ON SENSIBLE FUZZY IDEALS OF BCK-ALGEBRAS WITH RESPECT TO A $t$-CONORM

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Received 3 October 2005; Revised 29 May 2006; Accepted 5 June 2006

We introduce the notion of sensible fuzzy ideals of BCK-algebras with respect to a $t$-conorm and investigate some of their properties. We give the conditions for a sensible fuzzy subalgebra with respect to a $t$-conorm to be a sensible fuzzy ideal with respect to a $t$-conorm. Some properties of the direct product and $S$-product of fuzzy ideals of BCK-algebras with respect to a $t$-conorm are also discussed.

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1. Introduction

Imai and Iséki [3] introduced the class of logical algebras: BCK-algebras. This notion is originated from two different ways: one of the motivations is based on set theory, another motivation is from classical and nonclassical propositional calculus.

The notion of fuzzy sets was first introduced by Zadeh [8]. On the other hand, Schweizer and Sklar [5, 6] introduced the notions of triangular norm ($t$-norm) and triangular conorm ($t$-conorm). Triangular norm ($t$-norm) and triangular conorm ($t$-conorm) are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators, respectively. Thus, the $t$-norm generalizes the conjunctive (AND) operator and the $t$-conorm generalizes the disjunctive (OR) operator. In application, $t$-norm $T$ and $t$-conorm $S$ are two functions that map the unit square into the unit interval. Jun and Kim [4] introduced the notion of imaginable fuzzy ideals of BCK-algebras with respect to a $t$-norm. Cho et al. [1] have recently introduced the notion of sensible fuzzy subalgebras of BCK-algebras with respect to $s$-norm and studied some of their properties. In this paper, we introduce the notion of sensible fuzzy ideals of BCK-algebras with respect to a $t$-conorm and investigate some of their properties. We give conditions for a sensible fuzzy subalgebra with respect to a $t$-conorm to be a sensible fuzzy ideal with respect to a $t$-conorm. Some properties of the direct product and $S$-product of fuzzy ideals of BCK-algebras with respect to a $t$-conorm are also obtained.
2. Preliminaries

In this section, we review some definitions and results that will be used in the sequel.

An algebra \((X;\ast,0)\) of type \((2,0)\) is called a BCK-algebra if it satisfies the following conditions:

\[
\begin{align*}
(1) \quad & (x \ast y) \ast (x \ast z) = (z \ast y) \ast (x \ast z), \\
(2) \quad & x \ast (x \ast y) = y, \\
(3) \quad & x \ast x = 0, \\
(4) \quad & x \ast y = 0, \quad y \ast x = 0 \Rightarrow x = y, \\
(5) \quad & 0 \ast x = 0 \\
\end{align*}
\]

for all \(x, y, z \in X\). We can define a partial ordering relation \(\leq\) on \(X\) by letting \(x \leq y\) if and only if \(x \ast y = 0\). Let \(S\) be a nonempty subset of a BCK-algebra \(X\), then \(S\) is called a subalgebra of \(X\) if \(x \ast y \in S\) for all \(x, y \in S\). A mapping \(f : X \rightarrow Y\) of BCK-algebras is a homomorphism if \(f(x \ast y) = f(x) \ast f(y)\) for all \(x, y \in X\). A nonempty subset \(A\) of a BCK-algebra \(X\) is called an ideal of \(X\) if, for all \(x, y \in X\), it satisfies (I1) \(0 \in A\), (I2) \(x \ast y, y \in A \Rightarrow x \in A\). A mapping \(\mu : X \rightarrow [0,1]\), where \(X\) is an arbitrary nonempty set, is called a fuzzy set in \(X\). For any fuzzy set \(\mu\) in \(X\) and any \(\alpha \in [0,1]\), we define the set \(L(\mu;\alpha) = \{x \in X \mid \mu(x) \leq \alpha\}\), which is called lower level cut of \(\mu\).

**Definition 2.1** [2]. A fuzzy set \(\mu\) in a BCK-algebra \(X\) is called an antifuzzy ideal of \(X\) if

\[
\begin{align*}
\text{(AF1)} \quad & \mu(0) \leq \mu(x) \quad \text{for all } x \in X, \\
\text{(AF2)} \quad & \mu(x) \leq \max(\mu(x \ast y), \mu(y)) \quad \text{for all } x, y \in X.
\end{align*}
\]

**Definition 2.2** [7]. A triangular conorm \((t\text{-conorm } S)\) is a mapping \(S : [0,1] \times [0,1] \rightarrow [0,1]\) that satisfies the following conditions:

\[
\begin{align*}
\text{(S1)} \quad & S(x,0) = x, \\
\text{(S2)} \quad & S(x,y) = S(y,x), \\
\text{(S3)} \quad & S(x,S(y,z)) = S(S(x,y),z), \\
\text{(S4)} \quad & S(x,y) \leq S(x,z) \text{ whenever } y \leq z.
\end{align*}
\]

for all \(x, y, z \in [0,1]\).

