TOTALLY REAL SUBMANIFOLDS IN A COMPLEX PROJECTIVE SPACE

LIU XIMIN

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Abstract. In this paper, we establish the following result: Let $M$ be an $n$-dimensional complete totally real minimal submanifold immersed in $\mathbb{CP}^n$ with Ricci curvature bounded from below. Then either $M$ is totally geodesic or $\inf r \leq (3n + 1)(n - 2)/3$, where $r$ is the scalar curvature of $M$.

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1. Introduction. Let $\mathbb{CP}^n$ be the $n$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $c = 4$ and let $M$ be an $n$-dimensional totally real submanifold of $\mathbb{CP}^n$. Let $r$ be the scalar curvature of $M$. If $M$ is compact, then many authors studied them and obtained many beautiful results (for example [2, 4, 5]).

In this paper, we make use of Yau’s maximum principle to study the complete totally real minimal submanifold with Ricci curvature bounded from below and obtain the following result.

**Theorem 1.** Let $M$ be an $n$-dimensional complete totally real minimal manifold immersed in $\mathbb{CP}^n$ with Ricci curvature bounded from below. Then either $M$ is totally geodesic or $\inf r \leq (3n + 1)(n - 2)/3$.

2. Preliminaries. Let $M$ be an $n$-dimensional totally real minimal submanifold of $\mathbb{CP}^n$. We choose a local field of orthonormal frames $e_1, \ldots, e_n, e_1^* = Je_1, \ldots, e_n^* = Je_n$ ($J$ is the complex structure of $\mathbb{CP}^n$), such that, restricted to $M$, the vectors $e_1, \ldots, e_n$ are tangent to $M$. We make use of the following convention on the range of indices

$$A, B, C, \ldots = 1, \ldots, n, 1^*, \ldots, n^*; \quad i, j, k, \ldots = 1, \ldots, n. \quad (2.1)$$

With respect to the frame field of $\mathbb{CP}^n$, let $w^A$ be the field of dual frames. Then the structure equations of $\mathbb{CP}^n$ are given by

$$dw^A = - \sum w^A_B \wedge w^B, \quad w^B_A + w^B_B = 0, \quad (2.2)$$

$$dw^A_B = - \sum w^A_C \wedge w^C_B + \frac{1}{2} \sum R^A_{BCD} w^C \wedge w^D, \quad (2.3)$$

$$R^A_{BCD} = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} + J_{AC} J_{BD} - J_{AD} J_{BC} + 2 J_{AB} J_{CD}, \quad (2.4)$$
where $J = J_{AB}e_A \otimes e_B$, so that

$$\langle J_{AB} \rangle = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

(2.5)

where $I_n$ is the identity matrix of order $n$. We restrict these forms to $M$. Then from [2], we have

$$w^{i*} = 0, \quad w^i_j = w^{i*}_j, \quad w^{i*}_j = w^j_i,$$

(2.6)

$$w^k_i = \sum h^{k*}_{ij} w^j, \quad h^{k*}_{ij} = h^{i*}_{ij} = h^{j*}_{ik},$$

(2.7)

d$w^i_j = -\sum w^i_j \wedge w^j_i, \quad w^{i*}_j = w^j_i,$$

(2.8)

d$w^i_j = -\sum w^k \wedge w^i_k + \frac{1}{2} \sum R_{ijkl}^i w^k \wedge w^j_l,$$

(2.9)

$$R_{ijkl}^i = \tilde{R}_{ijkl}^i w^k + (h^{m*}_{ik} h^{m*_j} - h^{m*}_{il} h^{m*_k}),$$

(2.10)

$$R_{ijkl}^* = \tilde{R}_{ijkl}^* + \sum (h^{m*}_{km} h^{m*_l} - h^{m*}_{ml} h^{m*_k}).$$

(2.11)

The second fundamental form $h$ of $M$ in $CP^n$ is defined as $h = \sum h^{k*}_{ij} w^j \otimes e^k$, whose squared length is $\|h\|^2 = \sum (h^{k*}_{ij})^2$.

If $M$ is minimal in $CP^n$, i.e., trace $h = 0$, then from (2.4) and (2.10), we have

$$r = n(n-1) - \|h\|^2,$$

(2.13)

where $r$ is the scalar curvature of $M$.

Define $h^{m*}_{ij}$ and $h^{m*}_{ijkl}$ by

$$\sum h^{m*}_{ij} w^k = dh^{m*}_{ij} - \sum h^{m*}_{kj} w^i - \sum h^{m*}_{ik} w^j + \sum h^{m*}_{ij} w^{m*}_{k},$$

(2.14)

$$\sum h^{m*}_{ijkl} w^i = dh^{m*}_{ijk} - \sum h^{m*}_{ikj} w^i + \sum h^{m*}_{ijk} w^j - \sum h^{m*}_{ijkl} w^k + \sum h^{m*}_{ijk} w^{m*}_l,$$

(2.15)

respectively.

