We obtain the asymptotic distribution of the nonprincipal eigenvalues associated with the singular problem \( x'' + \lambda q(t)x = 0 \) on an infinite interval \([a, +\infty)\). Similar to the regular eigenvalue problem on compact intervals, we can prove a Weyl-type expansion of the eigenvalue counting function, and we derive the asymptotic behavior of the eigenvalues.

1. Introduction

In this work we study the second-order linear ordinary differential equation

\[ x'' + \lambda q(t)x = 0, \quad t \geq a, \quad (1.1) \]

with the boundary conditions

\[ x(a, \lambda) = 0, \quad \lim_{t \to \infty} [x(t, \lambda) - t] = 0, \quad \lim_{t \to \infty} t[x'(t, \lambda) - 1] = 0, \quad (1.2) \]

where \( \lambda \) is a real parameter and \( q(t) \) is a positive continuous function on \([a, \infty)\) satisfying

\[ \int_a^\infty t^2 q(t) dt < \infty. \quad (1.3) \]

A nonoscillatory solution \( x_0(t, \lambda) \) of (1.1) satisfying the boundary conditions (1.2) is called a nonprincipal eigenfunction if

\[ \int_a^\infty \frac{dt}{(x_1(t, \lambda))^2} < \infty, \quad (1.4) \]

and the corresponding value of \( \lambda \) is called a nonprincipal eigenvalue.
Concerning the existence and uniqueness of nonprincipal eigenvalues, the main result is due to Elbert et al. [2]. There exists a sequence of positive constants \( \{\lambda_k\}_k \), \( 0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots \rightarrow \infty \) such that, for each \( \lambda = \lambda_k \), (1.1) possesses a solution \( x_k(t, \lambda_k) \) satisfying the boundary condition (1.2) and having exactly \( k \) zeros in \((a, \infty)\), \( k = 0, 1, 2, \ldots\), imposing the integrability condition (1.3) on \( q(t) \).

We are interested in the distribution and asymptotic behavior of eigenvalues \( \{\lambda_k\}_k \). To this end, we study the spectral counting function \( N(\lambda) = \#\{k : \lambda_k \leq \lambda\} \).

It is well known that the eigenvalue problem in a closed interval \([a, b]\) has asymptotic distribution (see [1]):

\[
N(\lambda) \sim \frac{\lambda^{1/2}}{\pi} \int_a^b q^{1/2}(t) \, dt
\]

as \( \lambda \rightarrow \infty \), generalizing the Weyl formula. Here, \( f \sim g \) means that \( f/g \rightarrow 1 \).

Our main result is the following theorem.

**Theorem 1.1.** Let \( \{\lambda_k\} \) be the sequence of nonprincipal eigenvalues of problem (1.1)-(1.2), and let \( q(t) \) be a positive, continuous, and nonincreasing function satisfying (1.3). Then, the asymptotic expansion of \( N(\lambda) \) is given by

\[
N(\lambda) = \frac{\lambda^{1/2}}{\pi} \int_a^\infty q^{1/2}(t) \, dt + o(\lambda^{1/2})
\]

as \( \lambda \rightarrow \infty \). Also, the \( k \)-th eigenvalue has the following asymptotic behavior:

\[
\lambda_{k-1} = \left( \frac{\pi k}{\int_a^\infty q^{1/2}(t) \, dt} \right)^2 + o(k^2)
\]

as \( k \rightarrow \infty \).

The paper is organized as follows. In Section 2 we prove some auxiliary results, and the proof of Theorem 1.1 is given in Section 3.

**2. Sturm-Liouville bracketing of eigenvalues**

Let us observe that problem (1.1)-(1.2) is not a variational one, since \( x'(t) \sim 1 \) as \( t \rightarrow +\infty \) and \( x'(t) \notin L^2(0, +\infty) \). Hence, we need the following generalization of the Dirichlet-Neumann bracketing of Courant (see [1]) in order to prove Theorem 1.1.

**Theorem 2.1.** Let \( N(\lambda, I) \) be the spectral counting function on \( I = (a, b) \) of the problem

\[
-x'' = \lambda q(t) x, \quad x(a) = 0 = x(b).
\]

Let \( c \in (a, b) \). Then,

\[
N(\lambda, I) \sim N(\lambda, I_1) + N(\lambda, I_2)
\]

as \( \lambda \rightarrow \infty \), where \( I_1 = (a, c) \) and \( I_2 = (c, b) \).
Remark 2.2. For simplicity, we deal only with the Dirichlet boundary condition on a bounded interval. With minor modifications of the proof, the result is valid for different boundary conditions, including the case $b = +\infty$ and the boundary condition (1.2), since the proof is based on the Sturm-Liouville oscillation theory.

Let us sketch the proof of the Dirichlet Neumann bracketing for a second-order differential operator $L$ with variational structure in an interval $I$. The eigenvalues of $L$ are obtained minimizing a quadratic functional in a convenient subspace $H \subset H^1(I)$. We have

$$H_0^1(I_1) \oplus H_0^1(I_2) \subset H^1_0(I) \subset H \subset H^1(I) \subset H^1(I_1) \oplus H^1(I_2) \quad (2.3)$$

and we obtain the Dirichlet eigenvalues of $L$ in $I_1$ and $I_2$ as an upper bound of the eigenvalues of $L$ in $I$, and the Neumann eigenvalues of $I_1$ and $I_2$ as a lower bound.

In problem (1.1)-(1.2), the solutions and eigenvalues are obtained by a fixed point argument, instead of a minimization procedure, and we need a different argument to relate the eigenvalue of two intervals and those of the union of them. Since the eigenfunction $x_k$ has exactly $k$ zeros in $(a, b)$, it is possible to obtain the asymptotic distribution of eigenvalues from the asymptotic number of zeros of solutions, an idea which goes back at least to Hartman (see [3]). For the sake of self-completeness, we prove Theorem 2.1 here.

**Proof of Theorem 2.1.** Let us consider the following eigenvalue problems in $I_1$ and $I_2$, with the original boundary conditions in $a$ and $b$, and a Neumann boundary condition at $c$:

$$-u'' = \mu q(t)u, \quad t \in (a, c),$$

$$u(a) = 0, \quad u'(c) = 0,$$  \quad (2.4)

$$-v'' = \nu q(t)v, \quad t \in (c, b),$$

$$v'(c) = 0, \quad v(b) = 0.$$  \quad (2.5)

For each problem there exists a sequence of simple eigenvalues $\{\mu_k\}_k, \{\nu_k\}_k$ tending to infinity, and the $k$th eigenfunction $u_k$ corresponding to $\mu_k$ (resp., $v_k, \nu_k$) has exactly $k$ zeros.

Let $\lambda$ be fixed. Let $\lambda_n$ be the greater eigenvalue of problem (2.1) lower or equal than $\lambda$ and $x_n(t)$ the corresponding eigenfunction, which has $n$ zeros in $(a, b)$. Let $k$ be the number of zeros of $x_n$ in $(a, c)$, and let $n - k$ be the number of zeros in $(c, b)$.

Let $\mu_j$ be the greater eigenvalue of problem (2.4) lower or equal than $\lambda$, and let $u_j$ be the corresponding eigenfunction. We will show that $j$, the number of zeros of $u_j$, satisfies

$$k - 1 \leq j \leq k + 2. \quad (2.6)$$

Let us suppose first that $u_j$ has $k + 3$ zeros. Then, the Sturmian theory gives $\mu_j > \lambda_n$. Let $x_{\mu_j}(t)$ be the unique solution of (2.1) satisfying

$$x_{\mu_j}(c) = u_j(c),$$

$$x'_{\mu_j}(c) = u'_j(c). \quad (2.7)$$
4 Distribution of eigenvalues

Hence, \( x_{\mu_j} \equiv u_j \) in \((a, c)\), and \( x_{\mu_j}(t) \) has at least \( n - k - 1 \) zeros in \((c, b)\) (let us note that one of the original zeros of \( x_n(t) \) could cross the point \( c \) to the left). Thus, the solution \( x_{\mu_j}(t) \) has at least \( n + 2 \) zeros in \((a, b)\).

However, the eigenfunction \( x_{n+1}(t) \) of problem (2.1) corresponding to the eigenvalue \( \lambda_{n+1} \) has \( n + 1 \) zeros and satisfy \( \lambda_{n+1} < \mu_j \). Hence,

\[
\lambda_{n+1} < \mu_j \leq \lambda,
\]

which contradicts our assumption.

On the other hand, let us suppose that \( u_j \) has \( k - 2 \) zeros. Clearly, \( \mu_j < \lambda_n < \lambda \). Let \( u_{j+1} \) be the eigenfunction of problem (2.4) with \( k - 1 \) zeros in \((a, c)\), and let \( \mu_{j+1} \) be the corresponding eigenvalue. By using the Sturm-Liouville theory,

\[
\mu_{j+1} < \lambda_n < \lambda,
\]

because \( x_n(t) \) has \( k \) zeros in \((a, c)\), which contradicts the fact that \( \mu_j \) is the greater eigenvalue of problem (2.4) lower or equal than \( \lambda \).