Replacing 0 by 1 in condition S1, we obtain the concept of \(t\)-norm \(T\).

**Definition 2.3.** Given a \(t\)-norm \(T\) and a \(t\)-conorm \(S\), \(T\) and \(S\) are dual (with respect to the negation ‘) if and only if \((T(x,y))' = S(x',y')\).

**Proposition 2.4.** Conjointive (AND) operator is a \(t\)-norm \(T\) and disjunctive (OR) operator is its dual \(t\)-conorm \(S\).

**Proposition 2.5** [5]. For a \(t\)-conorm \(T\), the following statement holds:

\[
S(x,y) \geq \max(x,y), \quad \forall x, y \in [0,1].
\]

**Definition 2.6.** Let \(S\) be a \(t\)-conorm. A fuzzy set \(\mu\) in \(X\) is called sensible with respect to \(S\) if \(\text{Im } \mu \subseteq \Delta_S\), where \(\Delta_S = \{\alpha \in [0,1] \mid S(\alpha,\alpha) = \alpha\}\).

3. Fuzzy ideals with respect to a \(t\)-conorm

In what follows, let \(X\) denote a BCK-algebra unless otherwise specified.
Definition 3.1. Let $S$ be a $t$-conorm. A fuzzy set $\mu : X \to [0,1]$ is called a fuzzy ideal of $X$ with respect to $S$ if

(SF1) $\mu(0) \leq \mu(x)$,
(SF2) $\mu(x) \leq S(\mu(x \ast y) , \mu(y))$

for all $x, y \in X$.

Example 3.2. Let $X = \{0, a, b, 1\}$ be a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(x) = 0$ if $x \in \{0, a\}$ and $\mu(x) = 1$ for all $x \notin \{0, a\}$ and let $S_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $S_m(x, y) = \min(x + y, 1)$ which is a $t$-conorm for all $x, y \in [0,1]$. By routine calculations, it is easy to check that $\mu$ is a sensible fuzzy ideal of $X$ with respect to $S_m$.

Proposition 3.3. Let $S$ be a $t$-conorm. Then every sensible fuzzy ideal of $X$ with respect to $S$ is an antifuzzy ideal of $X$.

Proof. The proof is obtained dually by using the notion of $t$-conorm $S$ instead of $t$-norm $T$ in [4].

The converse of Proposition 3.3 is not true in general as seen in the following example.

Example 3.4. Let $X = \{0,1,2,3,4\}$ be a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>1</td>
<td>1</td>
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<td>1</td>
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<td>2</td>
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<td>2</td>
<td>0</td>
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<td>3</td>
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<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = 0.1$, $\mu(1) = \mu(2) = \mu(3) = 0.4$ and $\mu(4) = 0.7$ is an antifuzzy ideal of $X$. Let $\gamma \in (0,1)$ and define the binary operation $S_\gamma$ on $(0,1)$ as follows:

$$S_\gamma(\alpha, \beta) = \begin{cases} \max\{\alpha, \beta\} & \text{if } \min\{\alpha, \beta\} = 0, \\ 1 & \min\{\alpha, \beta\} > 0, \alpha + \beta \geq 1 + \gamma, \\ \gamma & \text{otherwise} \end{cases}$$  \hspace{1cm} (3.1)$$

for all $\alpha, \beta \in [0,1]$. Then $S_\gamma$ is a $t$-conorm. Thus $S_\gamma(\mu(0), \mu(0)) = S_\gamma(0.1, 0.1) = \gamma \neq \mu(0)$ whenever $\gamma < 0.8$. Hence $\mu$ is not a sensible fuzzy ideal of $X$ with respect to $S_\gamma$.

Theorem 3.5. Let $S$ be a $t$-conorm and $\mu$ a nonempty fuzzy set of $X$. Then $\mu$ is fuzzy ideal of $X$ with respect to $S$ if and only if each nonempty level subset $L(\mu; \alpha)$ of $\mu$ is an ideal of $X$. 

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Proof. Suppose that $\mu$ is a fuzzy ideal of $X$ with respect to $S$. Since $L(\mu, \alpha)$ is nonempty, there exists $x \in L(\mu, \alpha)$. Now, from (SF1), $\mu(0) \leq \mu(x) \leq \alpha$, we have $0 \in L(\mu, \alpha)$. Let $x, y \in X$ be such that $x \ast y \in L(\mu, \alpha)$ and $y \in L(\mu, \alpha)$. Then we have $\mu(x) \leq S(\mu(x \ast y), \mu(y)) \leq S(\alpha, \alpha) = \alpha$, and so $x \in L(\mu, \alpha)$. This shows that the level set $L(\mu, \alpha)$ is an ideal of $X$.

Conversely, assume that every nonempty level subset $L(\mu; \alpha)$ of $\mu$ is an ideal of $X$. Then it can be easily checked that $\mu$ satisfies (SF1). If there exist $x, y \in X$ such that $\mu(x) > S(\mu(x \ast y), \mu(y))$, then by taking $t_0 := (1/2)\{\mu(x) + S(\mu(x \ast y), \mu(y))\}$, we have $x \ast y \in L(\mu; t_0)$ and $y \in L(\mu; t_0)$. Since $\mu$ is an ideal of $X$, $x \in L(\mu; t_0)$, we have $\mu(x) \leq t_0$, a contradiction. Hence $\mu$ is a fuzzy ideal of $X$ with respect to $S$. \hfill $\square$

Definition 3.6. Let $X$ be a BCK-algebra and a family of fuzzy sets $\{\mu_i \mid i \in I\}$ in a BCK-algebra $X$. Then the union $\bigvee_{i \in I} \mu_i$ of $\{\mu_i \mid i \in I\}$ is defined by

$$\left(\bigvee_{i \in I} \mu_i\right)(x) = \sup \{\mu_i(x) \mid i \in I\} \quad (3.2)$$

for each $x \in X$.

Theorem 3.7. If $\{\mu_i \mid i \in I\}$ is a family of fuzzy ideals of a BCK-algebra $X$ with respect to $S$, then $\bigvee_{i \in I} \mu_i(x)$ is a fuzzy ideal of $X$ with respect to $S$.

Proof. Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy ideals of $X$ with respect to $S$. It is easy to see that $\mu_i(0) \leq \mu_i(x)$ for all $x \in X$. For $x, y \in X$, we have

$$\left(\bigvee_{i \in I} \mu_i\right)(x) = \sup \{\mu_i(x) \mid i \in I\} \leq \sup \{S(\mu_i(x \ast y), \mu_i(y)) \mid i \in I\}$$

$$= S(\sup \{\mu_i(x \ast y) \mid i \in I\}, \sup \{\mu_i(y) \mid i \in I\}) \quad (3.3)$$

$$= S\left(\bigvee_{i \in I} \mu_i(x \ast y), \bigvee_{i \in I} \mu_i(y)\right).$$

Hence $\bigvee_{i \in I}$ is a fuzzy ideal of $X$ with respect to $S$. \hfill $\square$

Proposition 3.8. Every sensible fuzzy ideal of $X$ with respect to $S$ is order preserving.

Proposition 3.9. Let $\mu$ be a sensible fuzzy ideal of $X$ with respect to $S$. If the inequality $x \ast y \leq z$ holds in $X$, then $\mu(x) \leq S(\mu(y), \mu(z))$ for all $x, y, z \in X$.

Definition 3.10 [1]. A fuzzy set $\mu$ is called a fuzzy subalgebra of $X$ with respect to a $t$-conorm $S$ if $\mu(x \ast y) \leq S(\mu(x), \mu(y))$ for all $x, y \in X$.

Theorem 3.11. Let $S$ be a $t$-conorm. Then every sensible fuzzy ideal of $X$ with respect to $S$ is a sensible fuzzy subalgebra of $X$ with respect to $S$.

Proof. Straightforward. \hfill $\square$

The converse of Theorem 3.11 is not true in general as seen in the following example.
Example 3.12. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

\[
\begin{array}{|c|cccc|}
\hline
* & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & 0 \\
b & b & b & 0 & b \\
c & c & c & c & 0 \\
\hline
\end{array}
\]

Define a fuzzy set $\mu : X \rightarrow [0, 1]$ by $\mu(0) = \mu(b) = \mu(c) = 0$ and $\mu(a) = 1$ and let $S_m : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a function defined by $S_m(x, y) = \min\{x + y, 1\}$ which is a $t$-conorm for all $x, y \in [0, 1]$. By routine computation, we can easily check that $\mu$ is a sensible fuzzy subalgebra of $X$ with respect to $S_m$. But $\mu$ is not a sensible fuzzy ideal of $X$ with respect to $S_m$ because $\mu(a) = 1 \geq 0 = S_m(\mu(a \ast b), \mu(b))$.

Remark 3.13. In Example 3.12, we observe that a sensible fuzzy subalgebra with respect to $S$ is not a sensible fuzzy ideal with respect to $S$. So, a question arises: under what condition(s) a sensible fuzzy subalgebra with respect to $S$ is a sensible fuzzy ideal with respect to $S$? We answer this question in the following theorems without proofs.

