ABOUT INTERPOLATION OF SUBSPACES OF REARRANGEMENT INVARIANT SPACES GENERATED BY RADEMACHER SYSTEM

SERGEY V. ASTASHKIN

(Received 1 August 2000 and in revised form 27 November 2000)

Abstract. The Rademacher series in rearrangement invariant function spaces “close” to the space $L_\infty$ are considered. In terms of interpolation theory of operators, a correspondence between such spaces and spaces of coefficients generated by them is stated. It is proved that this correspondence is one-to-one. Some examples and applications are presented.

2000 Mathematics Subject Classification. Primary 46B70; Secondary 46B42, 42A55.

1. Introduction. Let

$$r_k(t) = \text{sign} \sin \frac{2^{k-1} \pi t}{k} \quad (k = 1, 2, \ldots)$$

be the Rademacher functions on the segment $[0, 1]$. Define the linear operator

$$T a(t) = \sum_{k=1}^{\infty} a_k r_k(t) \quad \text{for } a = (a_k)_{k=1}^{\infty} \in l_2.$$ (1.2)

It is well known (cf. [23, pages 340–342]) that $T a$ is an almost everywhere finite function on $[0, 1]$. Moreover, from Khintchine’s inequality it follows that

$$\|T a\|_{L_p} \asymp \|a\|_2 \quad \text{for } 1 \leq p < \infty,$$ (1.3)

where $\|a\|_p = (\sum_{k=1}^{\infty} |a_k|^p)^{1/p}$. The symbol $\asymp$ means the existence of two-sided estimates with constants depending only on $p$. Also, it can easily be checked that

$$\|T a\|_{L_\infty} = \|a\|_1.$$ (1.4)

A more detailed information on the behaviour of Rademacher series can be obtained by treating them in the framework of general rearrangement invariant spaces.

Recall that a Banach space $X$ of measurable functions $x = x(t)$ on $[0, 1]$ is said to be a rearrangement invariant space (r.i.s.) if the inequality $x^*(t) \leq y^*(t)$, for $t \in [0, 1]$ and $y \in X$, implies $x \in X$ and $\|x\| \leq \|y\|$. Here and in what follows $z^*(t)$ is the nonincreasing rearrangement of a function $|z(t)|$ with respect to the Lebesgue measure denoted by meas [10, page 83].

Important examples of r.i.s.’s are Marcinkiewicz and Orlicz spaces. Let $\mathcal{P}$ denote the cone of nonnegative increasing concave functions on the semiaxis $(0, \infty)$.

If $\varphi \in \mathcal{P}$, then the Marcinkiewicz space $M(\varphi)$ consists of all measurable functions $x = x(t)$ such that

$$\|x\|_{M(\varphi)} = \sup \left\{ \frac{1}{\varphi(t)} \int_{0}^{t} x^*(s) \, ds : 0 < t \leq 1 \right\} < \infty.$$ (1.5)
If \( S(t) \) is a nonnegative convex continuous function on \([0, \infty)\), \( S(0) = 0 \), then the Orlicz space \( L_S \) consists of all measurable functions \( x = x(t) \) such that
\[
\|x\|_S = \inf \left\{ u > 0 : \int_0^1 S\left( \frac{|x(t)|}{u} \right) \, dt \leq 1 \right\} < \infty.
\]  
(1.6)

In particular, if \( S(t) = t^p \) \((1 \leq p < \infty)\), then \( L_S = L^p \).

For any r.i.s. \( X \) on \([0, 1]\) we have \( L_{\infty} \subset X \subset L_1 \) [10, page 124]. Let \( X^0 \) denote the closure of \( L_{\infty} \) in an r.i.s. \( X \).

In problems discussed below, a special role is played by the Orlicz space \( L_N \), where \( N(t) = \exp(t^2) - 1 \) or, more precisely, by the space \( G = L_0^N \). In [19], V. A. Rodin and E. M. Semenov proved a theorem about the equivalence of Rademacher system to the standard basis in the space \( l_2 \).

**Theorem 1.1.** Suppose that \( X \) is an r.i.s. Then
\[
\|Ta\|_X = \left\| \sum_{k=1}^\infty a_k r_k \right\|_X \asymp \|a\|_2
\]
(1.7)

if and only if \( X \supset G \).

By Theorem 1.1, the space \( G \) is the minimal space among r.i.s.’s \( X \) such that the Rademacher system is equivalent in \( X \) to the standard basis of \( l_2 \).

In this paper, we consider problems related to the behaviour of Rademacher series in r.i.s.’s intermediate between \( L_{\infty} \) and \( G \). Here a major role is played by concepts and methods of interpolation theory of operators.

For a Banach couple \((X_0, X_1)\), \( x \in X_0 + X_1 \) and \( t > 0 \), we introduce the Peetre \( \mathcal{K} \)-functional
\[
\mathcal{K}(t, x; X_0, X_1) = \inf \left\{ \|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1 \right\}.
\]
(1.8)

Let \( Y_0 \) be a subspace of \( X_0 \) and \( Y_1 \) a subspace of \( X_1 \). A couple \((Y_0, Y_1)\) is called a \( \mathcal{K} \)-subcouple of a couple \((X_0, X_1)\) if
\[
\mathcal{K}(t, y; Y_0, Y_1) \approx \mathcal{K}(t, y; X_0, X_1),
\]
(1.9)

with constants independent of \( y \in Y_0 + Y_1 \) and \( t > 0 \).

In particular, if \( Y_i = P(X_i) \), where \( P \) is a linear projector bounded from \( X_i \) into itself for \( i = 0, 1 \), then \((Y_0, Y_1)\) is a \( \mathcal{K} \)-subcouple of \((X_0, X_1)\) (see [3] or [21, page 136]). At the same time, there are many examples of subcouples that are not \( \mathcal{K} \)-subcouples (see [21, page 589], [22], and Remark 3.2 of this paper).

Let \( T(l_1) \) (respectively \( T(l_2) \)) denote the subspace of \( L_{\infty} \) (of \( G \)) consisting of all functions of the form \( x = Ta \), where \( T \) is given by (1.2) and \( a \in l_1 (\in l_2) \). From (1.4) and Theorem 1.1 it follows that
\[
\mathcal{K}(t, Ta; T(l_1), T(l_2)) \approx \mathcal{K}(t, a; l_1, l_2).
\]
(1.10)

In spite of the fact that \( T(l_1) \) is uncomplemented in \( L_{\infty} \) (see [17] or [11, page 134]) the following assertion holds.
Theorem 1.2. The couple \((T(l_1), T(l_2))\) is a \(\mathfrak{K}\)-subcouple of the couple \((L_\infty, G)\). In other words (see (1.10)),

\[
\mathfrak{K}(t, T_a; L_\infty, G) \simeq \mathfrak{K}(t, a; l_1, l_2),
\]

with constants independent of \(a = (a_k)_{k=1}^\infty \in l_2 \) and \(t > 0\).

We will use in the proof of Theorem 1.2 an assertion about the distribution of Rademacher sums. It was proved by S. Montgomery-Smith [13].

Theorem 1.3. There exists a constant \(A \geq 1\) such that for all \(a = (a_k)_{k=1}^\infty \in l_2 \) and \(t > 0\)

\[
\begin{align*}
\text{meas} \left\{ s \in [0,1] : \sum_{k=1}^\infty a_k r_k(s) > \varphi_a(t) \right\} &\leq \exp \left( -\frac{t^2}{2} \right), \\
\text{meas} \left\{ s \in [0,1] : \sum_{k=1}^\infty a_k r_k(s) > A^{-1} \varphi_a(t) \right\} &\geq A^{-1} \exp (-At^2),
\end{align*}
\]

where \(\varphi_a(t) = \mathfrak{K}(t, a; l_1, l_2)\).

Now we need some definitions from interpolation theory of operators. We say that a linear operator \(U\) is bounded from a Banach couple \(\bar{X} = (X_0, X_1)\) into a Banach couple \(\bar{Y} = (Y_0, Y_1)\) (in short, \(U : \bar{X} \to \bar{Y}\)) if \(U\) is defined on \(X_0 + X_1\) and acts as bounded operator from \(X_i\) into \(Y_i\) for \(i = 0, 1\).

Let \(\bar{X} = (X_0, X_1)\) be a Banach couple. A space \(X\) such that \(X_0 \cap X_1 \subset X \subset X_0 + X_1\) is called an interpolation space between \(X_0\) and \(X_1\) if each linear operator \(U : \bar{X} \to \bar{X}\) is bounded from \(X\) into itself.

To every r.i.s. \(X\) assign the sequence space \(F_X\) of Rademacher coefficients of functions of the form (1.2) from \(X\):

\[
\|(a_k)\|_{F_X} = \left\| \sum_{k=1}^\infty a_k r_k \right\|_X.
\]

Well-known properties of Rademacher functions imply that \(F_X\) is an r.i. sequence space [19]. Furthermore, Theorem 1.3 and properties of the \(\mathfrak{K}\)-functional show that \(F_X\) is an interpolation space between \(l_1\) and \(l_2\) (see the proof of Theorem 1.2 later).

For interpolation r.i.s. between \(L_\infty\) and \(G\) the correspondence \(X \to F_X\) can be defined by using the real interpolation method.

For every \(p \in [1, \infty]\), we denote by \(l_p(u_k)\), \(u_k \geq 0\) \((k = 0, 1, \ldots)\) the space of all two-sided sequences of real numbers \(a = (a_k)_{k=-\infty}^\infty\) such that the norm \(\|a\|_{l_p(u_k)} = \|(a_k u_k)\|_p\) is finite. Let \(E\) be a Banach lattice of two-sided sequences, \(\min(1, 2^k)_{k=-\infty}^\infty \in E\). If \((X_0, X_1)\) is a Banach couple, then the space of the real \(\mathfrak{K}\)-method of interpolation \((X_0, X_1)_E^{\mathfrak{K}}\) consists of all \(x \in X_0 + X_1\) such that

\[
\|x\| = \|(\mathfrak{K}(2^k, x; X_0, X_1))k\|_E < \infty.
\]

It is readily checked that the space \((X_0, X_1)_E^{\mathfrak{K}}\) is an interpolation space between \(X_0\) and \(X_1\) (cf. [15, page 422]). In the special case \(E = l_p(2^{-k\theta})\) \((0 < \theta < 1, 1 \leq p \leq \infty)\) we obtain the spaces \((X_0, X_1)_{\theta, p}\) (for the detailed exposition of their properties see [4]).
A couple \( \vec{X} = (X_0, X_1) \) is said to be a \( \mathcal{H} \)-monotone couple if for every \( x \in X_0 + X_1 \) and \( y \in X_0 + X_1 \) there exists a linear operator \( U : \vec{X} \to \vec{X} \) such that \( y = Ux \) whenever
\[
\mathcal{H}(t, y; X_0, X_1) \leq \mathcal{H}(t, x; X_0, X_1) \quad \forall t > 0. \tag{1.15}
\]

As it is well known (cf. [15, page 482]), any interpolation space \( X \) with respect to a \( \mathcal{H} \)-monotone couple \( (X_0, X_1) \) is described by the real \( \mathcal{H} \)-method. It means that for some \( E \)
\[
X = (X_0, X_1)^E. \tag{1.16}
\]

In particular, by the Sparr theorem [20] the couple \( (l_1, l_2) \) is a \( \mathcal{H} \)-monotone couple. Therefore, if \( F \) is an interpolation space between \( l_1 \) and \( l_2 \), there exists \( E \) such that
\[
F = (l_1, l_2)^E. \tag{1.17}
\]

Hence Theorem 1.2 allows to find an r.i.s. that contains Rademacher series with coefficients belonging to an arbitrary interpolation space between \( l_1 \) and \( l_2 \). In [19], the similar result was obtained for sequence spaces satisfying more restrictive conditions (see Remark 3.3).

**Theorem 1.4.** Let \( F \) be an interpolation sequence space between \( l_1 \) and \( l_2 \) and \( F = (l_1, l_2)^E \). Then for the r.i.s. \( X = (L_\infty, G)^E \) we have
\[
\left\| \sum_{k=1}^\infty a_k r_k \right\|_X \asymp \| a \|_F \tag{1.18}
\]
with constants independent of \( a = (a_k)_{k=1}^\infty \).

Combining Theorem 1.4 with the above remarks, we get the following assertion. If \( F \) is a sequence space, then
\[
\| (a_k) \|_F \asymp \left\| \sum_{k=1}^\infty a_k r_k \right\|_X \tag{1.19}
\]
if and only if \( F \) is an interpolation space between \( l_1 \) and \( l_2 \).

The last result shows that the restriction of the correspondence (1.13) to interpolation r.i.s. between \( L_\infty \) and \( G \) is bijective.

**Theorem 1.5.** Let r.i.s.’s \( X_0 \) and \( X_1 \) be two interpolation spaces between \( L_\infty \) and \( G \). If
\[
\left\| \sum_{k=1}^\infty a_k r_k \right\|_{X_0} \asymp \left\| \sum_{k=1}^\infty a_k r_k \right\|_{X_1}, \tag{1.20}
\]
then \( X_0 = X_1 \) and the norms of \( X_0 \) and \( X_1 \) are equivalent.

In [16, 19], the similar results were obtained by additional conditions with respect to spaces \( X_0 \) and \( X_1 \).
2. Proofs

**Proof of Theorem 1.2.** It is known [10, page 164] that the $\mathcal{K}$-functional of a couple of Marcinkiewicz spaces is given by the formula

$$\mathcal{K}(t,x;M(\varphi_0),M(\varphi_1)) = \sup_{0<u<1} \frac{\int_0^u x^*(s) \, ds}{\max(\varphi_0(u), \varphi_1(u)/t)}.$$  \hspace{1cm} (2.1)

If $N(t) = \exp(t^2) - 1$, then the Orlicz space $L_N$ coincides with the Marcinkiewicz space $M(\varphi_1)$, where $\varphi_1(u) = u \log_{1/2} (2/u)$ [12]. In addition, $L_\infty = M(\varphi_0)$, where $\varphi_0(u) = u$. Therefore,

$$\mathcal{K}(t,x;L_\infty,G) = \sup_{0<u<1} \left( \frac{1}{u} \int_0^u x^*(s) \, ds \min\left(1, t \log_{1/2} \left( \frac{2}{u} \right) \right) \right) \quad \text{for } x \in G. \hspace{1cm} (2.2)$$

Since $x^*(u) \leq 1/u \int_0^u x^*(s) \, ds$, then from (2.2) it follows that

$$\mathcal{K}(t,x;L_\infty,G) \geq \sup_{k=0,1,...} \{ x^*(2^{-k}) \min(1, t(k+1)^{-1/2}) \}. \hspace{1cm} (2.3)$$

Hence,

$$\mathcal{K}(t,x;L_\infty,G) \geq x^*(2^{-kt}) \quad \text{for } t \geq 1, \hspace{1cm} (2.4)$$

where $k_t = [t^2] - 1$ ([z] is the integral part of a number z).

Now let $a = (a_k)_{k=1}^\infty \in l_2$ and $x(t) = Ta(t) = \sum_{k=1}^\infty a_k r_k(t)$, where $r_k(t)$ is a nonincreasing rearrangement of the sequence $\{|a_k|\}_{k=1}^\infty$. By the Holmstedt formula [7],

$$\varphi_a(t) \leq [t^2] \sum_{k=1}^\infty a_k^2 + t \left\{ \sum_{k=[t^2]+1}^\infty (a_k^2)^{1/2} \right\} \leq B \varphi_a(t), \hspace{1cm} (2.5)$$

where $\varphi_a(t) = \mathcal{K}(t,a;l_1,l_2)$, $(a_k^*)_{k=1}^\infty$ is a nonincreasing rearrangement of the sequence $(|a_k|)_{k=1}^\infty$, and $B > 0$ is a constant independent of $a = (a_k)_{k=1}^\infty$ and $t > 0$.

Assume, at first, that $a \notin l_1$. Then inequality (2.5) shows that

$$\lim_{t \to 0^+} \varphi_a(t) = 0, \quad \lim_{t \to \infty} \varphi_a(t) = \infty. \hspace{1cm} (2.6)$$

The function $\varphi_a$ belongs to the class $\mathcal{P}$ [4, page 55]. Therefore it maps the semiaxis $(0,\infty)$ onto $(0,\infty)$ one-to-one, and there exists the inverse function $\varphi_a^{-1}$. By Theorem 1.3, we have

$$n_{|x|}(\tau) = \max\{s \in [0,1]: |x(s)| > \tau\} \geq \varphi(\tau) \quad \text{for } \tau > 0, \hspace{1cm} (2.7)$$

where $\varphi(\tau) = A^{-1} \exp\{-A[\varphi_a^{-1}(\tau A)]^2\}$. Passing to rearrangements we obtain

$$x^*(s) \geq \varphi^{-1}(s) \quad \text{for } 0 < s < A^{-1}. \hspace{1cm} (2.8)$$

Obviously, by condition $t \geq C_1 = C_1(A) = \sqrt{2 \log_2 (2A)}$, it holds

$$2^{-k_t/2} < A^{-1} \quad \text{for } k_t = [t^2] - 1. \hspace{1cm} (2.9)$$

Hence (2.4) and (2.8) imply

$$\mathcal{K}(t,x;L_\infty,G) \geq \varphi^{-1}(2^{-k_t}). \hspace{1cm} (2.10)$$
Combining the definition of the function $\psi$ with (2.9), we obtain
\[ \psi^{-1}(2^{-k_1}) = A^{-1} \varphi_a(A^{-1/2} \ln^{1/2} (A^{-1/2}k_1)) \geq A^{-1} \varphi_a \left( \sqrt{\frac{k_1 \ln^2 2}{2A}} \right) \]
\[ \geq A^{-3/2} \sqrt{\frac{\ln^2 2}{2}} \varphi_a \left( \sqrt{k_1} \right) \geq A^{-3/2} \sqrt{\frac{\ln^2 2}{2}} t^{-1} \sqrt{k_1} \varphi_a(t). \]
(2.11)

From the inequality $t \geq C_1 \geq \sqrt{2}$ it follows that
\[ \frac{\sqrt{k_1}}{t} \geq \frac{\sqrt{[t^2] - 1}}{\sqrt{[t^2] + 1}} \geq 3^{-1/2}. \]
(2.12)

Therefore, by (2.10), we have
\[ \mathcal{K}(t,x;L_\infty,G) \geq C_2 \varphi_a(t) \quad \text{for } t \geq C_1, \]
(2.13)
where $C_2 = C_2(A) = \sqrt{\ln 2/6} A^{-3/2}$.

