ON THE FEKETE-SZEGÖ PROBLEM

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Abstract. Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be an analytic function in the open unit disk. A sharp upper bound is obtained for $|a_3 - \mu a_2^2|$ by using the classes of strongly starlike functions of order $\beta$ and type $\alpha$ when $\mu \geq 1$.

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1. Introduction. Let $\mathcal{A}$ denote the family of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions which are univalent in $\mathbb{D}$. A function $f(z)$ belonging to $\mathcal{A}$ is said to be strongly starlike of order $\beta$ and type $\alpha$ in $\mathbb{D}$, and denoted by $\mathcal{S}_\alpha^*(\beta)$ if it satisfies

$$\left| \arg \left( \frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad (z \in \mathbb{D})$$

for some $\alpha (0 \leq \alpha < 1)$ and $\beta (0 < \beta \leq 1)$. If $f(z) \in \mathcal{A}$ satisfies

$$\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad (z \in \mathbb{D})$$

for some $\alpha (0 \leq \alpha < 1)$ and $\beta (0 < \beta \leq 1)$, then we say that $f(z)$ is strongly convex of order $\beta$ and type $\alpha$ in $\mathbb{D}$, and we denote by $\mathcal{K}_\alpha(\beta)$ the class of all such functions (see also Srivastava and Owa [16]). For the class $\mathcal{S}$ of analytic univalent functions, Fekete-Szegö [6] obtained the maximum value of $|a_3 - \mu a_2^2|$ when $\mu$ is real. For various functions of $\mathcal{S}$, the upper bound for $|a_3 - \mu a_2^2|$ is investigated by many different authors including [1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 14, 17].

In this paper, we obtain sharp upper bounds for $|a_3 - \mu a_2^2|$ when $f$ belonging to the classes of functions defined as follows.

**Definition 1.1.** Let $0 \leq \alpha < 1$, $\beta > 0$ and let $f \in \mathcal{A}$. Then $f \in \mathcal{M}(\alpha, \beta)$ if and only if there exist $g \in \mathcal{S}_\alpha^*(\beta)$ such that

$$\text{Re} \left( \frac{zf'(z)}{g(z)} \right) > 0 \quad (z \in \mathbb{D}),$$

and $f \in \mathcal{G}(\alpha, \beta)$ if and only if there exists $g \in \mathcal{K}_\alpha(\beta)$ and satisfy condition (1.4) with $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$. 


Note that $\mathcal{M}(0, \beta) = \mathcal{K}(\beta)$ is the class of close-to-convex functions defined in [3] and $\mathcal{M}(0, 1) = \mathcal{K}(1)$ is the class of normalized close-to-convex functions defined by Kaplan [9].

2. Main results. In order to derive our main results, we have to recall here the following lemma [15].

**Lemma 2.1.** Let $h \in \mathcal{C}$, that is, $h$ be analytic in $\mathbb{U}$ and be given by $h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, and $\text{Re} h(z) > 0$ for $z \in \mathbb{U}$, then

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1^2|}{2}. \quad (2.1)$$

**Theorem 2.2.** Let $f(z) \in \mathcal{M}(\alpha, \beta)$ and be given by (1.1). Then for $0 \leq \alpha < 1$, $\beta \geq 1$, and $\mu \geq 1$ we have the sharp inequality

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{2\beta^2 (\mu - 1) + \alpha \beta^2 (8 - 2\alpha - 3\mu)}{(1 - \alpha)^2 (2 - \alpha)} + \frac{2\beta + 1 - \alpha (3\mu - 2)}{3(1 - \alpha)}. \quad (2.2)$$

**Proof.** Let $f(z) \in \mathcal{M}(\alpha, \beta)$. It follows from (1.4) that

$$zf'(z) = g(z)q(z), \quad (2.3)$$

for $z \in \mathbb{U}$, with $q \in \mathcal{C}$ given by $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots$. Equating coefficients, we obtain

$$2a_2 = q_1 + b_2, \quad 3a_3 = q_2 + b_2 q_1 + b_3. \quad (2.4)$$

Also, it follows from (1.2) that

$$zg'(z) - \alpha g(z) = g(z) (p(z))^\beta, \quad (2.5)$$

where $z \in \mathbb{U}$, $p \in \mathcal{C}$, and $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$. Thus equating coefficients, we obtain

$$(1 - \alpha) b_2 = \beta p_1, \quad (2 - \alpha) b_3 = \beta \left( p_2 + \frac{\beta (3 - \alpha) + \alpha - 1}{2(1 - \alpha)} p_1^2 \right). \quad (2.6)$$