Let us consider now problem (2.5). Let \( \nu_h \) be the greater eigenvalue of problem (2.5) lower or equal than \( \lambda \), and let \( \nu_h \) be the corresponding eigenfunction. In much the same way, fixing the boundary condition at \( t = b \), we can show that \( h \), the number of zeros of \( \nu_h \), satisfy

\[
n - k - 2 \leq h \leq n - k + 1.
\]

Then, from inequalities (2.6) and (2.10),

\[
N(\lambda, I_1) + N(\lambda, I_2) - 3 \leq N(\lambda, I) \leq N(\lambda, I_1) + N(\lambda, I_2) + 3
\]

and the proof is finished. \( \square \)

3 Asymptotic of nonprincipal eigenvalues

In this section we prove Theorem 1.1. First, we need the following lemma.

Lemma 3.1. Let \( q(t) \) be a positive continuous function satisfying

\[
\int_a^\infty t^2 q(t) dt < \infty. \tag{3.1}
\]

Then,

\[
\int_a^\infty q^{1/2}(t) dt < \infty. \tag{3.2}
\]

Proof. It follows from Holder’s inequality:

\[
\int_a^\infty q^{1/2}(t) dt < \left( \int_a^\infty t^2 q(t) dt \right)^{1/2} \left( \int_a^\infty t^{-2} dt \right)^{1/2} < \infty. \tag{3.3}
\]

\( \square \)
We divide the proof of Theorem 1.1 in three parts. We obtain an optimal lower bound for \( N(\lambda) \); then we obtain an upper bound for \( N(\lambda) \); and finally, we improve the upper bound.

**Proposition 3.2.** Let \( N(\lambda) \) be the eigenvalue counting function of Theorem 1.1. The following inequality holds:

\[
\frac{\lambda^{1/2}}{\pi} \int_a^{+\infty} q^{1/2}(t) dt + o(\lambda^{1/2}) \leq N(\lambda). \tag{3.4}
\]

**Proof.** Let \( \varepsilon > 0 \) be fixed, there exist \( T_\varepsilon \) such that

\[
\frac{1}{\pi} \int_{T_\varepsilon}^{\infty} q^{1/2}(t) dt \leq \frac{\varepsilon}{2}. \tag{3.5}
\]

Let us consider the Dirichlet eigenvalue problem on \([a, T_\varepsilon]\):

\[
- y''(t) = \mu q(t) y(t), \quad (3.6)
\]
\[
y(a) = 0 = y(T_\varepsilon). \quad (3.7)
\]

It is well known that there exists a sequence of eigenvalues \( \{\mu_k\}_{k \geq 0} \), with associated eigenfunctions \( \{y_k\}_{k \geq 0} \). Each eigenvalue is isolated and \( y_k \) has exactly \( k \) zeros in the open interval \((a, T_\varepsilon)\).

The spectral counting function \( N_D(\lambda, [a, T_\varepsilon]) \) of problem (3.6) has the following asymptotic expansion:

\[
N_D(\lambda, [a, T_\varepsilon]) = \frac{\lambda^{1/2}}{\pi} \int_a^{T_\varepsilon} q^{1/2}(t) dt + o(\lambda^{1/2}). \tag{3.8}
\]

Therefore, for the same \( \varepsilon > 0 \), there exists \( \lambda(\varepsilon) \) such that

\[
\left| \frac{N_D(\lambda, [a, T_\varepsilon])}{\lambda^{1/2}} - \frac{1}{\pi} \int_a^{T_\varepsilon} q^{1/2}(t) dt \right| \leq \frac{\varepsilon}{2} \tag{3.9}
\]

for every \( \lambda \geq \lambda(\varepsilon) \).

By the Sturmian comparison theorem, we have the inequality \( \lambda_k \leq \mu_k \), which gives the lower bound for \( N(\lambda) \):

\[
N_D(\lambda, [a, T_\varepsilon]) \leq N(\lambda). \tag{3.10}
\]

Hence,

\[
\frac{N(\lambda)}{\lambda^{1/2}} \geq \frac{N_D(\lambda, [a, T_\varepsilon])}{\lambda^{1/2}} \geq \frac{1}{\pi} \int_a^{T_\varepsilon} q^{1/2}(t) dt - \frac{\varepsilon}{2} \geq \frac{1}{\pi} \int_a^{\infty} q^{1/2}(t) dt - \varepsilon \tag{3.11}
\]

for every \( \lambda \geq \lambda(\varepsilon) \), and the proof is finished. \( \square \)

**Remark 3.3.** Let us note that Proposition 3.2 is valid whenever \( \int_a^{\infty} q^{1/2}(t) dt < +\infty \), which is guaranteed by Lemma 3.1, without any monotonicity assumption.
Distribution of eigenvalues

**Proposition 3.4.** Let $N(\lambda)$ be the eigenvalue counting function of Theorem 1.1. The following inequality holds:

$$
\frac{4\lambda^{1/2}}{\pi} \int_{a}^{+\infty} q^{1/2}(t) dt + o(\lambda^{1/2}) \geq N(\lambda).
$$

(3.12)

**Proof.** We need a lower bound for eigenvalues due to Nehari [4]. Let $q(t)$ be a monotonic function, and $\mu_k$ the $k$th Dirichlet eigenvalue of (1.1) in $(a, b)$. Then,

$$
\mu_k \left( \int_{a}^{b} q^{1/2}(t) dt \right)^2 \geq \frac{\pi^2 k^2}{4}.
$$

(3.13)

Let $\{\lambda_k\}_{k \geq 0}$ be the nonprincipal eigenvalues of problem (1.1)-(1.2), and let $t_k$ be the $k$th zero of the associated eigenfunction $x_k(t)$. Clearly, $\lambda_k$ coincides with the $k$th Dirichlet eigenvalue in $(a, t_k]$.

Hence,

$$
\lambda_k \geq \frac{\pi^2 k^2}{4 \left( \int_{a}^{t_k} q^{1/2}(t) dt \right)^2} \geq \frac{\pi^2 k^2}{4 \left( \int_{a}^{\infty} q^{1/2}(t) dt \right)^2}.
$$

(3.14)

We obtain

$$
N(\lambda) = \# \{ k : \lambda_k \leq \lambda \}
$$

$$
\leq \# \left\{ k : \frac{\pi^2 k^2}{4 \left( \int_{a}^{\infty} q^{1/2}(t) dt \right)^2} \leq \lambda \right\}
$$

$$
= \# \left\{ k : k \leq 2 \frac{\lambda^{1/2}}{\pi} \int_{a}^{\infty} q^{1/2}(t) dt \right\}
$$

$$
\leq 2 \frac{\lambda^{1/2}}{\pi} \int_{a}^{\infty} q^{1/2}(t) dt + O(1),
$$

(3.15)

and the proof is finished.

Now we prove Theorem 1.1.

**Proof of Theorem 1.1.** Let be $T_\varepsilon$ such that

$$
\int_{T_\varepsilon}^{+\infty} q^{1/2}(t) dt < \varepsilon.
$$

(3.16)

Applying Theorem 2.1 we obtain

$$
N(\lambda) \sim N(\lambda, (a, T_\varepsilon)) + N(\lambda, (T_\varepsilon, \infty)).
$$

(3.17)

The asymptotic behavior of $N(\lambda, (a, T_\varepsilon))$ is obtained from the classical theory,

$$
N(\lambda, (a, T_\varepsilon)) \sim \frac{\lambda^{1/2}}{\pi} \int_{a}^{T_\varepsilon} q^{1/2}(t) dt.
$$

(3.18)
Hence, for $\lambda \geq \lambda(\varepsilon)$, we have

$$N(\lambda, (a, T_\varepsilon)) \leq \frac{\lambda^{1/2}}{\pi} \int_a^{T_\varepsilon} q^{1/2}(t) dt + \varepsilon \lambda^{1/2} \leq \frac{\lambda^{1/2}}{\pi} \int_a^{+\infty} q^{1/2}(t) dt + \varepsilon \lambda^{1/2}. \quad (3.19)$$

Now, $N(\lambda, (T_\varepsilon, \infty))$ can be bounded by using Proposition 3.4:

$$N(\lambda, (T_\varepsilon, \infty)) \leq \frac{2\lambda^{1/2}}{\pi} \int_{T_\varepsilon}^{+\infty} q^{1/2}(t) dt \leq \varepsilon \frac{2\lambda^{1/2}}{\pi}. \quad (3.20)$$

Hence,

$$N(\lambda) \leq \frac{\lambda^{1/2}}{\pi} \int_a^{+\infty} q^{1/2}(t) dt + \varepsilon \lambda^{1/2} + \varepsilon \frac{2\lambda^{1/2}}{\pi}. \quad (3.21)$$

Since $\varepsilon$ is arbitrarily small, and by using Proposition 3.2, we have the asymptotic expansion

$$N(\lambda) \sim \frac{\lambda^{1/2}}{\pi} \int_a^{+\infty} q^{1/2}(t) dt. \quad (3.22)$$

Finally, from (3.22), we have

$$k = N(\lambda_{k-1}) \sim \frac{\lambda_k^{1/2}}{\pi} \int_a^{+\infty} q^{1/2}(t) dt, \quad (3.23)$$

which gives the asymptotic behavior of the $k$th-eigenvalue,

$$\lambda_k = \left( \frac{\pi k}{\int_a^{+\infty} q^{1/2}(t) dt} \right)^2 + o(k^2). \quad (3.24)$$

This completes the proof. \qed

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References


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