If now $t \geq 1$, then the concavity of the $\mathcal{K}$-functional and the previous inequality yield
\[ \mathcal{K}(t,x;L_\infty,G) \geq C_1^{-1} \mathcal{K}(t_1,x;L_\infty,G) \geq C_2 \varphi_a(C_1 t) \geq C_2 \varphi_a(t). \]
(2.14)

Using the inequalities $\|a\|_2 \leq \|a\|_1$ ($a \in l_1$) and $\|x\|_G \leq \|x\|_\infty$ ($x \in L_\infty$), the definition of the $\mathcal{K}$-functional, and Theorem 1.1, we obtain
\[ \mathcal{K}(t,x;L_\infty,G) = t \|x\|_G \geq C_3 t \|a\|_2 = C_3 \varphi_a(t) \quad \text{for } 0 < t \leq 1. \]
(2.15)

Thus,
\[ \mathcal{K}(t,a;l_1,l_2) \leq C \mathcal{K}(t,Ta;l_\infty,G), \]
(2.16)
if $C = \max(C_2^{-1},C_1/C_2)$.

Suppose now $a \in l_1$. By (2.5), without loss of generality, we can assume that the function $\varphi_a$ maps the semiaxis $(0,\infty)$ injectively onto the interval $(0,\|a\|_1)$. Hence we can define the mappings $\varphi_a^{-1} : (0,\|a\|_1) \to (0,\infty)$, $\psi : (0,A^{-1}\|a\|_1) \to (0,A^{-1})$, and $\psi^{-1} : (0,A^{-1}) \to (0,A^{-1}\|a\|_1)$. Arguing as above, we get inequality (2.16).

The opposite inequality follows from Theorem 1.1 and relation (1.4). Indeed,
\[ \mathcal{K}(t,Ta;l_\infty,G) \leq \inf \{ \|Ta^0\|_\infty + t \|Ta^1\|_G : a = a^0 + a^1, \ a^0 \in l_1, \ a^1 \in l_2 \} \]
\[ \leq D \mathcal{K}(t,a;l_1,l_2). \]  
(2.17)

**PROOF OF THEOREM 1.4.** It is sufficient to use Theorem 1.2 and the definition of the real $\mathcal{K}$-method of interpolation. \(\square\)

For the proof of Theorem 1.5 we need some definitions and auxiliary assertions. These results are also of some independent interest.

Let $f(t)$ be a function defined on the interval $(0, l)$, where $l = 1$ or $l = \infty$. Then the dilation function of $f$ is defined as follows:
\[ M_f(t) = \sup \left\{ \frac{f(st)}{f(s)} : s, st \in (0, l) \right\}, \quad \text{if } t \in (0, l). \]  
(2.18)
Since this function is semimultiplicative, then there exist numbers

\[ y_f = \lim_{t \to 0^+} \frac{\ln M_f(t)}{\ln t}, \quad \delta_f = \lim_{t \to \infty} \frac{\ln M_f(t)}{\ln t}. \]  

(2.19)

A Banach couple \( \overline{X} = (X_0, X_1) \) is called a partial retract of a couple \( \overline{Y} = (Y_0, Y_1) \) if each element \( x \in X_0 + X_1 \) is orbitally equivalent to some element \( y \in Y_0 + Y_1 \). The last means that there exist linear operators \( U : \overline{X} \to \overline{Y} \) and \( V : \overline{Y} \to \overline{X} \) such that \( Ux = y \) and \( V y = x \).

**Proposition 2.1.** Suppose that \( M(\varphi) \) is a Marcinkiewicz space on \([0, 1]\). If \( \gamma_{\varphi} > 0 \), then \( \overline{X} = (L_\infty, M(\varphi)) \) is a \( \mathcal{H} \)-monotone couple.

**Proof.** It is sufficient to show that the couple \( \overline{X} \) is a partial retract of the couple \( \overline{Y} = (L_\infty, L_\infty(\tilde{\varphi})) \), where

\[ \|x\|_{L_\infty(\varphi)} = \sup_{0 < t \leq 1} \tilde{\varphi}(t) |x(t)|, \quad \tilde{\varphi}(t) = \frac{t}{\varphi}(t). \]  

(2.20)

Indeed, a partial retract of a \( \mathcal{H} \)-monotone couple is a \( \mathcal{H} \)-monotone couple [15, page 420], and by the Sparr theorem [20] \( \overline{Y} \) is a \( \mathcal{H} \)-monotone couple.

First note that the inclusion \( L_\infty \subset M(\varphi) \) implies \( L_\infty + M(\varphi) = M(\varphi) \). So, let \( x \in M(\varphi) \). Without loss of generality [10, page 87], assume that \( x(t) = x^*(t) \). Define the operator

\[ U_1 y(t) = \sum_{k=1}^{\infty} 2^{-k} \int_{2^{-k}}^{2^{-k+1}} y(s) ds x_{(2^{-k},2^{-k+1})}(t) \quad \text{for } y \in M(\varphi). \]  

(2.21)

Clearly, \( U_1 \) maps \( L_\infty \) into itself. In addition, the concavity of the function \( \varphi \) and properties of the nonincreasing rearrangement imply

\[ \|U_1 y\|_{L_\infty(\varphi)} \leq 2 \sup_{k=1,2,...} \left( \varphi(2^{-k+1}) \right)^{-1} \int_{0}^{2^{-k}} y^*(s) ds \leq 2 \|y\|_{M(\varphi)}. \]  

(2.22)

Hence \( U_1 : \overline{X} \to \overline{Y} \). Since \( x(t) \) is nonincreasing, then \( U_1 x(t) \geq x(t) \). Therefore the linear operator

\[ U y(t) = \frac{x(t)}{U_1 x(t)} U_1 y(t) \]  

(2.23)

is bounded from the couple \( \overline{X} \) into the couple \( \overline{Y} \). In addition, \( U x(t) = x(t) \).

