SUBGROUPS OF FINITE INDEX IN AN ADDITIVE GROUP OF A RING

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(Received 15 March 2000 and in revised form 10 June 2000)

Abstract. If $K$ is an infinite field and $G \subseteq K$ is a subgroup of finite index in an additive group, then $K^* = G^* G^{*-1}$ where $G^*$ denotes the set of all invertible elements in $G$ and $G^{*-1}$ denotes all inverses of elements of $G^*$. Similar results hold for various fields, division rings and rings.

2000 Mathematics Subject Classification. 16Kxx.

1. Introduction. Let $R$ be a ring (not necessarily commutative) with a unit element $1$ and $R^*$ denotes the multiplicative group of invertible elements of $R$. In [9] Leep and Shapiro proved that if $G$ is a subgroup of index $3$ in the multiplicative group $F^*$, then $G + G = F$. In [2] Berrizbetia proved that if $F$ is a field and $G \subseteq F^*$ is a subgroup of finite index $n$, then there is a positive integer $m$, that depends on $n$, so that if $\text{char } F = 0$ or $\text{char } F \geq m$, then $G - G = F$. In [1] Bergelson and Shapiro proved that, for various ring $R$, if $G$ is a subgroup of finite index of $R^*$, then $G - G = R$. In [14] Turnwald proved that if $G$ is a subgroup of finite index $n$ in the multiplicative group of a division ring $F$ then $G - G = F$ or $|F| < (n + 1)^4 + 4n$, and if $|F| > (n - 1)^2$ and $-1$ is a sum of elements of $G$ then every element of $F$ has this property; the bound $(n - 1)^2$ is optimal for infinitely many $n$. The theories which have important role in studying of the above were Ramsey theory, measure theory and number theory, (cf. [4, 7, 15]). Furthermore in [1] the roles of multiplication and addition were switched, and it was shown that

**Proposition 1.1** (see [1, Proposition 2.14]). Let $K$ be an infinite field and $G \subseteq K$ a subgroup of finite index in additive group. Then $G^* G^{*-1} = K^*$ where $G^* = G \setminus \{0\}$; that is, for every $c \in K^*$ there exist $g_1, g_2 \in G$ such that $c = g_1/g_2$.

**Corollary 1.2.** If $D$ is an infinite division ring then the above result is satisfied.

In this paper, the roles of multiplication and addition are switched and it is shown that **Proposition 1.1** and **Corollary 1.2** hold for various fields, division rings and rings.

Now let $G \subseteq R$ be a subgroup of finite index in an additive group, $G^*$ be the set of all invertible elements in $G$, $G^{*-1} = \{g^{-1} : g \in G^*\}$ and $G^* G^{*-1} = \{g_1 g_2^{-1} : g_1, g_2 \in G^*\}$.

2. $G^* G^{*-1}$. Let $K$ be a ring or field and $G \subseteq K$ be a subgroup of finite index in an additive group, then it is not necessary that $G^* G^{*-1} = R^*$ or even $G^*, G^{*-1}$, and $G^* G^{*-1}$ have group structure. Note the following statements.

(i) Let $F = F_{p^2}$, and $G = F_p$, then $G^* = F^*_p$ and $G^* G^{*-1} = F^*_p \neq F^*$.

(ii) Let $\alpha$ be a root of the polynomial $x^3 + x + 1$ over the splitting field $Z_2(\alpha) = F_8$, then $G^* G^{*-1} = (F_8)^*$. The following statements:

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where $F_8$ has 8 elements. Put $G = \{0, \alpha, \alpha^2, \alpha + \alpha^2\}$ so $G^* = \{\alpha, \alpha^2, \alpha + \alpha^2\}$. It is clear that $G^*$ is not a group, because $G^*$ does not contain the unit element 1. But $G^*G^{*-1}$ is a subgroup of the multiplicative group $F_8^*$. Furthermore, $G^*G^{*-1} = F_8^*$.

(iii) Let $\beta$ be a root of the polynomial $x^4 - x + 1$ over the splitting field $Z_2(\beta) = F_{16}$. Put $G = \{0, 1, \beta, 1 + \beta\}$, so $G^* = \{1, \beta, 1 + \beta\}, G^{*-1} = \{1, \beta^3 + 1, \beta^3 + \beta^2 + \beta\}$ and therefore $G^*G^{*-1} = \{1, \beta^3 + 1, \beta^3 + \beta^2 + \beta, 1 + \beta + \beta^2 + \beta^3, 1 + \beta, \beta^3\}$. It is clear that $G^*$ is not a group but $G^*G^{*-1}$ is a proper subgroup of $F_{16}$.

(iv) Let $R = \mathbb{Z}/n\mathbb{Z}$ where $n$ is not a prime number. If $G \subseteq R$ is a proper subgroup in an additive group then it is clear that $G^* = \emptyset$ and $G^*G^{*-1} = \emptyset \neq R^*$.

(v) Let $S = \mathbb{Z}/n\mathbb{Z}$ where $n$ is a natural number, $R = S[x]$ and $H$ is a proper subgroup of $S$. If $G = \{f(x) = a_0 + a_1x + \cdots + a_kx^k : a_i \in S, a_0 \in H\}$, so $G \subseteq R$ is a subgroup of finite index in an additive group and $G^* = \emptyset$. If the square of every prime number does not divide $n$ and $a_0 \in S$ but $a_t \in H$, for finitely many $t > 0$, then $G \subseteq R$ is a subgroup of finite index in an additive group, $G^* = R^*$ and $G^*G^{*-1} = R^*$.

(ii) Since $G = \{\mathbb{Z} : (n, 2) = 1\}$ is a valuation ring (cf. [3, 10, 11, 12] or [13]). If $G = \{2m/n : m, n \in \mathbb{Z} : (m, 2) = 1\}$, then $G$ is a subgroup of finite index 2 in an additive group where $0 + G$ and $1 + G$ are two distinct cosets $G$ in $R$. It is easy to see that $G^* = \emptyset$, $R^* = \{m/n : m = 2k + 1, n = 2l + 1\}$, and $G^*G^{*-1} \neq R^*$.

The above statement can be shown for any $p$-adic valuation ring in $\mathbb{Q}$.

By Proposition 1.1, Corollary 1.2, and the previous statements, the following question may be raised.

**Question 2.1.** If $F$ is a finite field or a ring and $G$ is a subgroup of finite index in an additive group, must $G^*G^{*-1} = F^*$?

We will answer the question for all finite fields and some rings.

If $F$ is a finite field and $|F|$ is sufficiently large to the index of $G$, in other words, $G$ is sufficiently large, then $G^*G^{*-1} = F^*$.

**Theorem 2.2.** (i) Let $D$ be a division ring with $\text{char} D = p$ and $G \subseteq D$ be a subgroup of index $p^k$ in an additive group. If $|D| \geq p^{2k+1}$, then $G^*G^{*-1} = D^*$.

(ii) If $G$ is a subgroup of finite index $n \geq p^k$ in a division ring $D$ and $|D| = p^{2k}$, then $G^*G^{*-1} + D^*$.

**Proof.** (i) Fix $c \in D^*$. Let the $g_i$’s be distinct elements in $G^* (|G| > p^{k+1})$. We form the cosets $(cg_1 + G), \ldots, (cg_{p^{k+1}} + G)$. By the pigeonhole principle at least two cosets are equal. So $cg_i + G = cg_i + G \Rightarrow c(g_i - g_j) \in G \Rightarrow cg' = g \Rightarrow c = g/g' \in G^*G^{*-1}$.

(ii) Since $|D| = p^{2k}$ hence $|D^*| = p^{2k-1}$. By hypothesis $|D : G| = |D|/|G| \geq p^k$, so $|G| \leq p^k$ and therefore, $|G^*| = |G^*| \geq p^k - 1$, so we have, $|G^*G^{*-1}| \leq (p^k - 1)^2 = p^{2k} - 2p^{k+1} + 1 < p^{2k} - 1 = |D^*|$ so $G^*G^{*-1} \neq D^*$.

**Remark 2.3.** Theorem 2.2(ii) gives a bound for $|D|$ in part (i) which is optimum.
We now give the result which generalizes Proposition 1.1, Corollary 1.2, and Theorem 2.2(i).

**Lemma 2.4.** Let \( R \) be a ring and let \( S \) be a subset of \( R \) with invertible differences, that is, \( a - b \in R^* \) for any distinct elements \( a, b \in S \).

(i) Suppose \( G \subseteq R \) is a subgroup of index \( n \) in an additive group. If \( |S| > n^2 \) then \( G^*G^{*−1} = R^* \).

(ii) If \( |S| = \infty \) then \( G^*G^{*−1} = R^* \).

**Proof.** (i) Let \( r \in R^* \) be any element. By the pigeonhole principle there exist \( s, t \in S \) such that \( s - t = a \) and \( rs - rt \) both lie in \( G^* \). So \( r = ba^{-1} \in G^*G^{*−1} \), as claimed.

(ii) This part is an immediate consequence of part (i). \(\square\)

Apply the lemma with \( S = K \) for the proof of Proposition 1.1, with \( S = D \) for the proof of Proposition 1.1 and Theorem 2.2(i).

We now state the following definition which is a key concept in the paper. This is the analog of [1, Definition 0.1].

**Definition 2.5.** A ring \( R \) is a \( G^*G^{*−1} \)-ring, if \( G^*G^{*−1} = R^* \) for every subgroup \( G \subseteq R \) of finite index in an additive group.

If \( R \) is a ring which is a divisible group, then \( R \) has no additive subgroups of finite index (cf. [6]). Combining this statement, Lemma 2.4, and Definition 2.5 we obtain the following result.

**Proposition 2.6.** If \( D \) is an infinite division ring, then every ring \( R \) which contains a copy of \( D \) is a \( G^*G^{*−1} \)-ring. In particular \( D[x], D[[x]], M_n(D), M_n(D[x]), \) and \( M_n(D[[x]]) \) are \( G^*G^{*−1} \)-rings.

**Proof.** If \( \text{char}(D) \) is zero then every ring that contains a copy of \( D \) is a divisible group and hence \( R^* = G^*G^{*−1} \). If \( \text{char}(D) \neq 0 \), Lemma 2.4 implies that \( R^* = G^*G^{*−1} \). \(\square\)

**Remark 2.7.** The converse of Definition 2.5 does not necessarily hold. Let \( R = Q[x], G = Q \), then \( G^*G^{*−1} = R^* \). But \( G \) is not of finite index in an additive group.

3. Properties of \( G^*G^{*−1} \)-ring. In this section, some properties of the \( G^*G^{*−1} \)-ring is verified.

**Proposition 3.1.** Let \( R \) be a commutative ring and \( I \) an ideal of \( R \) such that \( R/I \) is a \( G^*G^{*−1} \)-ring. If \( I \) does not contain any additive subgroup of finite index and every element of \( 1 + I \) is invertible, then \( R + I \) is a \( G^*G^{*−1} \)-ring.

**Proof.** Let \( G \subseteq R \) be a subgroup of finite index in an additive group. Since \( (G + I)/G \cong I/(G \cap I) \) so \( |I/(G \cap I)| < \infty \) and hence \( I \cap G = I \). Choose \( a \in R^* \), then \( x + I = (g_1 + I)(g_2 + I)^{-1} \). It is easily seen that \( g_1, g_2 \in G^* \). So \( x = g_1^a g_2^{-1} + a \) for some \( a \in I \). But \( x = (g_1 + ag_2)g_2^{-1} \) where \( g_1 + ag_2 \in G^* \), that is, \( R^* = G^*G^{*−1} \). \(\square\)

Let \( R \) be a commutative ring and \( R[x] \) the polynomial ring over \( R \). The element \( f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x] \) is invertible if and only if \( a_0 \in R^* \) and each \( a_i \) \((i > 0)\) is nilpotent. So by Proposition 3.1 we have the following result.
Theorem 3.2. Let $R$ be a commutative ring, $R[x]$ the polynomial ring and $I = \{g(x) = a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i (i \geq 1) \text{ is nilpotent}\}$. If $I$ does not contain any additive subgroup of finite index and $R$ is a $G^*G^{-1}$-ring, then $R[x]$ also has that property.

Proof. We have $(R[x]/I)^* = ((R + I)/I)^* = \{r + I \mid r \in R^*\}$. Suppose $G/I \subseteq R[x]/I$ is a subgroup of finite index in an additive group, then $G$ is an additive subgroup of finite index in $R[x]$ and $|R/(G \cap R)| = |(R + G)/G| < \infty$. By hypothesis for $r \in R^*$ there exist $g_1, g_2 \in G^* \cap R^*$ such that $r = g_1g_2^{-1}$. So $r + I = g_1g_2^{-1} + I = (g_1 + I)(g_2^{-1} + I) \in (G/I)^*(G/I)^{-1}$, that is, $(R[x]/I)^* = (G/I)^*(G/I)^{-1}$. Now Proposition 3.1 completes the proof.

As an immediate consequence we obtain the following result.

Corollary 3.3. Let $R$ be a commutative ring without any nonzero nilpotent elements. If $R$ is a $G^*G^{-1}$-ring then so is $R[x]$.

The converse of Theorem 3.2 holds in general, see the following result.

Theorem 3.4. Let $R$ be a commutative ring. If $R[x]$ is a $G^*G^{-1}$-ring then $R$ also has that property.

Proof. Let $G \subseteq R$ be a subgroup of finite index in an additive group. Put $H = \{a_0 + r_1x + \cdots + r_kx^k \mid k \text{ is a nonnegative integer, } a_0 \in G, r_i \in R, i > 0\}$. It is easily seen that, $H \subseteq R[x]$ is a subgroup of finite index in an additive group and $H^* \cap R^* = G^*$. Since $(R[x])^* = H^*H^{-1}$ therefore $R^* = (R[x])^* \cap R^* = (H^*H^{-1}) \cap R^* = G^*G^{-1}$. Thus the proof is complete.

Here, we give a necessary condition for infinite $R; G^*G^{-1} = R^*$, this condition is not sufficient. We also verify the behavior of $G^*G^{-1}$-ring under homomorphisms.

Theorem 3.5. If $R$ is a $G^*G^{-1}$-ring. Assume that $S$ is a nontrivial homomorphic image of $R$ with homomorphism $\varphi : R \to S$ then

(i) $S$ is infinite.

(ii) Assume $\varphi^{-1}\{1_S\} = \{1_R\}$. If $R^*$ is a $G^*G^{-1}$-ring, then $S^*$ also a $G^*G^{-1}$-ring.

Proof. (i) Suppose $S$ is a finite ring. Let $G = \ker \varphi \cdot G \subseteq R$ is a subgroup of finite index in an additive group, because $R/G \cong S$. Then $G^*G^{-1} = R^*$ therefore $1_R \in G^*G^{-1}$ and $1_S = \varphi(1_R) \in \varphi(G^*G^{-1}) = \varphi(G^*)\varphi(G^{-1}) \subseteq \varphi(G)\varphi(G^{-1}) = 0$ therefore $S = 0$ which is a contradiction, and the proof is complete.

(ii) Let $G \subseteq S$ be a subgroup of finite index in an additive group. Put $H = \varphi^{-1}(G)$, then $H$ is a subgroup of $R$. Now define the following homomorphism

$$\alpha : R \longrightarrow \frac{S}{G}, \quad \alpha(x) = \varphi(x) + G,$$

so $\alpha$ is also surjective and by the first isomorphism theorem $R/H \cong S/G$. Since $S/G$ is finite then so is $R/H$ and thus $H$ is of finite index. Then by hypothesis $H^*H^{-1} = R^*$ now we have $\varphi(H^*)\varphi(H^{-1}) = \varphi(H^*H^{-1}) = \varphi(R^*) = S^*$ so $G^*G^{-1} = S^*$, and this implies that $S^*$ is a $G^*G^{-1}$-ring.

Theorem 3.5 implies that if $R$ is a finite ring, then $R^*$ is not a $G^*G^{-1}$-ring.
We now verify the behavior of $G^*G^{-1}$-rings under products.

**Theorem 3.6.** Suppose $R = R_1 \times R_2$, if $R_1^*$ and $R_2^*$ is $G^*G^{-1}$-ring then so is $R^*$.

**Proof.** Suppose $G \subseteq R = R_1 \times R_2$ is a subgroup of finite index of $R$. Put $A_1 = \{a \in R_1 : (a,0) \in G\}$. Now define

$$\alpha : R \twoheadrightarrow \frac{R_1 \times R_2}{G}, \quad \alpha(a) = (a,0) + G,$$

(3.2)

so $A_1 = \ker \alpha$. It implies that $A_1 \subseteq R_1$ is a subgroup of finite index in an additive group. Therefore $A_1^*A_1^{-1} = R_1^*$. Similarly, we define $A_2$ in $R_2$, so $A_2^*A_2^{-1} = R_2^*$. Now we have $A_1 \times A_2 = \{(a,b) \mid (a,0),(0,b) \in G\} = \{(a,0) + (0,b) \mid (a,0),(0,b) \in G\} \subseteq G + G \subseteq G$ and also $(A_1 \times A_2^*)(A_1^{-1} \times A_2^*)^{-1} = (A_1^* \times A_2^*)(A_1^{-1} \times A_2^{-1}) = A_1^*A_1^{-1} \times A_2^*A_2^{-1} = R_1^* \times R_2^* = R^*$. Since $A_1 \times A_2 \subseteq G$ then $G^*G^{-1} = R^*$ and thus $R^*$ is a $G^*G^{-1}$-group. \qed

**Theorem 3.7.** Let $R$ be a ring, $I$ its ideal and every element of $1 + I$ is invertible. If $R$ is $G^*G^{-1}$-ring then $R/I$ is also $G^*G^{-1}$-ring.

**Proof.** Let $G/I \subseteq R/I$ be a subgroup of finite index in an additive group, then $G \subseteq R$ is a subgroup of finite index in an additive group. Choose $r + I \in (R/I)^*$ where $r \in R^*$ and $r = g_1g_2^{-1}$ where $g_i \in G^*$, $i = 1,2$. Therefore, $r + I = (g_1 + I)(g_2 + I)^{-1} \in (G/I)^*(G/I)^{-1}$. \qed

Theorems 3.5, 3.6, the properties of isomorphism, Proposition 2.6, Artin-Wedderburn theorem (cf. [8]), and Theorem 3.7 imply the following result.

**Corollary 3.8.** (i) If $R \cong R_1 \times R_2$ then $R_1^*$ and $R_2^*$ are $G^*G^{-1}$-rings if and only if $R^*$ is a $G^*G^{-1}$-ring.

(ii) Every semisimple ring which has no finite homomorphic image is a $G^*G^{-1}$-ring.

(iii) Let $R$ be a $G^*G^{-1}$-ring and $J$ the Jacobson radical of $R$. Then $S$ is a $G^*G^{-1}$-ring.

**Remark 3.9.** If $S$ is a $G^*G^{-1}$-ring and $R$ is a subring of $S$ then $R$ is not necessarily a $G^*G^{-1}$-ring. So if $\varphi : R \to S$ is a monomorphism and $S$ is a $G^*G^{-1}$-ring then $R$ is not necessarily a $G^*G^{-1}$-ring.

We end this section by verifying whether $D^* = G^*G^{-1}D$ is an infinite division ring and $G = F + [D,D]$ where $F$ denotes the center of $D$ and $[D,D]$ denotes the additive commutator subgroup of $D$, (cf. [5]). As an example see the following example.

**Example 3.10.** Suppose that $D = Q(i,j,k)$ is the rational quaternion, by a simple investigation one can see that $[D,D] = ai + bj + ck$ for $a,b,c \in Q$, therefore $G = F + [D,D] = D$ and so $G^*G^{-1} = D^*$.

This also holds for real quaternions. But in general we have the following result.

**Lemma 3.11.** Let $D$ be a finite-dimensional division (or, more generally, central simple) algebra with center $F$. Then $[D,D]$ coincides with the set of elements of $D$ of trace 0.
Proof. Let \(d_1, d_2, \ldots, d_n\) be a basis of \(D\) of \(F\) vector space; here \(n = \deg(D)\). Let \(T_0\) be the \(n^2 - 1\)-dimensional of \(F\)-subspace of \(D\) consisting of trace-zero elements. Clearly \([D, D] \subseteq T_0\). Thus it is enough to show that \(\dim_F[D, D] \geq n^2 - 1\). Let \(K\) be a splitting field of \(D\). Then \(D \otimes_F K = M_n(K)\). It is easy to see that \([M_n(K), M_n(K)]\) is precisely the set of \(n \times n\)-matrices of trace zero. On the other hand, this set is spanned by elements \([d_i, d_j]\), as \(i, j = 1, 2, \ldots, n^2\). Thus \(n^2 - 1\) of these elements are linearly independent over \(K\) and, hence, over \(F\). This proves the inequality \(\dim_F[D, D] \geq n^2 - 1\), as desired.

We therefore conclude that \(\dim_F[D, D] = n^2 - 1\), while \(\dim_F(D) = n^2\). Thus
\[
F + [D, D] = \begin{cases} D, & \text{if } \text{char}(F) \text{ does not divide } n, \\ [D, D], & \text{if } \text{char}(F) \text{ divides } n. \end{cases} \tag{3.3}
\]

If \(D\) is noncommutative (i.e., of degree \(n \geq 2\)) then the following lemma shows that \(D^* = (D, D)^* - 1\) in any characteristic.

**Lemma 3.12.** Let \(D\) be a finite-dimensional division algebra of degree \(n \geq 2\) with center \(F\) and let \(G\) be a \(d\)-dimensional \(F\)-vector subspace of \(D\).

(a) Assume \(2d > n^2\). Then \(D^* = G^*G^{* - 1}\).

(b) Assume \(G = [D, D]\). Then \(D^* = G^*G^{* - 1}\).

**Proof.** (a) Let \(a \in D^*\). Since \(2d > n^2\), the \(d\)-dimensional \(F\)-vector spaces \(G\) and \(aG\) have a nontrivial intersection in \(D\), that is, \(g_1 = ag_2\) for some \(g_1, g_2 \in G^*\). Then \(a = g_1g_2^{-1}\), as desired.

(b) By Lemma 3.12, \(d = \dim_F[D, D] = n^2 - 1\). Since \(D\) is noncommutative, \(n \geq 2\) and thus \(2d = 2n^2 - 2 > n^2\). Now apply part (a).

**Question 3.13.** (1) If \(R\) is not a finite homomorphic image, must \(R^*\) be infinite? Must \(R\) contain an infinite subset with invertible differences?

(2) Is there a ring with no finite homomorphic image, but with some finite index subgroup \(G\) avoiding all units: \(G^* = \emptyset\)?

(3) If \(R\) is a \(G^*G^{* - 1}\)-ring then must \(R^*\) be infinite? If \(R\) is a \(G^*G^{* - 1}\)-ring must \(R\) contain an infinite sets with invertible differences?

(4) If \(R\) is a \(G^*G^{* - 1}\)-ring, then must the matrix ring \(M_n(R)\) also have that property? Conversely, if \(M_n(R)\) is a \(G^*G^{* - 1}\)-ring, then must \(R\) be a \(G^*G^{* - 1}\)-ring?

(5) If \(R\) is a \(G^*G^{* - 1}\)-ring and \(R\) is a subring of a ring \(S\) then must \(S\) also be a \(G^*G^{* - 1}\)-ring?

(6) If \(R/I\) is a \(G^*G^{* - 1}\)-ring and \(1 + I\) is invertible elements then must \(R\) also have that property?

(7) Let \(D\) be an infinite (algebraic) division algebra over its center \(F\). If \(G = F + [D, D]\). Is \(D^* = G^*G^{* - 1}\)?

**Acknowledgement.** The authors would like to thank the referees for their helpful comments.

**References**

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