A SUFFICIENT CONDITION FOR STARLIKENESS OF ORDER $\alpha$

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(Received 28 January 2001 and in revised form 12 June 2001)

Abstract. We obtain a sufficient condition for starlikeness of order $\alpha$, $|f'(z) - \lambda (f(z)/z) + \lambda - 1| < M = M_n(\lambda, \alpha)$, where $\lambda \in [0, 1], \alpha \in (0, 1)$ and the function $f(z) = z + a_{n+1}z^{n+1} + \cdots$ is analytic in the unit disc $U$.

2000 Mathematics Subject Classification. 30C45.

1. Introduction and preliminaries. Denote by $U$ the unit disc of the complex plane

$$U = \{ z \in \mathbb{C} : |z| < 1 \}. \quad (1.1)$$

Let $\mathfrak{H}(U)$ be the space of holomorphic functions in $U$, and let

$$A_n = \{ f \in \mathfrak{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \cdots, \ z \in U \} \quad (1.2)$$

with $A_1 = A$.

Let $\mathfrak{H}(a, n)$ denote the class of analytic functions in the unit disc of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \ \ z \in U. \quad (1.3)$$

Let

$$S^*(\alpha) = \{ f \in A, \ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ z \in U \}, \quad 0 \leq \alpha < 1, \quad (1.4)$$

be the class of starlike functions of order $\alpha$ in $U$.

If $f$ and $g$ are analytic in $U$, then we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function $w$ analytic in $U$, with $w(0) = 0, |w(z)| < 1$, for any $z \in U$, such that $f(z) = g(w(z))$, for $z \in U$.

If $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We use the following subordination result due to Hallenbeck and Ruscheweyh [1, page 71].

**Lemma 1.1.** Let $h$ be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\text{Re} \gamma \geq 0$. If $p \in \mathfrak{H}(a, n)$ and

$$p(z) + \frac{1}{\gamma} zp'(z) < h(z), \quad (1.5)$$

then

$$p(z) < q(z), \quad (1.6)$$
where
\[ q(z) = \frac{Y}{nz^{1/n}} \int_0^z h(t)t^{y/n-1} \, dt, \quad q < h. \] (1.7)

2. Main results

**Theorem 2.1.** Let \( \lambda \in [0, 1] \), \( \alpha \in (0, 1) \), and
\[
M = M_n(\lambda, \alpha) = \frac{(1-\alpha)(n+1-\lambda)}{|\lambda - \alpha| + \sqrt{(1-\lambda)^2 + (n+1-\lambda)^2}}.
\] (2.1)

If \( f \in A_n \) satisfies the inequality
\[
\left| f''(z) - \frac{\lambda f(z)}{z} + \lambda - 1 \right| < M_n(\lambda, \alpha),
\] (2.2)
with \( M_n(\lambda, \alpha) \) given by (2.1), then \( f \in S^* (\alpha) \).

**Proof.** In the case \( \lambda = 1 \), the proof is given in [3]. We suppose that \( \lambda \in [0, 1) \). If we consider \( P(z) = f(z)/z \), then
\[
f(z) = zP(z), \quad f'(z) = P(z) + zP'(z),
\] (2.3)
and (2.2) can be written in the following form:
\[
\left| P(z) + \frac{zP'(z)}{1-\lambda} - 1 \right| < \frac{M}{1-\lambda}
\] (2.4)
which is equivalent to the differential subordination
\[
P(z) + \frac{zP'(z)}{1-\lambda} < 1 + \frac{M}{1-\lambda} z \equiv h(z),
\] (2.5)
and by using Lemma 1.1, we obtain
\[
P(z) < q(z) = \frac{Y}{nz^{1/n}} \int_0^z h(t)t^{y/n-1} \, dt = 1 + \frac{M}{1-\lambda + n} z.
\] (2.6)

Subordination (2.6) is equivalent to
\[
|P(z) - 1| < \frac{M}{1-\lambda + n} \equiv R.
\] (2.7)

After a simple computation, from (2.7) it follows that
\[
R < \frac{1-\alpha}{|\lambda - \alpha|}.
\] (2.8)

If we put
\[
\frac{zf'(z)}{f(z)} = (1-\alpha)p(z) + \alpha,
\] (2.9)
then
\[
f'(z) = P(z)\left[(1-\alpha)p(z) + \alpha\right]
\] (2.10)
and (2.2) can be written as
\[ |P(z)[(1 - \alpha)p(z) + \alpha - \lambda] + \lambda - 1| < M = (1 - \lambda + n)R. \tag{2.11} \]

We have to show that (2.11) implies \( \Re p(z) > 0 \) in \( U \). Suppose that this is false. Since \( p(0) = 1 \), there exist \( z_0 \in U \) and a real \( \rho \), such that \( p(z_0) = i\rho \).

Therefore, in order to show that (2.11) implies \( \Re p(z) > 0 \) in \( U \), it is sufficient to obtain the contradiction from the inequality
\[ |P(z_0)[(1 - \alpha)p(z_0) + \alpha - \lambda] + \lambda - 1| \geq (1 - \lambda + n)R. \tag{2.12} \]

If we let \( P(z_0) = P = u + iv \), then
\[ E = |P[(1 - \alpha)i\rho + \alpha - \lambda] + \lambda - 1|^2 \]
\[ = |P|^2[(1 - \alpha)^2\rho^2 + (\alpha - \lambda)^2] - 2(1 - \lambda)\Re |P(1 - \alpha)i\rho + \alpha - \lambda| + (1 - \lambda)^2 \tag{2.13} \]
\[ = (u^2 + v^2)(1 - \alpha)^2\rho^2 + 2(1 - \lambda)(1 - \alpha)v\rho + |P(\alpha - \lambda) - (1 - \lambda)|^2. \]

By using (2.7) and the well-known triangle inequality, one obtains
\[ |P(\alpha - \lambda) - (1 - \lambda)| = |P(\alpha - \lambda) + \alpha - \lambda - \alpha + \lambda - 1 + \lambda| \]
\[ = |(\alpha - \lambda)(P - 1) - (1 - \alpha)| \tag{2.14} \]
\[ \geq 1 - \alpha - |\lambda - \alpha|R \]
and we deduce
\[ E \geq (u^2 + v^2)(1 - \alpha)^2\rho^2 + 2(1 - \lambda)(1 - \alpha)v\rho + [(1 - \alpha) - (\lambda - \alpha)R]^2. \tag{2.15} \]

If we let
\[ F(\rho) = E - M^2 \]
\[ \geq (u^2 + v^2)(1 - \alpha)^2\rho^2 + 2(1 - \lambda)(1 - \alpha)v\rho \tag{2.16} \]
\[ + [(1 - \alpha) - |\lambda - \alpha|R] - (1 - \lambda + n)^2R^2, \]
then (2.12) holds if \( F(\rho) \geq 0 \), for any real number \( \rho \).

Because \((u^2 + v^2)(1 - \alpha)^2 > 0\), the inequality \( F(\rho) \geq 0 \) holds if the discriminant \( \Delta \) is negative, that is,
\[ \Delta = (1 - \alpha)^2\{(1 - \lambda)^2v^2 - (u^2 + v^2)[(1 - \alpha) - |\lambda - \alpha|R]^2 - (1 - \lambda + n^2R^2] \leq 0. \tag{2.17} \]

The last inequality is equivalent to
\[ v^2[(1 - \lambda)^2 - (1 - \alpha - |\lambda - \alpha|R)^2 + (1 - \lambda + n)^2R^2] \leq u^2[(1 - \alpha - |\lambda - \alpha|R)^2 - (1 - \lambda + n^2R^2]. \tag{2.18} \]

After an easy computation, by using (2.7) we obtain the inequality
\[ \frac{v^2}{u^2} \leq \frac{R^2}{1 - R^2} \leq \frac{(1 - \alpha - |\lambda - \alpha|R)^2 - (1 - \lambda + n)^2R^2}{(1 - \lambda)^2 - (1 - \alpha - |\lambda - \alpha|R)^2 + (1 - \lambda + n)^2R^2}. \tag{2.19} \]

which is equivalent to \( \Delta \leq 0 \). Therefore \( F(\rho) \leq 0 \), a contradiction of (2.11). It follows
that $\text{Re} p(z) > 0$, and
\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) = \text{Re}(1 - \alpha)p(z) + \alpha = (1 - \alpha)\text{Re} p(z) + \alpha \geq \alpha
\] (2.20)

hence $f \in S^*(\alpha)$.

If $\lambda = 0$ then
\[
M_n(0, \alpha) = \frac{(1 - \alpha)(n + 1)}{\alpha + \sqrt{(n + 1)^2 + 1}}
\] (2.21)

and we obtain the following corollary.

**Corollary 2.2.** If $f \in A_n$ and
\[
\left| f'(z) - 1 \right| < \frac{(1 - \alpha)(n + 1)}{\alpha + \sqrt{(n + 1)^2 + 1}}
\] (2.22)

then $f \in S^*(\alpha)$.

For $\alpha = 0$ this result was obtained in [2].

If $\lambda = 1$,
\[
M_n(1, \alpha) = \frac{n(1 - \alpha)}{n + 1 - \alpha}
\] (2.23)

and we obtain the following corollary.

**Corollary 2.3** (see [3]). If $f \in A_n$ and
\[
\left| f'(z) - \frac{f(z)}{z} \right| < \frac{n(1 - \alpha)}{n + 1 - \alpha}
\] (2.24)

then $f \in S^*(\alpha)$.

If $\lambda = \alpha$,
\[
M_n(\alpha, \alpha) = \frac{(1 - \alpha)(n + 1 - \alpha)}{\sqrt{(1 - \alpha)^2 + (1 - \alpha + n)^2}}
\] (2.25)

**Corollary 2.4.** If $f \in A_n$ and
\[
\left| f'(z) - \alpha \frac{f(z)}{z} + \alpha - 1 \right| < \frac{(1 - \alpha)(n + 1 - \alpha)}{\sqrt{(1 - \alpha)^2 + (1 - \alpha + n)^2}}
\] (2.26)

then $f \in S^*(\alpha)$.

**References**


Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

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