Let $H_{ij}$ and $\Delta$ denote the $(n \times n)$-matrix $(h^{i*}_{ij})$ and the Laplacian on $M$, respectively. By a simple calculation, we have (cf. [2])

$$\frac{1}{2} \Delta \|h\|^2 = \sum (h^{i*}_{ij})^2 + (n+1) \|h\|^2 + \sum \text{tr} (H_{ik} H_{j*} - H_{kj} H_{i*})^2$$

$$- \sum (\text{tr} H_{i*} \text{tr} H_{j*})^2.$$

(2.16)

The following lemma is important in this paper.

**Lemma 1** [6]. Let $M^n$ be a complete Riemannian manifold with Ricci curvature bounded from below and let $f$ be a $C^2$-function bounded from above on $M^n$, then for all $\epsilon > 0$, there exists a point $x \in M^n$ at which

(i) $\sup f - \epsilon < f(x)$;

(ii) $\|\nabla f(x)\| < \epsilon$;

(iii) $\Delta f(x) < \epsilon$. 


Proof of the main theorem. By \([3]\), we have \(\sum (\text{tr} H_i \ast H_j) = \sum (\text{tr} H_i^2)\). From \([1]\), we know that \(\sum \text{tr}(H_i \ast H_j - H_j \ast H_i)^2 - \sum (\text{tr} H_i^2)^2 \geq -3/2 \| h \|^4\). So, from (2.16), we obtain
\[
\frac{1}{2} \Delta \| h \|^2 \geq \| h \|^2 ((n + 1) - 3/2 \| h \|^2). \tag{2.17}
\]
We know that \(\| h \|^2 = n(n - 1) - r\). By the condition of the theorem, we conclude that \(\| h \|^2\) is bounded. We define \(f = \| h \|^2\) and \(F = (f + a)^{1/2}\) (where \(a > 0\) is any positive constant number). \(F\) is bounded. We have
\[
dF = \frac{1}{2} (f + a)^{-1/2} df, \tag{2.18}
\]
\[
\Delta F = \frac{1}{2} \left(- \frac{1}{2} (f + a)^{-3/2} \| df \|^2 + (f + a)^{-1/2} \Delta f\right) \tag{2.19}
\]
i.e.,
\[
\Delta F = \frac{1}{2F} \left(-2 \| dF \|^2 + \Delta f\right) (f + a)^{-1/2}, \tag{2.20}
\]
Hence, \(F \Delta F = -\| dF \|^2 + 1/2 \Delta f\) or \(1/2 \Delta f = F \Delta F + \| dF \|^2\).

Applying Lemma 1 to \(F\), we have for all \(\epsilon > 0\), there exists a point \(x \in M\) such that at \(x\)
\[
|dF(x) < \epsilon|; \tag{2.21}
\]
\[
\Delta F(x) < \epsilon; \tag{2.22}
\]
\[
F(x) > \sup F - \epsilon. \tag{2.23}
\]
From (2.21), (2.22), and (2.23), we have
\[
\frac{1}{2} \Delta f < \epsilon^2 + F \epsilon = \epsilon(\epsilon + F). \tag{2.24}
\]
We take a sequence \(\{\epsilon_m\}\) such that \(\epsilon_m \to 0(m \to \infty)\) and for all \(m\), there exists a point \(x_m \in M\) such that (2.21), (2.22), and (2.23) hold. Therefore, \(\epsilon_m (\epsilon_m + F(x_m)) \to 0(m \to \infty)\) (because \(F\) is bounded).

From (2.23), we have \(F(x_m) > \sup F - \epsilon_m\). Because \(\{F(x_m)\}\) is a bounded sequence. So we get \(F(x_m) \to F_0\) (if necessary, we can choose a subsequence). Hence, \(F_0 \geq \sup F\). So we have
\[
F_0 = \sup F. \tag{2.25}
\]
From the definition of \(F\), we get
\[
f(x_m) \to f = \sup f. \tag{2.26}
\]
(2.17) and (2.24) imply that
\[
f\left((n + 1) - \frac{3}{2} f\right) \leq \frac{1}{2} \Delta f \leq \epsilon(\epsilon + F), \tag{2.27}
\]
and
\[
f(x_m)\left((n + 1) - \frac{3}{2} f(x_m)\right) < \epsilon_m^2 + \epsilon_m F(x_m) \leq \epsilon_m^2 + \epsilon_m F_0 \tag{2.28}
\]
let $m \to \infty$, then $\epsilon_m \to 0$ and $f(x_m) \to f_0$. Hence,

$$f_0 \left( (n+1) - \frac{3}{2} f_0 \right) \leq 0. \quad (2.29)$$

(i) if $f_0 = 0$, we have $f = \|h\|^2 = 0$. Hence, $M$ is totally geodesic.

(ii) if $f_0 > 0$, we have $(n+1) - 3/2 f_0 \leq 0$ and $f_0 \geq 2/3(n+1)$, that is, $\sup \|h\|^2 \geq 2/3(n+1)$. Therefore, $\inf \ r \leq (3n+1)(n-2)/3$. This completes the proof. \qed

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**References**


**Ximin:** Department of Mathematics, Nankai University, Tianjin 300071, China

**Current address:** Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China

**E-mail address:** xmliu@dlut.edu.cn
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<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
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</table>

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José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk