ON INVERSION OF H-TRANSFORM IN \( E_{\nu,r} \)-SPACE

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ABSTRACT. The paper is devoted to study the inversion of the integral transform

\[ (H f)(x) = \int_0^\infty H_{p,q}^{m,n} \left[ \begin{array}{c} z(t) \\ \nu \\ \eta \end{array} \right] \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} f(t) dt \]

involving the \( H \)-function as the kernel in the space \( E_{\nu,r} \) of functions \( f \) such that

\[ \int_0^\infty \left| t^\nu f(t) \right| \frac{dt}{t} < \infty \quad (1 < r < \infty, \nu \in \mathbb{R}). \]

KEY WORDS AND PHRASES: \( H \)-function, Integral transform,

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1. INTRODUCTION

This paper deals with the integral transforms of the form

\[ (H f)(x) = \int_0^\infty H_{p,q}^{m,n} \left[ \begin{array}{c} z(t) \\ \nu \\ \eta \end{array} \right] \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} f(t) dt, \quad (1.1) \]

where \( H_{p,q}^{m,n} \left[ \begin{array}{c} z(t) \\ \nu \\ \eta \end{array} \right] \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \) is the \( H \)-function, which is a function of general hypergeometric type being introduced by S. Pincherle in 1888 (see [2, §1.19]). For integers \( m, n, p, q \) such that

\[ 0 \leq m \leq q, \quad 0 \leq n \leq p, \quad a_i, b_j \in \mathbb{C} \quad \text{and} \quad \alpha_i, \beta_j \in \mathbb{R}_+ = [0, \infty) \quad (1 \leq i \leq p, 1 \leq j \leq q), \]

it can be
written by
\[
H^{m,n}_{p,q} \left[ \frac{(a_1, \alpha_1)_{1,p}}{(b_1, \beta_1)_{1,q}} \right] = H^{m,n}_{p,q} \left[ \frac{(a_1, \alpha_1), \ldots, (a_p, \alpha_p)}{(b_1, \beta_1), \ldots, (b_q, \beta_q)} \right]
\]
\[
= \frac{1}{2\pi i} \int_L \mathcal{H}^{m,n}_{p,q} \left[ \frac{(a_1, \alpha_1)_{1,p}}{(b_1, \beta_1)_{1,q}} \right] z^{-s} ds,
\]

where
\[
\mathcal{H}^{m,n}_{p,q} \left[ \frac{(a_1, \alpha_1)_{1,p}}{(b_1, \beta_1)_{1,q}} \right] = \frac{\prod_{j=1}^p \Gamma(b_j + \beta_j s) \prod_{i=1}^m \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=m+1}^q \Gamma(1 - b_i - \beta_i s) \prod_{j=p+1}^q \Gamma(1 - a_i + \alpha_i s)}.
\]

the contour $L$ is specially chosen and an empty product, if it occurs, is taken to be one. The theory of this function may be found in Braaksma [1], Srivastava et al. [13, Chapter 1], Mathai and Saxena [8, Chapter 2] and Prudnikov et al. [9, §8.3]. We abbreviate the $H$-function (1.2) and the function (1.3) to $H(z)$ and $\mathcal{H}(s)$ when no confusion occurs. We note that the formal Mellin transform $\mathcal{M}$ of (1.1) gives the relation
\[
(\mathcal{M}Hf)(s) = \mathcal{H}(s)(\mathcal{M}f)(1 - s).
\]

Most of the known integral transforms can be put into the form (1.1), in particular, if $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$, (1.1) is the integral transform with Meijer's $G$-function in the kernel (Rooney [11], Sarmko et al. [12, §36]). The integral transform (1.1) with the $H$-function kernel or the $H$-transform was investigated by many authors (see Bibliography in Kilbas et al. [5-6]). In Kilbas et al. [5-7] we have studied it in the space $L_{n,r} (1 \leq r < \infty, \nu \in \mathbb{R})$ consisted of Lebesgue measurable complex valued functions $f$ for which
\[
\int_0^\infty |t^n f(t)| \frac{dt}{t} < \infty.
\]

We have investigated the mapping properties such as the boundedness, the representation and the range of the $H$-transform (1.1) on the space $L_{n,2}$ in Kilbas et al. [5] and on the space $L_{n,r}$ with any $1 \leq r < \infty$ in Kilbas et al. [6-7], provided that $\alpha^* \geq 0$, $\delta = 1$ and $\Delta = 0$ or $\Delta \neq 0$, respectively. In Glaeske et al. [3] the results were extended to any $\delta > 0$. Here
\[
\alpha^* = \sum_{i=1}^n \alpha_i - \sum_{i=m+1}^p \alpha_i - \sum_{j=m+1}^q \beta_j,
\]
\[
\delta = \prod_{i=1}^p \alpha_i - \prod_{j=1}^q \beta_j,
\]
\[
\Delta = \sum_{j=m+1}^q \beta_j - \sum_{i=1}^p \alpha_i.
\]

In particular, we have proved that for certain ranges of parameters, the $H$-transform (1.1) have the representations
\[
(Hf)(x) = hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H^{m,n+1}_{p+1,q+1} \left[ \frac{(a_1, \alpha_1, \ldots, a_p, \alpha_p)}{(b_1, \beta_1, \ldots, b_q, \beta_q)} \right] \left( -\lambda, h \right), \left( -\lambda - 1, h \right) \frac{f(t)}{t} dt (1.9)
\]
or
\[
(Hf)(x) = -hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H^{m+1,n}_{p+1,q+1} \left[ \frac{(a_1, \alpha_1, \ldots, a_p, \alpha_p)}{(b_1, \beta_1, \ldots, b_q, \beta_q)} \right] \left( -\lambda, h \right), \left( -\lambda - 1, h \right) \frac{f(t)}{t} dt (1.10)
\]
owing to the value of Re(λ), where λ ∈ ℂ and h ∈ ℜ \ {0}.

In this paper we apply the results of Kilbas et al. [5-7] and Glaeske et al. [3] to find the inverse of the integral transforms (1.1) on the space ℒ_r with 1 < r < ∞ and μ ∈ ℜ. Section 2 contains preliminary information concerning the properties of the H-transform (1.1) in the space ℒ_r and an asymptotic behavior of the H-function (1.2) at zero and infinity. In Sections 3 and 4 we prove that the inversion of the H-transform have the respective form (1.9) or (1.10):

$$ f(z) = h z^{1-(\lambda+1)/h} \frac{d}{dz} x^{(\lambda+1)/h} $$

or

$$ f(z) = -h z^{1-(\lambda+1)/h} \frac{d}{dz} x^{(\lambda+1)/h} $$

provided that a* = 0. Section 3 is devoted to treat on the spaces ℒ_r and ℒ_r with Δ = 0, while Section 4 on the space ℒ_r with Δ ≠ 0.


2. PRELIMINARIES

We give here some results from Kilbas et al. [5-6], Glaeske et al. [3] and from Kilbas and Saigo [4], Mathai and Saxena [8], Srivastava et al. [13] concerning the properties of H-transforms (1.1) in ℒ_r-spaces and the asymptotic behavior of the H-function at zero and infinity, respectively.

For the H-function (1.2), let a* and Δ be defined by (1.6) and (1.8) and let

$$ \alpha = \begin{cases} \max \left[ -\text{Re} \left( \frac{b_1}{\beta_1} \right), \ldots, -\text{Re} \left( \frac{b_m}{\beta_m} \right) \right] & \text{if } m > 0, \\ -\infty & \text{if } m = 0; \end{cases} $$

and

$$ \beta = \begin{cases} \min \left[ \frac{1-a_1}{\alpha_1}, \ldots, \frac{1-a_n}{\alpha_n} \right] & \text{if } n > 0, \\ \infty & \text{if } n = 0; \end{cases} $$

$$ a_i^* = \sum_{j=1}^{m} \beta_j - \sum_{i=m+1}^{p} \alpha_i; \quad a^* = \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j; \quad a_i^* + a^* = a^*; \quad (2.3) $$

$$ \mu = \sum_{j=1}^{q} b_i - \sum_{i=1}^{p} a_i + \frac{p-q}{2}. \quad (2.4) $$

For the function \( \mathcal{H}(s) \) given in (1.3), the exceptional set of \( \mathcal{H} \) is meant the set of real numbers \( \nu \) such that \( \alpha < 1 - \nu < \beta \) and \( \mathcal{H}(s) \) has a zero on the line \( \text{Re}(s) = 1 - \nu \) (see Rooney [11]). For two Banach space \( X \) and \( Y \) we denote by \( \|X, Y\| \) the collection of bounded linear operators from \( X \) to \( Y \).

**THEOREM 2.1.** [5, Theorem 3], [6, Theorem 3.3] Suppose that \( \alpha < 1 - \nu < \beta \) and that either \( a^* > 0 \) or \( a^* = 0 \), \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 0 \). Then

(a) There is a one-to-one transform \( H \in \mathcal{H}(s) \) so that (1.4) holds for \( f \in \mathcal{L}_{r,2} \) and \( \text{Re}(s) = 1 - \nu \). If \( a^* = 0 \), \( \Delta(1 - \nu) + \text{Re}(\mu) = 0 \) and \( \nu \) is not in the exceptional set of \( \mathcal{H} \), then the operator \( H \) transforms \( \mathcal{L}_{r,2} \) onto \( \mathcal{L}_{1-\nu,2} \).
(b) If \( f \in \mathcal{L}_{\nu,3} \) and \( \text{Re}(\lambda) > (1 - \nu)h - 1 \), \( Hf \) is given by (1.9). If \( f \in \mathcal{L}_{\nu,3} \) and \( \text{Re}(\lambda) < (1 - \nu)h - 1 \), then \( Hf \) is given by (1.10).

**Theorem 2.2.** [6, Theorem 4.1], [3, Theorem 1] Let \( a^* = \Delta = 0, \text{Re}(\mu) = 0 \) and \( \alpha < 1 - \nu < \beta \).

(a) The transform \( H \) is defined on \( \mathcal{L}_{\nu,3} \) and it can be extended to \( \mathcal{L}_{\nu,r} \) as an element of \([\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}]\) for \( 1 < r < \infty \).

(b) If \( 1 < r \leq 2 \), the transform \( H \) is one-to-one on \( \mathcal{L}_{\nu,r} \) and there holds the equality

\[
(\mathcal{M}Hf)(s) = \mathcal{H}(s)(\mathcal{M}f)(1 - s), \quad \text{Re}(s) > 0.
\]

(c) If \( f \in \mathcal{L}_{\nu,r} \) \((1 < r < \infty)\), then \( Hf \) is given by (1.9) for \( \text{Re}(\lambda) > (1 - \nu)h - 1 \), while \( Hf \) is given by (1.10) for \( \text{Re}(\lambda) < (1 - \nu)h - 1 \).

**Theorem 2.3.** [6, Theorem 5.1], [3, Theorem 3] Let \( a^* = 0, \Delta > 0, -\infty < \alpha < 1 - \nu < \beta, 1 < r < \infty \) and \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r) \), where

\[
\gamma(r) = \max \left[ \frac{1}{r}, \frac{1}{r'} \right] \quad \text{with} \quad \frac{1}{r} + \frac{1}{r'} = 1.
\]

(a) The transform \( H \) is defined on \( \mathcal{L}_{\nu,3} \), and it can be extended to \( \mathcal{L}_{\nu,r} \) as an element of \([\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}]\) for all \( s \) with \( r \leq s < \infty \) such that \( s' \geq [1/2 - \Delta(1 - \nu) - \text{Re}(\mu)]^{-1} \) with \( 1/s + 1/s' = 1 \).

(b) If \( 1 < r \leq 2 \), the transform \( H \) is one-to-one on \( \mathcal{L}_{\nu,r} \) and there holds the equality (2.5).

(c) If \( f \in \mathcal{L}_{\nu,r} \) and \( g \in \mathcal{L}_{\nu,s} \) with \( 1 < r < \infty, 1 < s < \infty, 1/r + 1/s \geq 1 \) and \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \max(\gamma(r), \gamma(s)) \), then the relation

\[
\int_0^\infty f(x)(Hg)(x)dx = \int_0^\infty g(x)(Hf)(x)dx
\]

holds.

The following two assertions give the asymptotic behavior of the the \( H \)-function (1.2) at zero and infinity provided that the poles of Gamma functions in the numerator of \( \mathcal{H}(s) \) do not coincide, i.e.

\[
\beta_j(a_i - 1 - k) \neq \alpha_i(b_j + l) \quad (i = 1, \ldots, n; j = 1, \ldots, m; k, l = 0, 1, 2, \ldots).
\]

**Theorem 2.4.** [8, §1.1.6], [13, §2.2] Let the condition (2.8) be satisfied and poles of Gamma functions \( \Gamma(b_j + \beta, s) \) \((j = 1, \ldots, m)\) be simple, i.e.

\[
\beta_j(b_j + k) \neq \beta_j(b_i + l) \quad (i \neq j; i, j = 1, \ldots, m; k, l = 0, 1, 2, \ldots).
\]

If \( \Delta \geq 0 \), then

\[
H_{\nu,3}^{m,n}(x) = O(x^\rho) \quad (|x| \to 0) \quad \text{with} \quad \rho = \min_{1 \leq i \leq m} \left[ \frac{\text{Re}(b_i)}{\beta_j} \right].
\]

**Theorem 2.5.** [4, Corollary 3] Let \( a^*, \Delta \) and \( \mu \) be given by (1.6), (1.8) and (2.4), respectively. Let the conditions in (2.8) be satisfied and poles of Gamma functions \( \Gamma(1 - a_i - \alpha_i s) \) \((i = 1, \ldots, n)\) be simple, i.e.

\[
\alpha_j(1 - a_i + k) \neq \alpha_i(1 - a_j + l) \quad (i \neq j; i, j = 1, \ldots, n; k, l = 0, 1, 2, \ldots).
\]
If \( a^* = 0 \) and \( \Delta > 0 \), then
\[
H^{m,n}_{p,q}(z) = O(z^\epsilon) \quad (|z| \to \infty) \quad \text{with} \quad \epsilon = \max \left[ \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(\alpha_i) - 1}{\alpha_i} \right], \frac{\text{Re}(\mu) + 1/2}{\Delta} \right].
\] (2.12)