From (2.4) and (2.6), we have

$$a_3 - \mu a_2^2 = \frac{1}{3} \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{2 - 3\mu}{12} q_1^2 + \frac{\beta}{3(2 - \alpha)} \left( p_2 - \frac{1}{2} p_1^2 \right) \quad + \frac{\beta^2 [6(1 - \mu) + \alpha(2\alpha + 3\mu - 8)]}{12(1 - \alpha)^2 (2 - \alpha)} p_1^2 + \frac{\beta (2 - 3\mu)}{6(1 - \alpha)} p_1 q_1. \quad (2.7)$$

Assume that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\text{Re}(a_3 - \mu a_2^2)$, so from (2.7) and by using Lemma 2.1 and letting $p_1 = 2re^{i\theta}$, $q_1 = 2Re^{i\phi}$, $0 \leq r \leq 1$, $0 \leq R \leq 1$, $0 \leq \theta < 2\pi$, and $0 \leq \phi < 2\pi$, we obtain
\[3\text{Re}(a_3 - \mu a_z^2) = \text{Re}\left(q_2 - \frac{1}{2} q_1^2\right) + \frac{2 - 3\mu}{4} \text{Re} q_1^2 + \frac{\beta}{(2 - \alpha)} \text{Re}\left(p_2 - \frac{1}{2} p_1^2\right) + \frac{\beta^2 [6 + 2\alpha^2 + 3\alpha \mu] - (6\mu + 8\alpha)}{(4(1 - \alpha)^2(2 - \alpha))} \text{Re} p_1 q_1.\]

\[\leq 2(1 - R^2) + (2 - 3\mu)R^2 \cos 2\phi + \frac{2\beta}{2 - \alpha} (1 - r^2)\]

\[+ \frac{\beta^2 [6 + 2\alpha^2 + 3\alpha \mu] - (6\mu + 8\alpha)}{(1 - \alpha)^2(2 - \alpha)} r^2 \cos 2\theta + \frac{2\beta(2 - 3\mu)}{1 - \alpha} r R \cos(\theta + \phi)\]

\[\leq (3\mu - 4)R^2 + \frac{2\beta(3\mu - 2)}{1 - \alpha} r R\]

\[+ \frac{6\beta^2(\mu - 1) + \alpha\beta^2(8 - 2\alpha - 3\mu) - 2\beta(1 - \alpha)^2}{(1 - \alpha)^2(2 - \alpha)} r^2 + \frac{2(\beta - \alpha) + 4}{2 - \alpha} = \Psi(r, R).\]  

(2.8)

Letting \(\alpha, \beta,\) and \(\mu\) fixed and differentiating \(\Psi(r, R)\) partially when \(0 \leq \alpha < 1, \beta \geq 1,\) and \(\mu \geq 1,\) we observe that

\[\Psi_r \Psi_{rr} - (\Psi_r)^2 = 4\beta[4\beta + 2 + \alpha(2\alpha\beta + 2\alpha - 4 - 7\beta)] - 3\beta \mu(6\beta + 2 + \alpha(2\alpha\beta + 2\alpha - 4 - 8\beta)].\]  

(2.9)

Therefore, the maximum of \(\Psi(r, R)\) occurs on the boundaries. Thus the desired inequality follows by observing that

\[\Psi(r, R) \leq \Psi(1, 1) = \frac{6\beta^2(\mu - 1) + \alpha\beta^2(8 - 2\alpha - 3\mu)}{(1 - \alpha)^2(2 - \alpha)} + \frac{(2\beta + 1 - \alpha)(3\mu - 2)}{1 - \alpha}.\]  

(2.10)

The equality for (2.2) is attained when \(p_1 = q_1 = 2i\) and \(q_1 = q_2 = -2.\)

Letting \(\alpha = 0\) in Theorem 2.2, we have the result given by Jahangiri [8].

**Corollary 2.3.** Let \(f(z) \in \mathcal{K}(\beta)\) and be given by (1.1). Then for \(\beta \geq 1,\) and \(\mu \geq 1,\) we have the sharp inequality

\[|a_3 - \mu a_z^2| \leq \beta^2(\mu - 1) + \frac{(2\beta + 1)(3\mu - 2)}{3}.\]  

(2.11)

**Theorem 2.4.** Let \(f(z) \in \mathcal{G}(\alpha, \beta)\) and be given by (1.1). Then for \(0 \leq \alpha < 1, \beta \geq 1,\) and \(\mu \geq 1,\) we have the sharp inequality

\[|a_3 - \mu a_z^2| \leq \frac{6\beta^2(3\mu - 4) + \alpha\beta^2(32 - 8\alpha - 9\mu)}{36(1 - \alpha)^2(2 - \alpha)} + \frac{(\beta + 1 - \alpha)(3\mu - 2)}{3(1 - \alpha)}.\]  

(2.12)

**Proof.** Let \(f(z) \in \mathcal{G}(\alpha, \beta)\). It follows from (1.3) that

\[z g''(z) + (1 - \alpha) g'(z) = g'(z)(p(z))^{\beta},\]  

(2.13)

where \(z \in \mathcal{U}, p \in \mathcal{P},\) and \(p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots.\) Thus equating coefficients, we obtain

\[2(1 - \alpha)b_2 = \beta p_1, \quad 3(2 - \alpha)b_3 = \beta \left(\frac{\beta(3 - \alpha) + \alpha - 1}{2(1 - \alpha)} p_1^2\right).\]  

(2.14)
From (2.4) and (2.14) and proceeding as in the proof of Theorem 2.2, we get

\[ 3\text{Re} \left( a_3 - \mu a_2^2 \right) \leq (3\mu - 4)R^2 + \frac{\beta(3\mu - 2)}{1 - \alpha} rR + \frac{2(\beta - 3\alpha) + 12}{3(2 - \alpha)} + \frac{6\beta^2(3\mu - 4) + \alpha\beta^2(32 - 8\alpha - 9\mu) - 8\beta(1 - \alpha)^2}{12(1 - \alpha)^2(2 - \alpha)} r^2 \]

\[ = \Phi(r,R). \]  

(2.15)

Letting \( \alpha, \beta \) and \( \mu \) fixed and differentiating \( \Phi(r,R) \) partially when \( 0 \leq \alpha < 1, \beta \geq 1, \) and \( \mu \geq 1, \) we have

\[ \Phi_{rr} \Phi_{RR} - (\Phi_{rR})^2 = 4\beta \left[ 18\beta + 8 + \alpha(8\alpha\beta + 8\alpha - 16 - 29\beta) \right] \]

\[ - 3\beta\mu \left[ 24\beta + 8 + \alpha(8\alpha\beta + 8\alpha - 16 - 32\beta) \right] < 0. \]  

(2.16)

Therefore, the maximum of \( \Phi(r,R) \) occurs on the boundaries. Thus the desired inequality (2.12) follows by observing that

\[ \Phi(r,R) \leq \Phi(1,1) = \frac{6\beta^2(3\mu - 4) + \alpha\beta^2(32 - 8\alpha - 9\mu)}{12(1 - \alpha)^2(2 - \alpha)} + \frac{(\beta + 1 - \alpha)(3\mu - 2)}{1 - \alpha}. \]

(2.17)

The equality in (2.12) is attained on choosing \( p_1 = q_1 = 2i \) and \( q_1 = q_2 = -2. \) This completes the proof of Theorem 2.4.

**Corollary 2.5.** Let \( f(z) \in \mathcal{G}(0, \beta) \) and be given by (1.1). Then for \( \beta \geq 1, \) and \( \mu \geq 1, \) we have the sharp inequality

\[ |a_3 - \mu a_2^2| \leq \frac{1}{12} \left[ (3\mu - 2)(\beta + 2)^2 - 2\beta^2 \right]. \]

(2.18)

**References**


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