Take for \( V \) the identity mapping, that is, \( V y(t) = y(t) \). Since \( y_f > 0 \), then, by [10, page 156], we have

\[ \|V y\|_{M(\varphi)} \leq C \sup_{0 < t \leq 1} \tilde{\varphi}(t) y^*(t) \leq C \sup_{0 < t \leq 1} \tilde{\varphi}(t) |y(t)| = C \|y\|_{L_\infty(\tilde{\varphi})}. \]  

(2.24)

Therefore \( V : \overline{Y} \to \overline{X} \) and \( V x = x \).

Thus an arbitrary element \( x \in M(\varphi) \) is orbitally equivalent to itself as to element of the space \( L_\infty + L_\infty(\tilde{\varphi}) \). This completes the proof. \( \square \)
COROLLARY 2.2. If \( \gamma_\varphi > 0 \), then \((L_\infty, M(\varphi)^0)\) is a \( \mathcal{K} \)-monotone couple.

PROOF. Assume that \( x \) and \( y \) belong to the space \( M(\varphi)^0 \) and

\[
\mathcal{K}(t, y; L_\infty, M(\varphi)^0) \leq \mathcal{K}(t, x; L_\infty, M(\varphi)^0) \quad \text{for} \ t > 0. \tag{2.25}
\]

If \( z \in M(\varphi)^0 \), then

\[
\mathcal{K}(t, z; L_\infty, M(\varphi)^0) = \mathcal{K}(t, z; L_\infty, M(\varphi)). \tag{2.26}
\]

Therefore,

\[
\mathcal{K}(t, y; L_\infty, M(\varphi)) \leq \mathcal{K}(t, x; L_\infty, M(\varphi)) \quad \text{for} \ t > 0. \tag{2.27}
\]

Hence, by Proposition 2.1, there exists an operator \( T : (L_\infty, M(\varphi)) \to (L_\infty, M(\varphi)) \) such that \( \gamma = T \). It is readily seen that \( M(\varphi)^0 \) is an interpolation space of the couple \((L_\infty, M(\varphi))\). Therefore \( T : (L_\infty, M(\varphi)^0) \to (L_\infty, M(\varphi)^0) \).

We define now two subcones of the cone \( \mathcal{P} \). Denote by \( \mathcal{P}_0 \) the set of all functions \( f \in \mathcal{P} \) such that \( \lim_{t \to 0^+} f(t) = \lim_{t \to \infty} f(t) = 0 \). If \( f \in \mathcal{P}_0 \), then \( 0 \leq \gamma_f \leq \delta_f \leq 1 \) [10, page 76]. Let \( \mathcal{P}^{++} \) be the set of all \( f \in \mathcal{P} \) such that \( 0 < \gamma_f < \delta_f < 1 \). It is obvious that \( \mathcal{P}^{++} \subset \mathcal{P}_0 \).

A couple \((X_0, X_1)\) is called a \( \mathcal{K}_0 \)-complete couple if for any function \( f \in \mathcal{P}_0 \) there exists an element \( x \in X_0 + X_1 \) such that

\[
\mathcal{K}(t, x; X_0, X_1) = f(t). \tag{2.28}
\]

In other words, the set \( \mathcal{K}(X_0 + X_1) \) of all \( \mathcal{K} \)-functionals of a \( \mathcal{K}_0 \)-complete couple \((X_0, X_1)\) contains, up to equivalence, the whole of the subcone \( \mathcal{P}_0 \).

PROPOSITION 2.3. The Banach couple \((L_1(0, \infty), L_2(0, \infty))\) is a \( \mathcal{K}_0 \)-complete couple.

PROOF. By the Holmstedt formula for functional spaces [7],

\[
\mathcal{K}(t, x, L_1, L_2) \asymp \max \left\{ \int_0^t x^*(s) ds, t \right\}^{1/2}. \tag{2.29}
\]

If \( f \in \mathcal{P}_0 \), then \( g(t) = f(t^{1/2}) \) belongs to \( \mathcal{P}_0 \). We denote \( x(t) = g'(t) \). Then \( x(t) = x^*(t) \) and

\[
\int_0^t x(s) ds = g(t). \tag{2.30}
\]

Assume that \( f \in \mathcal{P}^{++} \). If \( \delta_f < 1 \), then there exists \( \varepsilon > 0 \) such that for some \( C > 0 \)

\[
G(s) = f(s^{1/2}) \leq C \left( \frac{\varepsilon}{t} \right)^{1-\varepsilon} f(t^{1/2}), \quad \text{if} \ s \geq t. \tag{2.31}
\]

Since \( g \in \mathcal{P}_0 \), then \( g'(t) \leq g(t)/t \). Therefore for \( t > 0 \)

\[
\int_t^\infty (x(s))^2 ds \leq \int_t^\infty \frac{g^2(s)}{s^2} ds \leq C^2 t^{\varepsilon-1} (f(t^{1/2}))^2 \int_t^\infty s^{1-\varepsilon} ds = C^2 \varepsilon t^{-1} (g(t))^2. \tag{2.32}
\]

Combining this with (2.29) and (2.30), we obtain

\[
\mathcal{K}(t, x; L_1, L_2) \asymp g(t^2) = f(t). \tag{2.33}
\]
Thus $\mathcal{K}(L_1 + L_2) \supseteq \mathcal{P}^+$. Hence, in particular, the intersection $\mathcal{K}(X_0 + X_1) \cap \mathcal{P}^+$ is not empty. Therefore, by [6, Theorem 4.5.7], $(L_1, L_2)$ is a $\mathcal{K}_0$-complete Banach couple. This completes the proof.

Let $\mathcal{K}(l_1 + l_2)$ be the set of all $\mathcal{K}$-functionals corresponding to the couple $(l_1, l_2)$. By $\mathcal{F}$ we denote the set of all functions $f \in \mathcal{P}$ such that

$$ f(t) = f(1)t \quad \text{for } 0 < t \leq 1, \quad \lim_{t \to \infty} \frac{f(t)}{t} = 0. \quad (2.34) $$

**COROLLARY 2.4.** Up to equivalence,

$$ \mathcal{K}(l_1 + l_2) \supseteq \mathcal{F}. \quad (2.35) $$

**Proof.** It is well known (cf. [4, page 142]) that for $x \in L_1(0, \infty) + L_\infty(0, \infty)$ and $u > 0$

$$ \mathcal{K}(u, x; L_1, L_\infty) = \int_0^u x^*(s) \, ds. \quad (2.36) $$

In addition,

$$ L_1 = (L_1, L_\infty)_{l_\infty}^\mathcal{K}, \quad L_2 = (L_1, L_\infty)_{l_2(2^{-k/2})}^\mathcal{K}. \quad (2.37) $$

The spaces $l_\infty$ and $l_2(2^{-k/2})$ are interpolation spaces with respect to the couple $(l_\infty, l_2(2^{-k}))$ [4]. Therefore, by the reiteration theorem (see [5] or [14]),

$$ \mathcal{K}(t, x; L_1, L_2) \approx \mathcal{K}(t, \mathcal{K}(\cdot, x; L_1, L_\infty); l_\infty, l_2(2^{-k/2})) \quad \text{for } x \in L_1 + L_2. \quad (2.38) $$

Introduce the average operator:

$$ Qx(t) = \sum_{k=1}^\infty \int_{k-1}^{k} x(s) \, ds \chi(k-1,k](t), \quad \text{if } t > 0. \quad (2.39) $$

From (2.36) it follows that

$$ \mathcal{K}(t, Qx^*; L_1, L_\infty) = \mathcal{K}(t, x; L_1, L_\infty) \quad (2.40) $$

for all positive integers $t$. Both functions in (2.40) are concave. Therefore,

$$ \mathcal{K}(t, Qx^*; L_1, L_\infty) \approx \mathcal{K}(t, x; L_1 \cdot L_\infty) \quad \forall t \geq 1. \quad (2.41) $$

Hence (2.38) yields

$$ \mathcal{K}(t, Qx^*; L_1, L_2) \approx \mathcal{K}(t, x; L_1, L_2), \quad \text{if } t \geq 1. \quad (2.42) $$

Now let $f \in \mathcal{F}$. Since $\mathcal{F} \subset \mathcal{P}_0$, then, by Proposition 2.3, there exists a function $x \in L_1(0, \infty) + L_2(0, \infty)$ such that

$$ \mathcal{K}(t, x; L_1, L_2) \approx f(t). \quad (2.43) $$

Clearly, the operator $Q$ is a projector in the spaces $L_1$ and $L_2$ with norm 1. Moreover, $Q(L_1) = l_1$ and $Q(L_2) = l_2$. Hence, by the theorem about complemented subcouples
mentioned in Section 1 (see [3] or [21, page 136]),
\[ \mathcal{H}(t, Qx^s; L_1, L_2) \sim \mathcal{H}(t, a; l_1, l_2) \quad \text{for} \ t > 0, \]  
(2.44)
where \( a = (\int_{k-1}^{k} x^s(s) \, ds)_{k=1}^{\infty} \).

Thus (2.42) and (2.43) imply
\[ \mathcal{H}(t, a; l_1, l_2) \approx f(t) \quad \text{for} \ t \geq 1. \]  
(2.45)
The last relation also holds if \( 0 < t \leq 1 \). Indeed, in this case
\[ \mathcal{H}(t, a; l_1, l_2) = t \mathcal{H}(1, a; l_1, l_2) \times t f(1) = f(t). \]  
(2.46)

This completes the proof.

**Proof of Theorem 1.5.** As it was already mentioned in the proof of Theorem 1.2, the Orlicz space \( L_N, N(t) = \exp(t^2) - 1 \), coincides with the Marcinkiewicz space \( M(\varphi_1) \), for \( \varphi_1(u) = u \log^{1/2}(2/u) \). Since \( \gamma_{\varphi_1} = 1 \), then Corollary 2.2 implies that the couple \((L_\infty, G)\) is a \( \mathcal{K} \)-monotone couple. Hence,
\[ X_0 = (l_\infty, G)^{\mathcal{X}}_{E_0}, \quad X_1 = (l_\infty, G)^{\mathcal{X}}_{E_1}, \]  
(2.47)
for some parameters of the real \( \mathcal{K} \)-method of interpolation \( E_0 \) and \( E_1 \). By Theorem 1.4,
\[ \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_i} \approx \|(a_k)\|_{F_i}, \]  
(2.48)
where \( F_i = (l_1, l_2)^{\mathcal{X}}_{E_i} \) (\( i = 0, 1 \)). So
\[ (l_1, l_2)^{\mathcal{X}}_{E_0} = (l_1, l_2)^{\mathcal{X}}_{E_1}. \]  
(2.49)
Equation (2.49) means that the norms of spaces \( E_0 \) and \( E_1 \) are equivalent on the set \( \mathcal{K}(l_1 + l_2) \). It is readily to check that this set coincides, up to the equivalence, with the set \( \mathcal{K}(l_\infty + G) \) of all \( \mathcal{K} \)-functionals corresponding to the couple \((L_\infty, G)\). More precisely,
\[ \mathcal{K}(l_1 + l_2) = \mathcal{K}(l_\infty + G) = \mathcal{F}. \]  
(2.50)
In fact, by Theorem 1.2 and Corollary 2.2, \( \mathcal{F} \subset \mathcal{K}(l_1 + l_2) \subset \mathcal{K}(l_\infty + G) \). On the other hand, since \( l_\infty \subset G \) with the constant 1 and \( l_\infty \) is dense in \( G \), then \( \mathcal{K}(l_\infty + G) \subset \mathcal{F} \) [15, page 386].

Now let \( x \in X_0 \). By (2.47), we have \( \mathcal{H}(2^k, x; l_\infty, G) \) \( k \in X_0 \). Using (2.50), we can find \( a \in l_2 \) such that
\[ \mathcal{H}(2^k, a; l_1, l_2) \approx \mathcal{H}(2^k, x; l_\infty, G) \]  
(2.51)
for all positive integers \( k \). Since a parameter of \( \mathcal{K} \)-method is a Banach lattice, then this implies \( \mathcal{H}(2^k, a; l_1, l_2) \) \( k \in E_0 \). Therefore, by (2.49), \( \mathcal{H}(2^k, a; l_1, l_2) \) \( k \in E_1 \), that is, \( \mathcal{H}(2^k, x; l_\infty, G) \) \( k \in E_1 \) or \( x \in X_1 \). Thus \( X_0 \subset X_1 \). Arguing as above, we obtain the converse inclusion, and \( X_0 = X_1 \) as sets. Since \( X_0 \) and \( X_1 \) are Banach lattices, then their norms are equivalent. This completes the proof.
3. Final remarks and examples

**Remark 3.1.** Combining Theorems 1.2, 1.4, and 1.5 with results obtained in [8], we can prove similar assertions for lacunary trigonometric series. Moreover, taking into account the main result of [1], we can extend Theorems 1.2, 1.4, and 1.5 to Sidon systems of characters of a compact abelian group.

**Remark 3.2.** In Theorem 1.2, we cannot replace the space $G$ by $L_q$ with some $q < \infty$. Indeed, suppose that the couple $(T(l_1), T(l_2))$ is a $\mathcal{K}$-subcouple of the couple $(L_\infty, L_q)$, that is,

$$\mathcal{K}(t, a; l_1, l_2) \approx \mathcal{K}(t, Ta; L_\infty, L_q).$$

(3.1)

Let $E = l_p(2^{-\theta k})$, where $0 < \theta < 1$ and $p = q/\theta$. Applying the $\mathcal{K}$-method of interpolation $(\cdot, \cdot)_{E}^\infty$ to the couples $(l_1, l_2)$ and $(L_\infty, L_q)$, we obtain

$$\|Ta\|_p \approx \|a\|_{r, p} = \left\{ \sum_{k=1}^\infty (a_k^*)^p k^{p/r-1} \right\}^{1/p}.$$  

(3.2)

Since $r = 2/(2-\theta) < 2$ [4, page 142], then this contradicts with (1.3).

**Remark 3.3.** Clearly, a partial retract of a couple $(Y_0, Y_1)$ is a $\mathcal{K}$-subcouple of $Y$. The opposite assertion is not true, in general (nevertheless, some interesting examples of $\mathcal{K}$-subcouples and partial retracts simultaneously are given in [9]). Indeed, by Theorem 1.2, the subcouple $(l_1, l_2)$ is a $\mathcal{K}$-subcouple of the couple $(L_\infty, G)$. Assume that $(l_1, l_2)$ is a partial retract of this couple. Then (see the proof of Proposition 2.1) $(l_1, l_2)$ is a partial retract of the couple $(L_\infty, L_\infty (\log^{1/2} 2/t))$, as well. Therefore, by Lemma 1 from [2] and [4, page 142] it follows that

$$[l_1, l_2]_\theta = (l_1, l_2)_{\theta, \infty} = l_p, \infty,$$

(3.3)

where $[l_1, l_2]_\theta$ is the space of the complex method of interpolation [4], $0 < \theta < 1$, and $p = 2/(2-\theta)$. On the other hand, it is well known [4, page 139] that

$$[l_1, l_2]_\theta = l_p \quad \text{for} \quad p = \frac{2}{2-\theta}.$$  

(3.4)

This contradiction shows that the couple $(l_1, l_2)$ is not a partial retract of the couple $(L_\infty, G)$.

Using Theorem 1.4, we can find coordinate sequence spaces of coefficients of Rademacher series belonging to certain r.i.s.’s.

**Example 3.4.** Let $X$ be the Marcinkiewicz space $M(\varphi)$, where $\varphi(t) = t \log_2 \log_2 (16/t)$, $0 < t \leq 1$. Show that

$$\left\| \sum_{k=1}^\infty a_k r_k \right\|_{M(\varphi)} \approx \|a\|_{l_1 (\log)},$$

(3.5)

where $l_1 (\log)$ is the space of all sequences $a = (a_k)_{k=1}^\infty$ such that the norm

$$\|a\|_{l_1 (\log)} = \sup_{k=1, 2, \ldots} \log_2^{-1} (2k) \sum_{i=1}^k a_i^*$$

(3.6)
is finite. Taking into account Theorem 1.4, it is sufficient to check that

\[ \left( l_1, l_2 \right)_F^X = l_1(\log), \]  
\[ (l_\infty, G)_F^X = M(\varphi), \]  
(3.7)  
(3.8)

for some parameter \( F \) of the \( X \)-method of interpolation. More precisely, we will prove that (3.7) and (3.8) are true for \( F = l_\infty(u_k) \), where \( u_k = 1/(k + 1) \) \( (k \geq 0) \) and \( u_k = 1 \) \( (k < 0) \).

By the Holmstedt formula (2.5),

\[ \varphi_a(2^k) \leq \sum_{i=1}^{2^k} a_i^* + 2^k \left[ \sum_{i=2^k+1}^\infty (a_i^*)^2 \right]^{1/2} \leq B\varphi_a(2^k) \quad \text{for } k = 0, 1, 2, \ldots, \]  
(3.9)

where, as before, \( \varphi_a(t) = \mathcal{X}(t, a; l_1, l_2) \). Without loss of generality, assume that \( a_i = a_i^* \). If \( \|a\|_{l_1(\log)} = R < \infty \), then by (3.6),

\[ \sum_{i=1}^{2^k} a_i^* \leq 2R(k + 1). \]  
(3.10)

In particular, this implies \( a_{2^k} \leq 2^{-(k+1)}R(k + 1) \), for nonnegative integer \( k \). Using (3.10), we obtain

\[ \sum_{i=2^k+1}^\infty a_i^2 = \sum_{j=k}^{2^{(j+1)}} \sum_{i=2^{(j+1)}+1} a_i^2 \leq 3 \sum_{j=k}^{2^{(j+1)}} 2^ja_{2j}^2 \leq 12R^2 \sum_{j=k}^{2^{(j+1)-2j}(j + 1)^2} \leq 192R^2 \int_{k+1}^\infty x^2 2^{-2x} \, dx \leq 144R^2(k + 1)^2 2^{-2k}. \]  
(3.11)

Hence the second term in (3.9) does not exceed \( 12R(k + 1) \). Therefore, if \( E = \left( l_1, l_2 \right)_F^X \), then (3.10) implies

\[ \|a\|_E = \sup_{k=0,1,\ldots} \frac{\varphi_a(2^k)}{k + 1} \leq 14\|a\|_{l_1(\log)}. \]  
(3.12)

Conversely, if \( 2^{j} + 1 \leq k \leq 2^{j+1} \) for some \( j = 0, 1, 2, \ldots \), then from (3.9) it follows that

\[ \sum_{i=1}^{2^{(j+1)}} a_i \leq B\varphi_a(2^{j+1}) \leq \sum_{i=1}^{2^{(j+1)}} a_i \leq B\|a\|_E(j + 2) \leq 2B\log_2(2k)\|a\|_E. \]  
(3.13)

Therefore, \( \|a\|_{l_1(\log)} \leq 2B\|a\|_E \) and (3.7) is proved.

We pass now to function spaces. At first, we introduce one more interpolation method which is, actually, a special case of the real method of interpolation. For a function \( \varphi \in \mathcal{P} \) and an arbitrary Banach couple \( (X_0, X_1) \) define generalized Marcinkiewicz space as follows:

\[ \mathcal{M}_\varphi(X_0, X_1) = \left\{ x \in X_0 + X_1 : \sup_{t>0} \frac{\mathcal{X}(t, x; X_0, X_1)}{\varphi(t)} < \infty \right\}. \]  
(3.14)
Let \( q_0(t) = \min(1, t), \ q_1(t) = \min(1, t \log_2^{1/2} [\max(2, 2/t)]) \), and \( N(t) = \exp(t^2) - 1 \), as before. By equation (2.36), we have
\[
L_\infty = M_{q_0}(L_1, L_\infty), \quad L_N = M_{q_1}(L_1, L_\infty),
\]
(here \( L_\infty \) and \( L_N \) are functional spaces on the segment \([0, 1]\)). In addition, using similar notation, it is easy to check that
\[
(X_0, X_1)_F^\infty = M_p(X_0, X_1),
\]
for an arbitrary Banach couple \((X_0, X_1)\) and \( \rho(t) = \log_2(4 + t) \). Hence, by the reiteration theorem for generalized Marcinkiewicz spaces [15, page 428], we obtain
\[
(L_\infty, L_N)_F^\infty = M_p(M_{q_0}(L_1, L_\infty), M_{q_1}(L_1, L_\infty)) = M_{q_\rho}(L_1, L_\infty) = M(\rho),
\]
where \( q_\rho(t) = q_0(t) \rho(q_1(t)/q_0(t)) \). A simple calculation gives \( q_\rho(t) \approx q(t) \), if \( t > 0 \). Thus,
\[
(L_\infty, L_N)_F^\infty = M(\rho).
\]
It is readily seen that \( \mathcal{H}(t, x; L_\infty, G) = \mathcal{H}(t, x; L_\infty, L_N) \), for all \( x \in G \). Therefore, for such \( x \) the norm \( \|x\|_{M(\rho)} \) is equal to the norm \( \|x\|_Y \), where \( Y = (L_\infty, G)_F^\infty \). On the other hand, for \( x \in M(\rho) \)
\[
\frac{1}{t \log_2^{1/2} (2/t)} \int_0^t x^*(s) \, ds \leq \|x\|_{M(\rho)} \frac{\log_2 \log_2 (16/t)}{\log_2 \log_2 (2/t)} \to 0 \quad \text{as} \quad t \to 0 +.
\]
This implies that \( M(\rho) \subset G \) [10, page 156]. Thus \( Y = M(\rho) \), and (3.8) is proved. Equivalence (3.5) follows now, as already stated, from (3.7) and (3.8).

**Remark 3.5.** Theorems 1.4 and 1.5 strengthen results of [18, 19], where similar assertions are obtained for sequence spaces \( F \) satisfying more restrictive conditions. For instance, we can readily show that the norm of the dilation operator
\[
\sigma_n a = \left( \frac{a_1, \ldots, a_3, a_2, \ldots, a_2}{n} \right)
\]
in the space \( l_1(\ln) \) (see Example 3.6) is equal to \( n \). Therefore, condition (11) from [19] fails for this space and the theorems obtained in [18, 19] cannot be applied to it. Similarly, the Marcinkiewicz space \( M(\rho) \) from Example 3.4 does not satisfy the conditions of Theorem 8 of [19].

Using Theorems 1.4 and 1.5, we can derive certain interpolation relations.

**Example 3.6.** Let \( \varphi \in \mathcal{P} \) and \( 1 \leq p < \infty \). Recall that the Lorentz space \( \Lambda_p(\varphi) \) consists of all measurable functions \( x = x(s) \) such that
\[
\|x\|_{\varphi, p} = \left( \int_0^1 (x^*(s))^p \, d \varphi(s) \right)^{1/p} < \infty.
\]
In [19], V. A. Rodin and E. M. Semenov proved that
\[
\sum_{k=1}^{\infty} a_k r_k \bigg|_{\varphi,p} \approx \| (a_k) \|_{\varphi,p},
\] (3.22)
where \( \varphi(s) = \log_2^{1-p}(2/s) \) and \( 1 < p < 2 \). Moreover, the space \( \Lambda_p(\varphi) \) is the unique r.i.s. having this property. Note that \( L_p = (L_1, L_2)_{\vartheta,p} \), where \( \vartheta = 2(p - 1)/p \) [4, page 142]. Therefore, by Theorem 1.4, we obtain
\[
(L_{\infty}, G)_{\vartheta,p} = \Lambda_p(\varphi)
\] (3.23)
for the same \( p \) and \( \vartheta \).

**Acknowledgement.** The author is grateful to Prof. S. Montgomery-Smith for useful advices and to referees for their suggestions and remarks.

**References**


ABOUT INTERPOLATION OF SUBSPACES OF REARRANGEMENT …


SERGEY V. ASTASHKIN: DEPARTMENT OF MATHEMATICS, SAMARA STREET UNIVERSITY, ACADEMIC PAVLOV STREET, 1, SAMARA, 443011, RUSSIA

E-mail address: astashkn@ssu.samara.ru
Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>December 1, 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

**Guest Editors**

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk

Hindawi Publishing Corporation
http://www.hindawi.com