REMARK 2.1. It was proved in Kilbas and Saigo [4, §6] that if poles of Gamma functions \( \Gamma(1 - \alpha_i, \cdot) \) (\( i = 1, \cdots, n \)) are not simple (i.e. conditions in (2.11) are not satisfied), then the \( H \)-function (1.1) have power-logarithmic asymptotics at infinity. In this case the logarithmic multiplier \( [\log(z)]^N \) with \( N \) being the maximal number of orders of the poles may be added to the power multiplier \( z^\epsilon \) and hence the asymptotic estimate \( O(z^\epsilon) \) in (2.12) may be replaced by \( O(z [\log(z)]^N) \). The same result is valid in the case of the asymptotics of the \( H \)-function (1.1) at zero, and the estimate \( O(z^\epsilon) \) in (2.10) may be replaced by \( O(z^\epsilon [\log(z)]^M) \), where \( M \) is the maximal number of orders of the points at which the poles of \( \Gamma(b_j + \beta_j, \cdot) \) (\( j = 1, \cdots, m \)) coincide.

3. INVERSION OF \( H \)-TRANSFORM IN \( \mathbb{L}_{\nu,2} \) AND \( \mathbb{L}_{\nu,\nu} \) WHEN \( \Delta = 0 \)

In this and next sections we investigate that \( H \)-transform will have the inverse of the form (1.11) or (1.12). If \( f \in \mathbb{L}_{\nu,2} \), and \( H \) is defined on \( \mathbb{L}_{\nu,\nu} \), then according to Theorem 2.2, the equality (2.5) holds under the assumption there. This fact implies the relation
\[
(\mathcal{M}f)(s) = \frac{(\mathcal{MF})(1 - s)}{\mathcal{H}(1 - s)}
\] (3.1)
for \( \text{Re}(s) = \nu \). By (1.3) we have
\[
\frac{1}{\mathcal{H}(1 - s)} = \mathcal{K}_{p,q}^{m,n} \left( \frac{(1 - a_i - \alpha_i, \cdot)_{n+1,p}, (1 - a_i - \alpha_i, \cdot)_{1,n}}{(1 - b_j - \beta_j, \cdot)_{m+1,q}, (1 - b_j - \beta_j, \cdot)_{1,m}} \right) \equiv \mathcal{K}_0(s),
\] (3.2)
and hence (3.1) takes the form
\[
(\mathcal{M}f)(s) = (\mathcal{MF})(1 - s)\mathcal{K}_0(s) \quad (\text{Re}(s) = \nu).
\] (3.3)
We denote by \( \alpha_0, \beta_0, \alpha_0^*, \alpha_0^*, \alpha_0^*, \delta_0, \Delta_0 \) and \( \mu_0 \) for \( \mathcal{K}_0 \) instead of those for \( \mathcal{K} \). Then we find
\[
\alpha_0 = \left\{ \begin{array}{ll}
\max & \left[ \frac{\text{Re}(b_{m+1}) - 1}{\beta_{m+1}} + 1, \cdots, \frac{\text{Re}(b_q) - 1}{\beta_q} + 1 \right] & \text{if } q > m, \\
-\infty & \text{if } q = m;
\end{array} \right.
\] (3.4)
\[
\beta_0 = \left\{ \begin{array}{ll}
\min & \left[ \frac{\text{Re}(a_{n+1})}{\alpha_{n+1}} + 1, \cdots, \frac{\text{Re}(a_p)}{\alpha_p} + 1 \right] & \text{if } p > n, \\
\infty & \text{if } p = n;
\end{array} \right.
\] (3.5)
\[
a_0^* = -a_0^*; \quad a_0^* = -a_0^*; \quad a_0^* = -a_0^*; \quad \delta_0 = \delta; \quad \Delta_0 = \Delta; \quad \mu_0 = -\mu - \Delta.
\] (3.6)
We also note that if \( \alpha_0 < \nu < \beta_0, \nu \) is not in the exceptional set of \( \mathcal{K}_0 \).

First we consider the case \( r = 2 \).

THEOREM 3.1. Let \( \alpha < 1 - \nu < \beta, \alpha_0 < \nu < \beta_0, a^* = 0 \) and \( \Delta(1 - \nu) + \text{Re}(\mu) = 0 \). If \( f \in \mathbb{L}_{\nu,2} \), the relation (1.11) holds for \( \text{Re}(\lambda) > \nu h - 1 \) and the relation (1.12) holds for \( \text{Re}(\lambda) < \nu h - 1 \).
**PROOF.** We apply Theorem 2.1 with $\mathcal{K}$ being replaced by $\mathcal{K}_0$ and $\nu$ by $1 - \nu$. By the assumption and (3.6) we have

\[
a_*^* = -a^* = 0, \tag{3.7}
\]

\[
\Delta_0[1 - (1 - \nu)] + \text{Re}(\mu_0) = \Delta_\nu - \text{Re}(\mu) - \Delta = -[\Delta(1 - \nu) + \text{Re}(\mu)] = 0 \tag{3.8}
\]

and $\alpha_0 < 1 - (1 - \nu) < \beta_0$, and thus Theorem 2.1(a) applies. Then there is a one-to-one transform $H_0 \in \mathcal{L}_{1-\nu,2}$ so that the relation

\[
(\mathcal{M}H_0f)(s) = \mathcal{K}_0(s)(\mathcal{M}f)(1 - s) \tag{3.9}
\]

holds for $f \in \mathcal{L}_{1-\nu,2}$ and $\text{Re}(s) = \nu$. Further if $f \in \mathcal{L}_{\nu,2}$, $Hf \in \mathcal{L}_{1-\nu,2}$ and it follows from (3.9), (1.4) and (3.2) that

\[
(\mathcal{M}H_0f)(s) = \mathcal{K}_0(s)(\mathcal{M}f)(1 - s) = \mathcal{K}_0(s)\mathcal{K}(1 - s)(\mathcal{M}f)(s) = (\mathcal{M}f)(s),
\]

if $\text{Re}(s) = \nu$. Hence $\mathcal{M}H_0Hf = \mathcal{M}f$ and

\[
H_0Hf = f \quad \text{for } f \in \mathcal{L}_{\nu,2}. \tag{3.10}
\]

Applying Theorem 2.1(b) with $\mathcal{K}$ being replaced by $\mathcal{K}_0$ and $\nu$ by $1 - \nu$, we obtain for $f \in \mathcal{L}_{1-\nu,2}$ that

\[
(H_0f)(x) = h^{-1} \frac{d}{dx} z^{-(\lambda+1)/h}
\]

\[
\cdot \int_0^\infty H_{\nu, \lambda-1/2}^\nu \mathcal{K}_{x-1/2}^\nu \left[ \frac{(-\lambda, h)}{(1 - b_j - \beta_j)_m, 1, m, (1 - b_j - \beta_j), 1, m} \right] f(t) dt, \tag{3.11}
\]

if $\text{Re}(\lambda) > [1 - (1 - \nu)]h - 1$ and

\[
(H_0f)(x) = -h^{-1} \frac{d}{dx} z^{-(\lambda+1)/h}
\]

\[
\cdot \int_0^\infty H_{\nu, \lambda-1/2}^\nu \mathcal{K}_{x-1/2}^\nu \left[ \frac{(1 - b_j - \beta_j)_m, 1, m, (1 - b_j - \beta_j), 1, m} \right] f(t) dt, \tag{3.12}
\]

if $\text{Re}(\lambda) < [1 - (1 - \nu)]h - 1$. Replacing $f$ by $Hf$ and using (3.10) we have the relations (1.11) and (1.12) for $f \in \mathcal{L}_{\nu,2}$, if $\text{Re}(\lambda) > \nu h - 1$ and $\text{Re}(\lambda) < \nu h - 1$, respectively, which completes the proof of theorem.

Next results is the extension of Theorem 3.1 to $\mathcal{L}_{r,\nu}$-spaces for any $1 < r < \infty$, provided that $\Delta = 0$ and $\text{Re}(\mu) = 0$.

**THEOREM 3.2.** Let $\alpha < 1 - \nu < \beta$, $\alpha_0 < \nu < \beta_0$, $a^* = 0$, $\Delta = 0$ and $\text{Re}(\mu) = 0$. If $f \in \mathcal{L}_{r,\nu}$, $1 < r < \infty$, the relation (1.11) holds for $\text{Re}(\lambda) > \nu h - 1$ and the relation (1.12) holds for $\text{Re}(\lambda) < \nu h - 1$.

**PROOF.** We apply Theorem 2.2 with $\mathcal{K}$ being replaced by $\mathcal{K}_0$ and $\nu$ by $1 - \nu$. By the assumption and (3.6), we have $a_*^* = 0$, $\Delta_0 = 0$, $\text{Re}(\mu_0) = 0$ and $\alpha_0 < 1 - (1 - \nu) < \beta_0$, and thus Theorem 2.2(a) can be applied. In accordance with this theorem, $H_0$ can be extended to $\mathcal{L}_{1-\nu,\nu}$ as an element of $H_0 \in \mathcal{L}_{1-\nu,\nu}$. By virtue of (3.10) $H_0H$ is identical operator in $\mathcal{L}_{\nu,2}$. By Rooney [11, Lemma 2.2] $\mathcal{L}_{\nu,2}$ is dense in $\mathcal{L}_{\nu,\nu}$ and since $H \in \mathcal{L}_{\nu,\nu}$ and $H_0 \in \mathcal{L}_{1-\nu,\nu}$, the operator $H_0H$ is identical in $\mathcal{L}_{\nu,\nu}$ and hence

\[
H_0Hf = f \quad \text{for } f \in \mathcal{L}_{\nu,\nu}. \tag{3.13}
\]
Applying Theorem 2.2(c) with $\mathcal{K}$ being replaced by $\mathcal{K}_0$ and $\nu$ by $1 - \nu$, we obtain that the relations (3.11) and (3.12) hold for $f \in \mathcal{L}_{1-\nu,r}$, when $\text{Re}(\lambda) > [1 - (1 - \nu)]h - 1$ and $\text{Re}(\lambda) < [1 - (1 - \nu)]h - 1$, respectively. Replacing $f$ by $Hf$ and using (3.13), we arrive at (1.11) and (1.12) for $f \in \mathcal{L}_{1-\nu,r}$, if $\text{Re}(\lambda) > \nu h - 1$ and $\text{Re}(\lambda) < \nu h - 1$, respectively, which completes the proof of theorem.

REMARK 3.1. If $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$ which means that the $H$-function (1.2) is Meijer's $G$-function, then $\Delta = q - p$ and Theorems 8.1 and 8.2 in Rooney [11] follow from Theorems 3.1 and 3.2.

4. INVERSION OF $H$-TRANSFORM IN $\mathcal{L}_{\nu,r}$ WHEN $\Delta \neq 0$

We now investigate under what condition the $H$-transform with $\Delta \neq 0$ will have the inverse of the form (1.11) or (1.12). First, we consider the case $\Delta > 0$. To obtain the inversion of the $H$-transform on $\mathcal{L}_{\nu,r}$ we use the relation (2.7).

THEOREM 4.1. Let $r < r < \infty$, $-\infty < \alpha < 1 - \nu < \beta$, $\alpha_0 < \nu < \min\{\beta_0, [\text{Re}(\mu + 1/2)/\Delta] + 1\}, \alpha^* = 0, \Delta > 0$ and $\Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r)$, where $\gamma(r)$ is given in (2.6). If $f \in \mathcal{L}_{\nu,r}$, then the relations (1.11) and (1.12) hold for $\text{Re}(\lambda) > \nu h - 1$ and for $\text{Re}(\lambda) < \nu h - 1$, respectively.

PROOF. According to Theorem 2.3(a), the $H$-transform is defined on $\mathcal{L}_{\nu,r}$. First we consider the case $\text{Re}(\lambda) > \nu h - 1$.

If we denote by $\tilde{\alpha}^*, \tilde{\beta}, \tilde{\Delta}$ and $\tilde{\mu}$ for $H_1$ instead of those for $H$, then

$$\tilde{\alpha}^* = -\alpha^* = 0; \quad \tilde{\beta} = \beta; \quad \tilde{\Delta} = \Delta > 0; \quad \tilde{\mu} = -\mu - \Delta - 1. \quad (4.2)$$

We prove that $H_1 \in \mathcal{L}_{\nu,r}$ for any $s (1 \leq s < \infty)$. For this, we first apply Theorems 2.4 and 2.5 and Remark 2.1 to $H_1(t)$ to find its asymptotic behavior at zero and infinity. According to (3.4), (3.5) and the assumptions, we find

$$\frac{\text{Re}(b_j)}{b_j} + 1 \leq \frac{\text{Re}(a_i)}{a_i} + 1 \quad (j = m + 1, \cdots, q; \quad i = n + 1, \cdots, p);$$

$$\frac{\text{Re}(b_j)}{b_j} + 1 \geq \frac{\text{Re}(\lambda) + 1}{\mu_h} \quad (j = m + 1, \cdots, q).$$

Then it follows from here that the poles

$$a_{ik} = \frac{a_i + k}{\alpha_i} + 1 \quad (i = n + 1, \cdots, p; \quad k = 0, 1, 2, \cdots), \quad \lambda_n = \frac{\lambda + 1 + n}{h} \quad (n = 0, 1, 2, \cdots)$$

of Gamma functions $\Gamma(a_i + \alpha_i - \alpha, s)$ ($i = n + 1, \cdots, p$) and $\Gamma(1 + \lambda - hs)$, and the poles

$$b_{jl} = \frac{b_j - 1}{\beta_j} + 1 \quad (j = m + 1, \cdots, q; \quad l = 0, 1, 2, \cdots)$$

of Gamma functions $\Gamma(1 - b_j - \beta_j + \beta, s)$ ($j = m + 1, \cdots, q$) do not coincide. Hence by Theorem 2.4, (4.1) and Remark 2.1, we have

$$H_1(t) = O(t^{\rho_1}) \quad (|t| \to 0) \quad \text{with} \quad \rho_1 = \min_{m+1 \leq j \leq q} \left[ \frac{1 - \text{Re}(b_j)}{\beta_j} \right] - 1 = -\alpha_0.$$
for \( \alpha_0 \) being given in (3.4), or
\[
H_1(t) = O(t^{-\alpha_0}) \quad (t \to 0)
\]
with an additional logarithmic multiplier \( \log t \) possibly, if Gamma functions \( \Gamma(1 - b_j - \beta_j + \beta_j s) \) \((j = m + 1, \cdots, q)\) have general poles of order \( N \geq 2 \) at some points.

Further by Theorem 2.5, (4.1) and Remark 2.1,
\[
H_1(t) = O(t^{-\alpha_0}) \quad (t \to \infty) \quad \text{with} \quad \gamma_1 = \max \left\{ \beta_0, \frac{-\text{Re}(\mu) - 1/2}{\Delta} - 1, \frac{-\text{Re}(\lambda) - 1}{h} \right\}
\]
for \( \beta_0 \) being given by (3.5), or
\[
H_1(t) = O(t^{-\alpha_0}) \quad (|t| \to \infty) \quad \text{with} \quad \gamma_0 = \min \left\{ \beta_0, \frac{-\text{Re}(\mu) + 1/2}{\Delta} + 1, \frac{-\text{Re}(\lambda) + 1}{h} \right\}
\]
and with an additional logarithmic multiplier \( \log(t) \) possibly, if Gamma functions \( \Gamma(1 + \lambda - h s), \Gamma(\alpha_i + \alpha_i - \alpha_i s) \) \((i = n + 1, \cdots, p)\) have general poles of order \( M \geq 2 \) at some points.

Let Gamma functions \( \Gamma(1 - b_j - \beta_j + \beta_j s) \) \((j = m + 1, \cdots, q)\) and \( \Gamma(1 + \lambda - h s), \Gamma(\alpha_i + \alpha_i - \alpha_i s) \) \((i = n + 1, \cdots, p)\) have simple poles. Then from (4.3) and (4.4) we see that for \( 1 \leq s < \infty \), \( H_1(t) \in \mathcal{L}_{\nu,s} \) if and only if, for some \( R_1 \) and \( R_2 \), \( 0 < R_1 < R_2 < \infty \), the integrals
\[
\int_0^{R_1} t^{(\nu-\alpha_0)-1} dt, \quad \int_{R_2}^{\infty} t^{(\nu-\gamma_0)-1} dt
\]
are convergent. Since by the assumption \( \nu > \alpha_0 \), the first integral in (4.5) converges. In view of our assumptions
\[
\nu < \beta_0, \quad \nu < \frac{-\text{Re}(\mu) + 1/2}{\Delta} + 1, \quad \nu < \frac{-\text{Re}(\lambda) + 1}{h}
\]
we find \( \nu - \gamma_0 < 0 \) and the second integral in (4.5) converges, too.

If Gamma functions \( \Gamma(1 - b_j - \beta_j + \beta_j s) \) \((j = m + 1, \cdots, q)\) or \( \Gamma(1 + \lambda - h s), \Gamma(\alpha_i + \alpha_i - \alpha_i s) \) \((i = n + 1, \cdots, p)\) have general poles, then the logarithmic multipliers \( \log(t)^N \) \((N = 1, 2, \cdots)\) may be added in the integrals in (4.5), but they do not influence on the convergence of them. Hence, under the assumptions we have
\[
H_1(t) \in \mathcal{L}_{\nu,s} \quad (1 \leq s < \infty).
\]

Let \( a \) be a positive number and \( \Pi_a \) denote the operator
\[
(\Pi_a f)(x) = f(ax) \quad (x > 0)
\]
for a function \( f \) defined almost everywhere on \((0, \infty)\). It is known in Rooney [11, p.268] that \( \Pi_a \) is a bounded isomorphism of \( \mathcal{L}_{\nu,r} \) onto \( \mathcal{L}_{\nu,r} \), and if \( f \in \mathcal{L}_{\nu,r} \) \((1 \leq r \leq 2)\), there holds the relation for the Mellin transform \( \mathcal{M} \)
\[
(\mathcal{M} \Pi_a f)(s) = a^{-s} (\mathcal{M} f) \left( \frac{s}{a} \right) \quad (\text{Re}(s) = \nu).
\]

By virtue of Theorem 2.3(c) and (4.6), if \( f \in \mathcal{L}_{\nu,r} \) and \( H_1 \in \mathcal{L}_{\nu,r} \) (and hence \( \Pi_a H_1 \in \mathcal{L}_{\nu,r} \)), then
\[
\int_0^{\infty} H_1(xt)(Hf)(t)dt = \int_0^{\infty} (\Pi_a H_1)(t)(Hf)(t)dt = \int_0^{\infty} (H \Pi_a H_1)(t)(f)(t)dt.
\]
From the assumption \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r) \leq 0 \), Theorem 2.3(b) and (4.8) imply that
\[
(\mathcal{M} H \Pi_a H_1)(s) = \mathcal{H}(s)(\mathcal{M} \Pi_a H_1)(1-s) = x^{-(1-s)} \mathcal{H}(s)(\mathcal{M} H_1)(1-s)
\]
for $\Re(s) = 1 - \nu$. Now from (4.6), $H_1(t) \in \mathcal{L}_{\nu,1}$. Then by the definitions of the $H$-function (1.2), (1.3) and the direct and inverse Mellin transforms (see, for example, Samko et al. [12, (1.112), (1.113)]), we have

$$\mathcal{M}H_1(s) = \mathcal{M}_{p+1,q+1}^{q-m,p-n+1} \begin{pmatrix} (-\lambda, h), (1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_{1,n} \\ (1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m} \\
(-\lambda - 1, h) \end{pmatrix}$$

Then by the definitions of the $H$-function (1.2), (1.3) and the direct and inverse Mellin transforms (see, for example, Samko et al. [12, (1.112), (1.113)]), we have

$$\mathcal{M}H_1(s) = \mathcal{H}_0(s) \frac{1}{1 + \lambda - h s}$$

for $\Re(s) = \nu$, where $\mathcal{H}_0$ is given by (3.2). It follows from here that for $\Re(s) = 1 - \nu$,

$$\mathcal{M}H_1(1 - s) = \frac{\mathcal{H}_0(1 - s)}{1 + \lambda - h (1 - s)} = \frac{1}{\mathcal{H}(s)[1 + \lambda - h (1 - s)]}.$$  

Substituting this into (4.10) we obtain

$$\mathcal{M}H_1(x) = \frac{x^{-(1-s)}}{1 + \lambda - h (1 - s)} (\Re(s) = 1 - \nu). \quad (4.11)$$

For $x > 0$ let us denote by $g_x(t)$ a function

$$g_x(t) = \begin{cases} \frac{1}{h} t^{(\lambda+1)/h-1} & \text{if } 0 < t < x, \\ 0 & \text{if } t > x, \end{cases} \quad (4.12)$$

then

$$\mathcal{M}g_x(s) = \frac{x^{-(\lambda+1)/h}}{1 + \lambda - h (1 - s)}$$

and (4.11) takes the form

$$\mathcal{M}(\mathcal{H} H_1) = \mathcal{M}[x^{-(\lambda+1)/h} g_x]],$$

which implies

$$(\mathcal{H} H_1)(t) = x^{-(\lambda+1)/h} g_x(t). \quad (4.13)$$

Substituting (4.13) into (4.9), we have

$$\int_0^\infty H_1(xt)(\mathcal{H} f)(t)dt = x^{-(\lambda+1)/h} \int_0^\infty g_x(t)f(t)dt$$

or, in accordance with (4.12),

$$\int_0^x t^{(\lambda+1)/h-1} f(t)dt = h x^{(\lambda+1)/h} \int_0^\infty H_1(xt)(\mathcal{H} f)(t)dt.$$  

Differentiating this relation we obtain

$$f(x) = h x^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H_1(xt)(\mathcal{H} f)(t)dt$$

which shows (1.11).

If $\Re(\lambda) < \nu h - 1$, the relation (1.12) is proved similarly to (1.11), by taking the function

$$H_2(t) = \mathcal{M}_{p+1,q+1}^{q-m,p-n+1} \begin{pmatrix} (1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_{1,n}, (-\lambda, h) \\
(-\lambda - 1, h), (1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m} \end{pmatrix} \quad (4.14)$$
instead of the function $H_1(t)$ in (4.1). This completes the proof of the theorem.

In the case $\Delta < 0$ the following statement gives the inversion of $H$-transform on $L_{\alpha,\gamma}$.

**THEOREM 4.2.** Let $1 \leq r < \infty, \alpha < 1 - \nu < \beta < \infty, \max[\alpha_0, \{\text{Re}(\mu + 1/2)/\Delta + 1\}] < \nu < \beta_0, a^* = 0, \Delta < 0$ and $\Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r)$, where $\gamma(r)$ is given by (2.6). If $f \in L_{\alpha,\gamma}$, then the relations (1.11) and (1.12) holds for $\text{Re}(\lambda) > \nu h - 1$ and for $\text{Re}(\lambda) < \nu h - 1$, respectively.

This theorem is proved similarly to Theorem 4.1, if we apply Theorem 5.2 from Kilbas et al. [6] instead of Theorem 2.3 and take into account the asymptotics of the $H$-function at zero and infinity (see Srivastava et al. [13, §2.2] and Kilbas and Saigo [4, Corollary 4]).

**REMARK 4.1.** If $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$, then Theorems 8.3 and 8.4 in Rooney [11] follow from Theorems 4.1 and 4.2.

**REFERENCES**


Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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