Theorem 3.14. Let $S$ be a $t$-conorm. A sensible fuzzy subalgebra $\mu$ of $X$ with respect to $S$ is a sensible fuzzy ideal of $X$ with respect to $S$ if and only if for all $x, y, z \in X$, the inequality $x \ast y \leq z$ implies that $\mu(x) \leq S(\mu(y), \mu(z))$.

Theorem 3.15. Let $S$ be a $t$-conorm and let $X$ be a BCK-algebra in which the equality $x = (x \ast y) \ast y$ holds for all distinct elements $x$ and $y$ of $X$. Then every sensible fuzzy subalgebra of $X$ with respect to $S$ is a sensible fuzzy ideal of $X$ with respect to $S$.

Definition 3.16. Let $f : X \rightarrow Y$ be a mapping, where $X$ and $Y$ are nonempty sets, and $\mu$ is fuzzy set of $Y$. The preimage of $\mu$ under $f$ written $\mu^f$ is a fuzzy set of $X$ defined by $\mu^f(x) = \mu(f(x))$ for all $x \in X$.

Theorem 3.17. Let $f : X \rightarrow Y$ be a homomorphism of BCK-algebras. If $\mu$ is a fuzzy ideal of $Y$ with respect to $S$, then $\mu^f$ is a fuzzy ideal of $X$ with respect to $S$.

Proof. For any $x \in X$, we have $\mu^f(x) = \mu(f(x)) \geq \mu(\hat{0}) = \mu(f(0)) = \mu^f(0)$. Let $x, y \in X$. Then we have

\[
S(\mu^f(x \ast y), \mu^f(y)) = S(\mu(f(x \ast y)), \mu(f(y)))
\]

\[
= S(\mu(f(x) \ast f(y)), \mu(f(y)))
\]

\[
\leq \mu(f(x)) = \mu^f(x).
\]

Hence $\mu^f$ is a fuzzy ideal of $X$ with respect to $S$.

Theorem 3.18. Let $f : X \rightarrow Y$ be an epimorphism of BCK-algebras. If $\mu^f$ is a fuzzy ideal of $X$ with respect to $S$, then $\mu$ is a fuzzy ideal of $Y$ with respect to $S$.

Proof. Let $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Then $\mu(y) = \mu(f(x)) = \mu^f(x) \geq \mu^f(0) = \mu(f(0)) = \mu(\hat{0})$, where $\hat{0} = f(0)$. Let $x, y \in Y$. Then there exist $a, b \in X$ such that
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Let \( f(a) = x \) and \( f(b) = y \). It follows that

\[
\mu(x) = \mu(f(a)) = \mu(f(a)) = \mu(\mu(f(a))) \\
\leq S(\mu^*(a \ast b), \mu^*(b)) = S(\mu(f(a \ast b)), \mu(f(b))) \\
= S(\mu(f(a) \ast f(b)), \mu(f(b))) = S(\mu(x \ast y), \mu(y)).
\]

(3.5)

Hence \( \mu \) is a fuzzy ideal of \( Y \) with respect to \( S \).

**Proof.** The proof is obtained dually by using the notion of t-conorm \( S \) instead of t-norm \( T \) in [4].

**Theorem 3.20.** Let \( S \) be a t-conorm and let \( f : X \rightarrow Y \) be an epimorphism of BCK-algebras, \( \nu \) sensible fuzzy ideal of \( Y \) with respect to \( S \) and \( \mu \), the preimage of \( \nu \) under \( f \). Then \( \mu \) is a sensible fuzzy ideal of \( X \) with respect to \( S \).

**Proof.** Since 0 is a fuzzy ideal of \( S \), it follows that either \( x \) or \( y \) is a fuzzy ideal of \( X \). Thus

\[
\mu(x) \leq \alpha_n = S(\mu(x \ast y), \mu(y)).
\]

(3.6)

If \( x \ast y \notin A^*_n \) and \( y \notin A^*_n \), then the following four cases arise:

1. \( x \ast y \in X \setminus A_n \) and \( y \in X \setminus A_n \),
2. \( x \ast y \in A_{n-1} \) and \( y \in A_{n-1} \),
3. \( x \ast y \in X \setminus A_n \) and \( y \in A_{n-1} \),
4. \( x \ast y \in A_{n-1} \) and \( y \in X \setminus A_n \).

But, in either case, we know that

\[
\mu(x) \leq S(\mu(x \ast y), \mu(y)).
\]

(3.7)

If \( x \ast y \in A^*_n \) and \( y \notin A^*_n \), then either \( y \in A_{n-1} \) or \( y \in X \setminus A_n \). It follows that either \( x \in A_n \) or \( x \in X \setminus A_n \). Thus

\[
\mu(x) \leq S(\mu(x \ast y), \mu(y)).
\]

(3.8)
If \( x \neq y \notin A^*_n \) and \( y \in A^*_n \), then by similar process, we have

\[
\mu(x) \leq S(\mu(x \ast y), \mu(y)).
\]

(3.9)

This completes the proof. \( \square \)

**Definition 3.22** [9]. A BCK-algebra \( X \) is said to satisfy the ascending (resp., descending) chain condition (ACC (resp., DCC)) if for every ascending (resp., descending) sequence \( A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \) (resp., \( A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \)) of ideals of \( X \) there exists a natural number \( n \) such that \( A_n = A_k \) for all \( n \geq k \). If \( X \) satisfies DCC, \( X \) is an Artin BCK-algebras.

**Theorem 3.23.** Let \( S \) be a \( t \)-conorm. If \( \mu \) is a fuzzy ideal of \( X \), with respect to \( S \), having finite image, then \( X \) is an Artin BCK-algebra.

**Proof.** Suppose that there exists a strictly descending chain \( A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \) of fuzzy ideals of \( X \) which does not terminate at finite step. Define a fuzzy set \( \mu \) in \( X \) by

\[
\mu(x) := \begin{cases} 
\frac{1}{n+1} & \text{if } x \in A_n \setminus A_{n+1}, \ n = 0, 1, 2, \ldots, \\
0 & \text{if } x \in \bigcap_{n=0}^{\infty} A_n, 
\end{cases}
\]  

(3.10)

where \( A_0 = X \). We prove that \( \mu \) is a fuzzy ideal of \( X \) with respect to \( S \). Clearly, \( \mu(0) \leq \mu(x) \) for all \( x \in X \). Let \( x, y \in X \). Assume that \( x \ast y \in A_n \setminus A_{n+1} \) and \( y \in A_k \setminus A_{k+1} \) for \( n = 0, 1, 2, \ldots; \ k = 0, 1, 2, \ldots \). Without loss of generality, we may assume that \( n \leq k \). Then obviously \( y \in A_n \), and so \( x \in A_n \) because \( A_n \) is a fuzzy ideal of \( X \). Hence

\[
\mu(x) \leq \frac{1}{n+1} = S(\mu(x \ast y), \mu(y)).
\]

(3.11)

If \( x \ast y, y \in \bigcap_{n=0}^{\infty} A_n \), then \( x \in \bigcap_{n=0}^{\infty} A_n \). Thus

\[
\mu(x) = 0 = S(\mu(x \ast y), \mu(y)).
\]

(3.12)

If \( x \ast y \notin \bigcap_{n=0}^{\infty} A_n \) and \( y \in \bigcap_{n=0}^{\infty} A_n \), then there exists \( k \in \mathbb{N} \) such that \( x \ast y \in A_k \setminus A_{k+1} \). It follows that \( x \in A_k \) so that

\[
\mu(x) \leq \frac{1}{k+1} = S(\mu(x \ast y), \mu(y)).
\]

(3.13)

Finally, suppose that \( x \ast y \in \bigcap_{n=0}^{\infty} A_n \) and \( y \notin \bigcap_{n=0}^{\infty} A_n \). Then \( y \in A_r \setminus A_{r+1} \) for some \( r \in \mathbb{N} \). Hence \( x \in A_r \), and so

\[
\mu(x) \leq \frac{1}{r+1} = S(\mu(x \ast y), \mu(y)).
\]

(3.14)

Consequently, we conclude that \( \mu \) is a fuzzy ideal of \( X \) with respect to \( S \) and \( \mu \) has infinite number of different values. This is a contradiction, and the proof is complete. \( \square \)

**Theorem 3.24.** Let \( S \) be a \( t \)-conorm. The following statements are equivalent:

(i) every ascending chain of ideals of \( X \) with respect to \( S \) terminates at finite step,

(ii) the set of values of any fuzzy ideal with respect to \( S \) is a well-ordered subset of \([0, 1]\).
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Proof. Let $\mu$ be a fuzzy ideal of $X$ with respect to $S$. Suppose that the set of values of $\mu$ is not a well-ordered subset of $[0,1]$. Then there exists a strictly increasing sequence $\{\alpha_n\}$ such that $\mu(x) = \alpha_n$. Let $G_n := \{x \in X \mid \mu(x) \leq \alpha\}$. Then

$$G_1 \subset G_2 \subset G_3 \subset \cdots$$

(3.15)

is a strictly ascending chain of ideals of $X$ which is not terminating. This is a contradiction.

Conversely, suppose that there exists a strictly ascending chain

$$G_1 \subset G_2 \subset G_3 \subset \cdots$$

(*)

of ideals of $X$ with respect to $S$ which does not terminate at finite step. Define a fuzzy set $\mu$ in $X$ by

$$\mu(x) := \begin{cases} 1/k, & \text{where } k = \max\{n \in \mathbb{N} \mid x \in G_n\}, \\ 1 & \text{if } x \in G_n, \end{cases}$$

(3.16)

where $G = \bigcup_{n \in \mathbb{N}} G_n$. Since $0 \in G_n$ for all $n = 0,1,\ldots$, therefore, $\mu(0) \leq \mu(x)$ for all $x \in X$. Let $x, y \in X$. If $x \star y, y \in G_n \setminus G_{n-1}$ for $n = 2,3,\ldots$, then $x \in G_n$. Thus, we obtain

$$\mu(x) \leq \frac{1}{n} = S(\mu(x \star y), \mu(y)).$$

(3.17)

Assume that $x \star y \in G_n$ and $y \in G_n \setminus G_m$ for all $m < n$. Since $\mu$ is an ideal of $X$, therefore, $x \in G_n$. Thus

$$\mu(x) \leq \frac{1}{n} \leq \frac{1}{m+1} \leq \mu(y),$$

(3.18)

and hence

$$\mu(x) \leq S(\mu(x \star y), \mu(y)).$$

(3.19)

Similarly, for the case $x \star y \in G_n \setminus G_m$ and $y \in G_n$, we have

$$\mu(x) \leq S(\mu(x \star y), \mu(y)).$$

(3.20)

Hence $\mu$ is an ideal of $X$ with respect to $t$-conorm $S$. Since the chain (*) is not terminating, $\mu$ has strictly descending sequence of values. This contradicts that the value of any set of fuzzy ideal with respect to $S$ is well ordered. This ends the proof. □

Lemma 3.25. Let $T$ be a $t$-norm. Then $t$-conorm $S$ can be defined as

$$S(x,y) = 1 - T(1-x,1-y).$$

(3.21)

Proof. Straightforward. □

Theorem 3.26. A fuzzy set $\mu$ of a BCK-algebra $X$ is a $T$-fuzzy ideal of $X$ if and only if its complement $\mu^c$ is an $S$-fuzzy ideal of $X$. 

Proof. Straightforward. □
Proof. Let $\mu$ be a $T$-fuzzy ideal of $X$. For $x, y \in X$, we have

$$
\begin{align*}
\mu^c(0) &= 1 - \mu(0) \leq 1 - \mu(x) = \mu^c(x), \\
\mu^c(x) &= 1 - \mu(x) \leq 1 - T\mu((x \ast y), \mu(y)) \\
&= 1 - T(1 - \mu^c((x \ast y), 1 - \mu^c(y))) \\
&= S(\mu^c(x \ast y), \mu^c(y)).
\end{align*}
$$

(3.22)

Hence $\mu^c$ is an S-fuzzy ideal of $X$. The converse is proved similarly.

\section{S-product and direct product with respect to a t-conorm}

In this section, we discuss properties of S-product and direct product of fuzzy ideals of a BCK-algebra with respect to a t-conorm.

\begin{definition}
Let $S$ be a t-conorm and let $\lambda$ and $\mu$ be two fuzzy sets in $X$. Then the S-product of $\lambda$ and $\mu$ is denoted by $[\lambda \cdot \mu]_S$ and defined by $[\lambda \cdot \mu]_S(x) = S(\lambda(x), \mu(x))$, for all $x \in X$.
\end{definition}

\begin{theorem}
Let $\lambda$ and $\mu$ be two fuzzy ideals of $X$ with respect to $S$. If a t-conorm $S^*$ dominates $S$, that is, if $S^*(S(\alpha, \gamma), S(\beta, \delta)) \leq S(S^*(\alpha, \beta), S^*(\gamma, \delta))$ for all $\alpha, \beta, \gamma, \delta \in [0, 1]$, then $S^*$-product $[\lambda \cdot \mu]_{S^*}$ is a fuzzy ideal of $X$ with respect to $S$.
\end{theorem}

\begin{proof}
For any $x \in X$, we have

$$
[\lambda \cdot \mu]_S(0) = S^*(\lambda(0), \mu(0)) \leq S^*(\lambda(x), \mu(x)) = [\lambda \cdot \mu]_{S^*}(x).
$$

(4.1)

Let $x, y \in X$. Then

$$
[\lambda \cdot \mu]_S(x) = S^*(\lambda(x), \mu(x)) \\
\leq S^*(S(\lambda(x \ast y), \lambda(y)), S(\mu(x \ast y), \mu(y))) \\
\leq S(S^*(\lambda(x \ast y), \mu(x \ast y)), S^*(\lambda(y), \mu(y))) \\
= S([\lambda \cdot \mu]_S(x \ast y), [\lambda \cdot \mu]_{S^*}(y)).
$$

(4.2)

Hence $[\lambda \cdot \mu]_{S^*}$ is a fuzzy ideal of $X$ with respect to $S$.
\end{proof}

\begin{theorem}
Let $S$ and $S^*$ be t-conorms in which $S^*$ dominates $S$. Let $f : X \to Y$ be an epimorphism of BCK-algebras. If $\lambda$ and $\mu$ are fuzzy ideals of $Y$ with respect to $S$, then $f^{-1}([\lambda \cdot \mu]_{S^*}) = [f^{-1}(\lambda), f^{-1}(\mu)]_{S^*}$.
\end{theorem}

\begin{proof}
For any $x \in X$, we have

$$
f^{-1}([\lambda \cdot \mu]_S(x)) = [\lambda \cdot \mu]_S(f(x)) = S^*(\lambda(f(x)), \mu(f(x))) \\
= S^*([f^{-1}(\lambda)](x), [f^{-1}(\mu)](x)) = [f^{-1}(\lambda), f^{-1}(\mu)]_{S^*}(x).
$$

(4.3)

\end{proof}

\begin{theorem}
Let $S$ be a t-conorm. Let $X_1$ and $X_2$ be BCK-algebras and let $X = X_1 \times X_2$ be the direct product BCK-algebra of $X_1$ and $X_2$. Let $\lambda$ be a fuzzy ideal of a BCK-algebra $X_1$ with

...
respect to $S$ and let $\mu$ be a fuzzy ideal of a BCK-algebra $X_2$ with respect to $S$. Then $\nu = \lambda \times \mu$ is a fuzzy ideal of $X = X_1 \times X_2$ with respect to $S$ defined by

$$\nu(x_1, x_2) = (\lambda \times \mu)(x_1, x_2) = S(\lambda(x_1), \mu(x_2)). \quad (4.4)$$

**Proof.** For any $(x, y) \in X_1 \times X_2 = X$, we have

$$\nu(0, 0) = (\lambda \times \mu)(0, 0) = S(\lambda(0), \mu(0)) \leq S(\lambda(x), \mu(y)) = (\lambda \times \mu)(x, y) = \nu(x, y). \quad (4.5)$$

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in X_1 \times X_2 = X$. Then we have

$$\nu(x) = (\lambda \times \mu)(x) = (\lambda \times \mu)(x_1, x_2) = S(\lambda(x_1), \mu(x_2))$$

$$\leq S(S(\lambda(x_1 \ast y_1), \lambda(y_1)), S(\mu(x_2 \ast y_2), \mu(y_2)))$$

$$= S(S(\lambda(x_1 \ast y_1), \mu(x_2 \ast y_2)), S(\lambda(y_1), \mu(y_2)))$$

$$= S((\lambda \times \mu)(x_1 \ast y_1, x_2 \ast y_2), (\lambda \times \mu)(y_1, y_2))$$

$$= S((\lambda \times \mu)((x_1, x_2) \ast (y_1, y_2)), (\lambda \times \mu)(y_1, y_2))$$

$$= S((\lambda \times \mu)(x \ast y), (\lambda \times \mu)(y)) = S(\nu(x \ast y), \nu(y)).$$

Hence $\nu$ is a fuzzy ideal of $X$ with respect to $S$. \hfill \Box

The relationship between fuzzy ideals $\mu_1 \times \mu_2$ and $[\mu_1 \ast \mu_2]_S$ with respect to $S$ can be viewed via the following diagram:

\[ X \xrightarrow{d} X \times X \]

\[ \begin{array}{c}
| \mu_1 \times \mu_2 \\
I \xleftarrow{S} I \\
\mu_1 \\
\mu_2 \\
\end{array} \]

where $I = [0, 1]$ and $d : X \to X \times X$ is defined by $d(x) = (x, x)$. It is easy to see that $[\mu_1 \ast \mu_2]_S$ is the preimage of $\mu_1 \times \mu_2$ under $d$.

Converse of Theorem 4.4 may not be true as seen in the following example.

**Example 4.5.** Let $X$ be a BCK-algebra and let $s, t \in [0, 1]$. Define fuzzy sets $\mu_1$ and $\mu_2$ in $X$ by $\mu_1(x) = 1$ and

$$\mu_2(x) = \begin{cases} 
1 & \text{if } x = 0, \\
0 & \text{otherwise} 
\end{cases} \quad (4.8)$$

for all $x \in X$, respectively.

If $x = 0$, then $\mu_2(x) = 1$, and thus

$$(\mu_1 \times \mu_2)(x, x) = S(\mu_1(x), \mu_2(x)) = S(1, 1) = 1. \quad (4.9)$$
If \( x \neq 0 \), then \( \mu_2(x) = t \), and thus

\[
(\mu_1 \times \mu_2)(x, x) = S(\mu_1(x), \mu_2(x)) = S(1, t) = 1. \tag{4.10}
\]

That is, \( \mu_1 \times \mu_2 \) is a constant function and so \( \mu_1 \times \mu_2 \) is a fuzzy ideal of \( X_1 \times X_2 \). Now \( \mu_1 \) is a fuzzy ideal of \( X \), but \( \mu_2 \) is not a fuzzy ideal of \( X \) since for \( x \neq 0 \), we have \( \mu_2(0) = 1 > t = \mu_2(x) \).

Now we generalize the product of two fuzzy ideals with respect to \( S \) to the product of \( n \) fuzzy ideals with respect to \( S \). We first need to generalize the domain of \( t \)-conorm \( S \) to \( \prod_{i=1}^{n}[0,1] \) as follows.

**Definition 4.6.** The function \( S_n : \prod_{i=1}^{n}[0,1] \to [0,1] \) is defined by

\[
S_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = S(\alpha_i, S_{n-1}(\alpha_1, \alpha_2, \ldots, \alpha_{i+1}, \ldots, \alpha_n)) \tag{4.11}
\]

for all \( 1 \leq i \leq n, n \geq 2, S_2 = S, \) and \( S_1 = \text{identity} \).

**Lemma 4.7.** For a \( t \)-conorm \( S \) and every \( \alpha_i, \beta_i \in [0,1] \), where \( 1 \leq i \leq n, n \geq 2 \),

\[
S_n(S(\alpha_1, \beta_1), S(\alpha_2, \beta_2), \ldots, S(\alpha_n, \beta_n)) = S(S_n(\alpha_1, \alpha_2, \ldots, \alpha_n), S_n(\beta_1, \beta_2, \ldots, \beta_n)). \tag{4.12}
\]

**Theorem 4.8.** Let \( S \) be a \( t \)-conorm and let \( X = \prod_{i=0}^{n} X_i \) be the direct product of BCK-algebras. If \( \mu_i \) is a fuzzy ideal of \( X_i \) with respect to \( S \), where \( 1 \leq i \leq n \), then \( \mu = \prod_{i=1}^{n} \mu_i \) defined by

\[
\mu(x) = \left( \prod_{i=1}^{n} \mu_i \right) (x_1, x_2, \ldots, x_n) = S_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)) \tag{4.13}
\]

for all \( x = (x_1, x_2, \ldots, x_n) \in X \) is a fuzzy ideal of \( X \) with respect to \( S \).

**Proof.** Clearly, \( \mu(0) \leq \mu(x) \) for all \( x = (x_1, x_2, \ldots, x_n) \in X = \prod_{i=1}^{n} X_i \).

Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) be the elements of \( X = \prod_{i=1}^{n} X_i \). Then

\[
\mu(x) = \left( \prod_{i=1}^{n} \mu_i \right) (x_1, x_2, \ldots, x_n) = S_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n))
\]

\[
\leq S_n(S(\mu_1(x_1 \ast y_1), \mu(y_1)), S(\mu_2(x_2 \ast y_2), \mu(y_2)), \ldots, S(\mu_n(x_n \ast y_n), \mu(y_n)))
\]

\[
= S(S_n(\mu_1(x_1 \ast y_1), \mu_2(x_2 \ast y_2), \ldots, \mu_n(x_n \ast y_n)), S_n(\mu(y_1), \mu(y_2), \ldots, \mu(y_n)))
\]

\[
= S\left( \left( \prod_{i=1}^{n} \mu_i \right) (x_1 \ast y_1, x_2 \ast y_2, \ldots, x_n \ast y_n), \left( \prod_{i=1}^{n} \mu_i \right) (y_1, y_2, \ldots, y_n) \right)
\]

\[
= S(\mu(x \ast y), \mu(y)). \tag{4.14}
\]

Hence \( \mu = \prod_{i=1}^{n} \mu_i \) is a fuzzy ideals of \( X \) with respect to \( S \). \( \square \)
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Acknowledgments

The authors are deeply grateful to the Editor-in-Chief and referees for their valuable comments and suggestions for improving the paper. The research work of the first author is supported by Punjab University College of Information Technology (PUCIT).

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