WHITEHEAD GROUPS OF EXCHANGE RINGS
WITH PRIMITIVE FACTORS ARTINIAN

HUANYIN CHEN and FU-AN LI

(Received 8 October 2000)

Abstract. We show that if $R$ is an exchange ring with primitive factors artinian then $K_1(R) \cong U(R)/V(R)$, where $U(R)$ is the group of units of $R$ and $V(R)$ is the subgroup generated by $\{(1+ab)(1+ba)^{-1} \mid a, b \in R \text{ with } 1+ab \in U(R)\}$. As a corollary, $K_1(R)$ is the abelianized group of units of $R$ if $1/2 \in R$.

2000 Mathematics Subject Classification. 16E50, 19B10.

Very recently, Ara et al. [2] showed that the natural homomorphism $GL_1(R) \to K_1(R)$ is surjective provided that $R$ is a separative exchange ring. A natural problem is the description of the kernel of the epimorphism $GL_1(R) \to K_1(R)$. In [9, Theorems 1.2 and 1.6], Menal and Moncaı̈ showed that if $R$ satisfies unit 1-stable range or is unit-regular, then $K_1(R) \cong U(R)/V(R)$. Here $U(R)$ is the group of units of $R$ while $V(R)$ is a subgroup described later. In [7], Goodearl and Menal remarked that if for each $x, y \in R$ there exists a unit $u \in R$ such that $x-u$ and $y-u^{-1}$ are both units, then $K_1(R) \cong U(R)^{ab}$. In this paper, we investigate the above kernel for exchange rings with primitive factors artinian.

Recall that $R$ is called an exchange ring if for every right $R$-module $A$ and two decompositions $A = M \oplus N = \oplus_{i \in I} A_i$, where $M_R \cong R$ and the index set $I$ is finite, there exist submodules $A'_i \subseteq A_i$ such that $A = M \oplus (\oplus_{i \in I} A'_i)$. It is well known that regular rings, $\pi$-regular rings, semiperfect rings, left or right continuous rings, clean rings and unit $C^*$-algebras of real rank zero [1] are all exchange rings.

Many authors have studied exchange rings with primitive factors artinian. Fisher and Snider proved that every regular ring with primitive factors artinian is unit-regular (see [6, Theorem 6.10]). Moreover, Menal [8, Thereom B] proved that every $\pi$-regular ring with primitive factors artinian has stable range one. Recently, Yu [13, Thereom 1] extended these results to exchange rings and showed that every exchange ring with primitive factors artinian has stable range one. On the other hand, Pardo [10] investigated the Grothendieck group $K_0$ of exchange rings. In this paper, we show that the Whitehead group $K_1(R) \cong U(R)/V(R)$ for an exchange ring $R$ with primitive factors artinian. We refer the reader to [11] for the general theory of Whitehead groups.

Throughout this paper, all rings are associative with identity. The set $U(R)$ denotes the set of all units of $R$, $V(R)$ denotes the subgroup generated by $\{p(a,b)p(b,a)^{-1} \mid p(a,b) \in U(R), a, b \in R\}$, and $J(R)$ denotes the Jacobson radical of $R$ (see below for the definition of $p(a,b)$ and other continuant polynomials). If $G$ is a group and $G'$ its
commutator subgroup, then \( G^{ab} \) denotes \( G/G' \). Let \( \text{GL}_n(R) \) be the group of units of \( M_n(R) \), the ring of all \( n \times n \) matrices over \( R \).

We start with the following new element-wise property of exchange rings with primitive factors artinian.

**Lemma 1.** Let \( R \) be an exchange ring with primitive factors artinian. Then, for any \( x, y \in R \), there exists a unit-regular \( w \in R \) such that \( 1 + xy - xw \in U(R) \).

**Proof.** Assume that there are some \( x, y \in R \) such that \( 1 + xy - xw \notin U(R) \) for any unit-regular \( w \in R \). Let \( \Omega \) be the set of all two-sided ideals \( A \) of \( R \) such that \( 1 + xy - xw \notin U(R) \) is a unit modulo \( A \) for any unit-regular \( w + A \in R/A \). Clearly, \( \Omega \neq \emptyset \).

Given any ascending chain \( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots \) in \( \Omega \), set \( M = \bigcup_{1 \leq i \leq n} A_i \). Then \( M \) is a two-sided ideal of \( R \). Assume that \( M \) is not in \( \Omega \). Then there exists a unit-regular \( w + M \in R/M \) such that \( (1 + xy - xw) + M \) is a unit of \( R/M \). Since \( w + M \) is unit-regular in \( R/M \), we have \( e + M = (e + M)^2 \), \( u + M \in U(R/M) \) such that \( w + M = (e + M)(u + M) \). As \( R/M \) is also exchange, we may assume that \( e = e^2 \in R \). Thus we can find positive integers \( n_i \) (\( 1 \leq i \leq 5 \)) such that \( (1 + xy - xw)s - 1 \in A_{n_1}, s(1 + xy - xw) - 1 \in A_{n_2}, w - eu \in A_{n_3}, ut - 1 \in A_{n_4}, \) and \( tu - 1 \in A_{n_5} \) for some \( s, t \in R \). Let \( n = \max\{n_1, n_2, n_3, n_4, n_5\} \). Then \( 1 + xy - xw + A_n \) is a unit of \( R/A_n \) for a unit-regular \( w + A_n \in R/A_n \). This contradicts the choice of \( A_n \). So \( M \in \Omega \). That is, \( \Omega \) is inductive. By using Zorn’s lemma, we have a two-sided ideal \( Q \) of \( R \) such that it is maximal in \( \Omega \).

Let \( S = R/Q \). If \( J(R/Q) \neq 0 \), then \( J(R/Q) = K/Q \) for some \( K \in \Omega \). Clearly, \( S/J(S) \cong R/K \). By the maximality of \( Q \), we can find a unit-regular \( (v + Q) + J(S) \) such that \( ((1 + xy - xw) + Q) + J(S) \) is a unit of \( S/J(S) \). Since idempotents and units of \( S/J(S) \) can be lifted modulo \( J(S) \), we may assume that \( v + Q \) is unit-regular in \( S \). On the other hand, \( (1 + xy - xw) + Q = (m + Q) + (r + Q) \) for some \( m + Q \in U(S) \), \( r + Q \in J(S) \). Thus, \( 1 + xy - xw + Q \) is a unit of \( S \). This gives a contradiction, whence \( J(R/Q) = 0 \).

By the maximality of \( Q \), one easily checks that \( R/Q \) is an indecomposable ring. According to [14, Lemma 3.7], \( R/Q \) is a simple artinian ring. Clearly, \( (1 + xy - xw) + Q = 1 + Q \) is a unit of \( R/Q \). Since \( R/Q \) is unit-regular, \( y + Q \) is a unit-regular element of \( R/Q \). This contradicts the choice of \( Q \). Therefore the proof is complete. \( \Box \)

We note that the following conditions for a ring \( R \) are equivalent.

1. For any \( x, y \in R \), there exists unit-regular \( w \in R \) such that \( 1 + x(y - w) \in U(R) \).
2. Given \( aR + bR = R \), then \( a+bw \in U(R) \) for some unit-regular \( w \in R \).
3. Given \( ax + b = 1 \), then \( aw + b \in U(R) \) for some unit-regular \( w \in R \) (cf. [5, Theorem 2.1]).

Clearly, the conditions above are stronger than the stable range one condition and can be viewed as a generalization of rings with many idempotents [4]. Call \( R \) a ring with many unit-regular elements if the equivalent conditions above hold. In [5, Theorem 3.1], the authors showed that such rings are all \( GE_2 \). Furthermore, Chen [3] proved that if \( R \) has many unit-regular elements then so does \( M_n(R) \). Now we include the fact that every exchange ring with primitive factors artinian has many unit-regular elements to make our paper self-contained.

Recall that \( p(a) = a, p(a,b) = 1 + ab \), and \( p(a,b,c) = a + c + abc \) for any \( a, b, c \in R \). It is easy to verify that \( p(a,b,c) = p(a,b)c + p(a), p(a,b,c)p(b,a) = p(a,b)p(c,b,a) \).
\textbf{Lemma 2.} Let \(a, b, c \in R\) with \(p(a, b, c) \in U(R)\). If \(a\) is unit-regular, then \(p(a, b, c) \equiv p(c, b, a) \pmod{V(R)}\).

\textbf{Proof.} Since \(a\) is unit-regular, there is an idempotent \(e\) and a unit \(u\) such that \(a = eu\). So we have \(p(a, b, c) = eu + c + eubc\), and then \(p(a, b, c)u^{-1} = e + cu^{-1} + eubcu^{-1}\). Obviously, \(p(e, -ub(1-e)) = p(e, -ub(1-e))(p(-ub(1-e), e))^{-1} \in V(R)\).

Thus we see that
\[ p(a, b, c)u^{-1} = p(e, -ub(1-e))p(a, b, c)u^{-1} \pmod{V(R)} \]
\[ = (1 - eub(1-e))(e + cu^{-1} + eubcu^{-1}) \]
\[ = e + cu^{-1} + eubc - e = e_{ubc}c^{-1} \pmod{V(R)}. \]

On the other hand, we can verify that
\[ p(c, b, a)u^{-1} = e + cu^{-1} + cbe \equiv (e + cu^{-1} + cu^{-1}ube)p(1 - e, -ube) \pmod{V(R)} \]
\[ = e + cu^{-1} + cu^{-1}eube = 1 + (c^{-1} - (1-e))(1 + eube) \]
\[ \equiv 1 + (1 + eube)(c^{-1} - (1-e)) \pmod{V(R)} \]
\[ \equiv p(a, b, c)u^{-1} \pmod{V(R)}. \]

Consequently, \(p(a, b, c) \equiv p(c, b, a) \pmod{V(R)}\), as asserted. \qed

In [4, Theorem 16], Chen showed that \(K_1(R) \cong U(R)/V(R)\) provided that \(R\) has idempotent 1-stable range. Now we extend this fact to exchange rings with primitive factors artinian.

\textbf{Theorem 3.} Let \(R\) be an exchange ring with primitive factors artinian. Then \(K_1(R) \cong U(R)/V(R)\).

\textbf{Proof.} Given \(a, b, c \in R\) with \(p(a, b, c) \in U(R)\), we have \(p(c, b, a) \in U(R)\). From Lemma 1, we can find some unit-regular \(w \in R\) such that \(1 + b(c - w) \in U(R)\). Let \(c - w = t\). Then \(c = t + w\) and \(1 + bt \in U(R)\). We check that
\[ p(a, b, c) = a + c + abc = (a + t + ab) + (1 + ab)w \]
\[ = (a + t + ab) + (1 + ab + tb + abtb)(1 + tb)^{-1}w \]
\[ = (a + t + ab) + (1 + tb)^{-1}w + (a + t + ab)b(1 + tb)^{-1}w \]
\[ \equiv (1 + tb)^{-1}((1 + tb)(a + t + ab) + w + (1 + tb)(a + t + ab)b(1 + tb)^{-1}w) \]
\[ = (1 + tb)^{-1}p((1 + tb)(a + t + ab), b(1 + tb)^{-1}, w). \]

In view of Lemma 2, we know that
\[ p(a, b, c) \equiv (1 + tb)^{-1}p(w, b(1 + tb)^{-1}, (1 + tb)(a + t + ab)) \pmod{V(R)} \]
\[ = (1 + tb)^{-1}(p(w, b(1 + tb)^{-1})(1 + tb)(a + t + ab) + p(w)) \]
\[ = (1 + tb)^{-1}(p(w, b(1 + tb)^{-1})p(t, b)p(a, b, t) + p(w)) \]
\[ = (1 + tb)^{-1}(p(w, b(1 + tb)^{-1})p(t, b, a)p(b, t) + p(w)). \]
On the other hand,

\[ p(c, b, a) = p(w, (1 + bt)^{-1}b, (t + a + tba)(1 + bt)(1 + bt)^{-1} \]

\[ = (p(w, (1 + bt)^{-1}b)(t + a + tba)(1 + bt) + p(w))(1 + bt)^{-1} \]

\[ = (p(w, (1 + bt)^{-1}b)p(t, b, a)p(b, t) + p(w))(1 + bt)^{-1}. \quad (5) \]

Since \( b(1 + tb)^{-1} = (1 + bt)^{-1}b, (1 + tb)p(a, b, c) \equiv p(c, b, a)(1 + bt) \pmod{V(R)}. \) Hence \( p(a, b, c)p(c, b, a)^{-1} \in V(R) \) because \( U(R)/V(R) \) is abelian. By virtue of [13, Theorem 1], \( R \) has stable range one. It follows from [9, Theorem 1.2] that \( K_1(R) \cong U(R)/W(R) \cong U(R)/V(R), \) where the notation \( W(R) \) denotes the subgroup of \( U(R) \) generated by \( \{p(a, b, c)p(c, b, a)^{-1} | p(a, b, c) \in U(R), a, b, c \in R\} \). So the proof is complete.

Recall that an exchange ring is said to be of bounded index if there exists some positive integer \( n \) such that \( x^n = 0 \) for all nilpotent \( x \in R \). We can derive the following corollary.

**Corollary 4.** Let \( R \) be an exchange ring of bounded index. Then \( K_1(R) \cong GL_3(R)^{ab} \cong U(R)/V(R). \)

**Proof.** It is easy to obtain the first isomorphism by an argument of Menal. In view of [13, Theorem 3], \( R \) is an exchange ring with primitive factors artinian. Therefore we complete the proof by Theorem 3. \( \blacksquare \)

Write \( L_1(R) \) for the subgroup generated by the elements in \( U(R)' \cup L \), where \( L \) is the subgroup generated by all \( 1 + er(1 - e) \) with \( e = e^2, r \in R \). Clearly, \( U(R)' \subseteq L_1(R) \subseteq V(R) \).

**Lemma 5.** Let \( a, b \in R \) with \( 1 + ab \in U(R) \). If \( a \) is unit-regular, then \( p(a, b) \equiv p(b, a) \pmod{L_1(R)} \).

**Proof.** Since \( a \) is unit-regular, there is an idempotent \( e \) and a unit \( u \) such that \( a = ue \). So we have

\[ p(a, b) = 1 + ueb = u(1 + ebu)u^{-1} \]

\[ = u(1 + ebu)u^{-1}(1 + ebu)^{-1}(1 + ebu) \]

\[ \equiv (1 + ebu) \pmod{L_1(R)} \]

\[ \equiv (1 - ebu(1 - e))(1 + ebu) \pmod{L_1(R)} \]

\[ = 1 + ebue = (1 + bue)(1 - (1 - e)bue) \]

\[ \equiv (1 + bue) \pmod{L_1(R)} = p(b, a), \] as required. \( \blacksquare \)

**Lemma 6.** Let \( R \) be an exchange ring with primitive factors artinian. If \( R \) does not have \( \mathbb{Z}/2\mathbb{Z} \) as a homomorphic image, then, for any \( x, y \in R \), there exists a \( w \in U(R) \) such that \( 1 + x(y - w) \in U(R) \) and \( y - w \) is unit-regular.

**Proof.** Assume that there exist some \( x, y \in R \) such that \( 1 + x(y - w) \notin U(R) \) or \( y - w \) is not unit-regular for any unit \( w \in R \). Let \( \Omega \) be the set of all two-sided ideals \( A \) of \( R \) such that \( 1 + x(y - w) \) is not a unit or \( y - w \) is not unit-regular modulo \( A \) for any unit \( w + A \in R/A \). Obviously, \( \Omega \neq \emptyset \).
Analogously to the discussion in Lemma 1, we can find a two-sided ideal Q of R such that it is maximal in \( \Omega \). Set \( S = R/Q \). If \( J(S) \neq 0 \), then \( J(S) = K/Q \) for some \( K \supseteq Q \). Obviously, \( S/J(S) \cong R/K \). By the maximality of \( Q \), we have a unit \( (v + Q) + J(S) \) such that \( ((1 + x(y - v)) + Q) + J(S) \) is a unit of \( S/J(S) \) and \( ((y - v) + Q) + J(S) \) is unit-regular in \( S/J(S) \). Since units of \( S/J(S) \) can be lifted modulo \( J(S) \) because \( S \) is an exchange ring, we may assume that \( v + Q \) is unit of \( S \). Similarly to Lemma 1, we see that \( (1 + x(y - v)) + Q = (m + Q) + (r + Q) \) for some \( m + Q \in U(S), r + Q \in J(S) \). Thus, \( (1 + x(y - v)) + Q \) is a unit of \( S \). On the other hand, if \( S/J(S) \) can be lifted modulo \( J(S) \) because \( S \) is an exchange ring. So we may assume that \( ((y - v) + Q) + J(S) = ((f + Q) + J(S))((u + Q) + J(S)) \) with \( f + Q = (f + Q)^2 \in R/Q, u + Q \in U(S) \). Thus we can find some \( t \in R \) with \( t + Q \in J(S) \) such that \( (y - (v-t)) + Q = (f + Q)(u + Q) \) is unit-regular in \( S \). From \( (1 + x(y - (v-t)) + Q = ((1 + x(y - v)) + Q) + (x + Q)(t + Q) \) with \( (1 + x(y - v)) + Q \in U(S) \) and \( t + Q \in J(S) \), one easily checks that \( (1 + x(y - (v-t)) + Q \) is a unit of \( S \). This contradicts the choice of \( Q \). Therefore \( J(S) = 0 \). The maximality of \( Q \) implies that \( S \) is indecomposable as a ring. By virtue of [14, Lemma 3.7], \( S \) is a simple artinian ring. Assume that \( R/Q \cong M_n(D) \), where \( n \) is a positive integer and \( D \) is a division ring. Since \( R \) does not have \( \mathbb{Z}/2\mathbb{Z} \) as a homomorphic image, we claim that \( n \geq 2 \), or \( n = 1 \), and \( D \ncong \mathbb{Z}/2\mathbb{Z} \). Thus \( R/Q \) satisfies unit 1-stable range. From \( (1 + xy) + Q)S + (x + Q)S = S \), we have a unit \( s + Q \) of \( R/Q \) such that \( (1 + x(y - s)) + Q = ((1 + xy) + Q) + (x + Q)(s + Q) \) is a unit of \( S \). Since \( S \) is simple artinian, \( (y - s) + Q \) is unit-regular in \( S \), a contradiction. Therefore the proof is complete.

**Theorem 7.** Let \( R \) be an exchange ring with primitive factors artinian. If \( R \) does not have \( \mathbb{Z}/2\mathbb{Z} \) as a homomorphic image, then \( K_1(R) \cong U(R)/L_1(R) \).

**Proof.** Since \( R \) is an exchange ring with primitive factors artinian, by virtue of Theorem 3, we know that \( K_1(R) \cong U(R)/V(R) \). Let \( b, c \in R \) with \( p(b, c) \in U(R) \). In view of Lemma 6, there exists some \( w \in U(R) \) such that \( 1 + b(c - w) \in U(R) \) and \( c - w = s \) unit-regular. Observe that

\[
p(b, c) = 1 + bc = 1 + bs + bw = 1 + bs + b(1 + sb)(1 + sb)^{-1}w \\
= (1 + bs)(1 + b(1 + sb)^{-1}w) = p(b, s)p(b, 1 + sb)^{-1}, w). \tag{7}
\]

Likewise, we see that

\[
p(c, b) = p(w, b(1 + sb)^{-1})(1 + sb) = p(w, b(1 + sb)^{-1})p(s, b). \tag{8}
\]

By Lemma 5 and the fact that units commute mod \( L_1(R) \), we see that

\[
p(b, c) = p(b, s)p(b, 1 + sb)^{-1}, w) \equiv p(w, b(1 + sb)^{-1})p(s, b) = p(c, b) \pmod{L_1(R)}. \tag{9}
\]

Hence, \( L_1(R) = V(R) \) and we conclude that \( K_1(R) \cong U(R)/L_1(R) \), as asserted.

**Corollary 8.** Let \( R \) be an exchange ring with primitive factors artinian. If \( 2 \in U(R) \), then \( K_1(R) \cong U(R)^{ab} \).

**Proof.** Let \( e = e^2 \in R \). Since \( 2 \in U(R) \), we have \( e = 1/2 + (2e - 1)/2 \). Obviously, \(((2e-1)/2)(4e-2) = 1\). By [9, Lemma 1.5], \( 1 + eR(1 - e) \subseteq U(R)' \), and then \( L_1(R) = U(R)' \).
On the other hand, $2 \in U(R)$ implies that $R$ does not have $\mathbb{Z}/2\mathbb{Z}$ as a homomorphic image. Using Theorem 7, we conclude that $K_1(R) \cong U(R)/L_1(R) \cong U(R)/U(R)' \cong U(R)^{ab}$, as desired.

**Corollary 9.** Let $R$ be an exchange ring of bounded index of nilpotence. If $2 \in U(R)$, then $K_1(R) \cong U(R)^{ab}$.

**Proof.** By [12, Proposition 2.1], we see that the primitive factors of $R$ are artinian. Thus the result follows from Corollary 8.

**Acknowledgements.** It is a pleasure to thank the referee for thoroughly reading the article and for helpful suggestions.

This work is supported by the National Science Foundation of China, No. 19801012.

**References**


Huanyin Chen: Department of Mathematics, Hunan Normal University, Changsha 410006, China  
E-mail address: chyzx1@sparc2.hunnu.edu.cn

Fu-an Li: Institute of Mathematics, Chinese Academy of Sciences, Beijing 100080, China  
E-mail address: fal@math08.math.ac.cn
Special Issue on
Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today’s economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems).

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

<table>
<thead>
<tr>
<th>Deadline</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

**Lean Yu,** Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

**Shouyang Wang,** Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

**K. K. Lai,** Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk