) - 1 \right) - 1 \right] \right| \leq \frac{(A-B)r}{1-B^2r^2}.
\]  

(3.3)

After brief calculations from (3.3), we obtain (3.1). \(\square\)

**Theorem 3.2.** The radius of starlikeness for the class \(C(A,B,b)\) is

\[
r_s = \frac{4}{|b|(A-B) + \sqrt{|b|^2(A-B)^2 + 8[2B^2 + (AB-B^2)\text{Re} b]}}.
\]

(3.4)

This radius is sharp, because the extremal function is

\[
f_*(z) = \begin{cases} 
\int_0^z (1 + B\zeta)^{b(A-B)/B} d\zeta, & B \neq 0, \\
\int_0^z e^{Ab\zeta} d\zeta, & B = 0.
\end{cases}
\]

(3.5)

**Proof.** After the brief calculations from inequality (3.1), we get

\[
\text{Re} \left( z^2 \frac{f'(z)}{f(z)} \right) \geq \frac{2 - |b|(A-B)r - [2B^2 + (AB - B^2)\text{Re} b]r^2}{1 - B^2r^2}.
\]

(3.6)

Hence for \(r < r_s\), the right-hand side of inequality (3.6) is positive. This implies that (3.4) holds.

Also note that inequality (3.6) becomes an equality for the function \(f_*(z)\). It follows that (3.4) holds. \(\square\)

**Corollary 3.3.** If \(A = 1, B = -1, b = 1\), then \(r_s = 1\). This is the radius of starlikeness of convex functions which is well known (see [1, Volume II, page 88]).

**References**


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