CERTAIN CONVEX HARMONIC FUNCTIONS

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We define and investigate a family of complex-valued harmonic convex univalent functions related to uniformly convex analytic functions. We obtain coefficient bounds, extreme points, distortion theorems, convolution and convex combinations for this family.

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1. Introduction. A continuous complex-valued function \( f = u + iv \) defined in a simply connected complex domain \( \mathbb{D} \subset \mathbb{C} \) is said to be harmonic in \( \mathbb{D} \) if both \( u \) and \( v \) are real harmonic in \( \mathbb{D} \). Consider the functions \( U \) and \( V \) analytic in \( \mathbb{D} \) so that \( u = \Re U \) and \( v = \Im V \). Then the harmonic function \( f \) can be expressed by

\[
 f(z) = h(z) + \overline{g(z)}, \quad z \in \mathbb{D},
\]

where \( h = (U + V)/2 \) and \( g = (U - V)/2 \). We call \( h \) the analytic part and \( g \) the coanalytic part of \( f \). If the coanalytic part of \( f \) is identically zero then \( f \) reduces to the analytic case.

The mapping \( z \mapsto f(z) \) is sense-preserving and locally one-to-one in \( \mathbb{D} \) if and only if the Jacobian of \( f \) is positive (see [1]), that is, if and only if

\[
 J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0, \quad z \in \mathbb{D}.
\]

Let \( \mathcal{H} \) denote the family of functions \( f = h + \overline{g} \) which are harmonic, sense-preserving, and univalent in the open unit disk \( \Delta = \{ z : |z| < 1 \} \) with \( h(0) = f(0) = f_z(0) = 0 \). Thus, we may write

\[
 h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.
\]

Also let \( \mathcal{K} \) denote the subclass of \( \mathcal{H} \) consisting of functions \( f = h + \overline{g} \) so that the functions \( h \) and \( g \) take the form

\[
 h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = -\sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1.
\]

Recently, Kanas and Wisniowska [5] (see also Kanas and Srivastava [4]), studied the class of \( k \)-uniformly convex analytic functions, denoted by \( k-\mathcal{UC} \), \( 0 \leq k < \infty \), so that \( h \in k-\mathcal{UC} \) if and only if

\[
 \Re \left\{ 1 + (z - \zeta) \frac{h''(z)}{h'(z)} \right\} \geq 0, \quad |\zeta| \leq k, \quad z \in \Delta.
\]
For real $\phi$ we may let $\zeta = -k z e^{i\phi}$. Then condition (1.5) can be written as

$$\Re \left\{ 1 + (1 + ke^{i\phi}) \frac{zh''(z)}{h''(z)} \right\} \geq 0. \quad (1.6)$$

Now considering the harmonic functions $f = h + \bar{g}$ of the form (1.3) we define the family $\mathcal{HE}(k,\alpha)$, $0 \leq \alpha < 1$, so that $f = h + \bar{g} \in \mathcal{HE}(k,\alpha)$ if and only if

$$\Re \left\{ 1 + (1 + ke^{i\phi}) \frac{z^{2}h''(z) + 2zg'(z) + z^{2}g''(z)}{zh'(z) - zg'(z)} \right\} \geq \alpha, \quad 0 \leq \alpha < 1. \quad (1.7)$$

Finally, we let $\mathcal{HE}(k,\alpha) \equiv \mathcal{HE}(k,\alpha) \cap \mathcal{H}$.

Notice that if $g \equiv 0$ and $\alpha = 0$ then the family $\mathcal{HE}(k,\alpha)$ defined by (1.7) reduces to the class $k^{-}\mathcal{U}$ of $k$-uniformly convex analytic functions defined by (1.5). If we, further, let $k = 1$ in (1.5), we obtain the class of uniformly convex analytic functions defined by Goodman [2]. A geometric characterization of the general family $\mathcal{HE}(k,\alpha)$ is an open question.

In Section 2, we introduce sufficient coefficient bounds for functions to be in $\mathcal{HE}(k,\alpha)$ and show that these bounds are also necessary for functions in $\mathcal{HE}(k,\alpha)$. In Section 3, the inclusion relation between the classes $k^{-}\mathcal{U}$ and $\mathcal{HE}(k,\alpha)$ is examined. Extreme points and distortion bounds for $\mathcal{HE}(k,\alpha)$ are given in Section 4. Finally, in Section 5, we show that the class $\mathcal{HE}(k,\alpha)$ is closed under convolution and convex combinations.

Here we state a result due to Jahangiri [3], which we will use throughout this paper.

**Theorem 1.1.** Let $f = h + \bar{g}$ with $h$ and $g$ of the form (1.3). If

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} |b_n| \leq 1, \quad 0 \leq \alpha < 1, \quad (1.8)$$

then $f$ is harmonic, sense-preserving, univalent in $\Delta$, and $f$ is convex harmonic of order $\alpha$ denoted by $\mathcal{H}(\alpha)$. Condition (1.8) is also necessary if $f \in \mathcal{H}(\alpha) \equiv \mathcal{H}(\alpha) \cap \mathcal{H}$.

**2. Coefficient bounds.** First we state and prove a sufficient coefficient bound for the class $\mathcal{HE}(k,\alpha)$.

**Theorem 2.1.** Let $f = h + \bar{g}$ be of the form (1.3). If $0 \leq k < \infty$, $0 \leq \alpha < 1$, and

$$\sum_{n=2}^{\infty} \frac{n(n+nk-k-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+nk+k+\alpha)}{1-\alpha} |b_n| \leq 1, \quad (2.1)$$

then $f$ is harmonic, sense-preserving, univalent in $\Delta$, and $f \in \mathcal{HE}(k,\alpha)$.

**Proof.** Since $n - \alpha \leq n + nk - k - \alpha$ and $n + \alpha \leq n + nk + k + \alpha$ for $0 \leq k < \infty$, it follows from Theorem 1.1 that $f \in \mathcal{H}(\alpha)$ and hence $f$ is sense-preserving and convex univalent in $\Delta$. Now, we only need to show that if (2.1) holds then

$$\Re \left\{ \frac{zh'(z) + (1 + ke^{i\phi})z^{2}h''(z) + (1 + 2ke^{i\phi})zg'(z) + (1 + ke^{i\phi})z^{2}g''(z)}{zh'(z) - zg'(z)} \right\} = \Re \frac{A(z)}{B(z)} \geq \alpha. \quad (2.2)$$
Using the fact that \( \Re (w) \geq \alpha \) if and only if \( |1 - \alpha + w| \geq |1 + \alpha - w| \) it suffices to show that

\[
|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \geq 0,
\]

(2.3)

where \( A(z) = zh'(z) + (1 + ke^{i\phi})z^2h''(z) + (1 + 2ke^{i\phi})zg'(z) + (1 + ke^{i\phi})z^2g''(z) \) and \( B(z) = zh'(z) - zg''(z) \). Substituting for \( A(z) \) and \( B(z) \) in (2.3), we obtain

\[
\begin{align*}
|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| &= (2 - \alpha)z + \sum_{n=2}^{\infty} n[n + 1 - \alpha + k(n - 1)e^{i\phi}]a_n z^n \\
&+ \sum_{n=1}^{\infty} n[n - 1 + \alpha + k(n + 1)e^{i\phi}]\bar{b}_n \bar{z}^n \\
- |\alpha z + \sum_{n=2}^{\infty} n[n - 1 - \alpha + k(n - 1)e^{i\phi}]a_n z^n \\
&+ \sum_{n=1}^{\infty} n[n + 1 + \alpha + k(n + 1)e^{i\phi}]\bar{b}_n \bar{z}^n |
\end{align*}
\]

\[
\geq (2 - \alpha)|z| - \sum_{n=2}^{\infty} n(kn + 1 + 1 - k - \alpha) |a_n| |z|^n
\]

(2.4)

\[
- \sum_{n=1}^{\infty} n[n(k + 1) + 1 + k + \alpha] |b_n| |z|^n
\]

\[
- \alpha|z| - \sum_{n=2}^{\infty} n[n(k + 1) + 1 - k - \alpha] |a_n| |z|^n
\]

\[
- \sum_{n=1}^{\infty} n[n(k + 1) + 1 + k + \alpha] |b_n| |z|^n
\]

\[
\geq 2(1 - \alpha)|z| \left\{ 1 - \sum_{n=2}^{\infty} \frac{n(k + 1) - k - \alpha}{1 - \alpha} |a_n| \\
- \sum_{n=1}^{\infty} \frac{n(k + 1) + k + \alpha}{1 - \alpha} |b_n| \right\} \geq 0, \quad \text{by (2.1)}. \]

The harmonic functions

\[
f(z) = z + \sum_{n=2}^{\infty} \frac{1 - \alpha}{n(nk + n - k - \alpha)} x_n z^n + \sum_{n=1}^{\infty} \frac{1 - \alpha}{n(nk + n + k + \alpha)} \bar{y}_n \bar{z}^n,
\]

(2.5)

where \( \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1 \), show that the coefficient bound given in Theorem 2.1 is sharp.

The functions of the form (2.5) are in \( \mathcal{H}(k, \alpha) \) because

\[
\sum_{n=2}^{\infty} \frac{n(n + nk - k - \alpha)}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n + nk + k + \alpha)}{1 - \alpha} |b_n| = \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1. \quad \text{(2.6)}
\]
Next we show that the bound (2.1) is also necessary for functions in \( \overline{\mathcal{H}}(k, \alpha) \).

**Theorem 2.2.** Let \( f = h + g \) with \( h \) and \( g \) of the form (1.4). Then \( f \in \overline{\mathcal{H}}(k, \alpha) \) if and only if

\[
\sum_{n=2}^{\infty} \frac{n(n + nk - k - \alpha)}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n + nk + k + \alpha)}{1 - \alpha} |b_n| \leq 1. \tag{2.7}
\]

**Proof.** In view of Theorem 2.1, we only need to show that \( f \notin \overline{\mathcal{H}}(k, \alpha) \) if condition (2.7) does not hold. We note that a necessary and sufficient condition for \( f = h + g \) given by (1.4) to be in \( \overline{\mathcal{H}}(k, \alpha) \) is that the coefficient condition (1.7) to be satisfied. Equivalently, we must have

\[
\Re \left[ (1 - \alpha)zh'(z) + (1 + ke^{i\phi})z^2h''(z) + (1 + \alpha + 2ke^{i\phi})z^2g'(z) + (1 + ke^{i\phi})z^2g''(z) \right] \geq 0.
\tag{2.8}
\]

Upon choosing the values of \( z \) on the positive real axis where \( 0 \leq z = r < 1 \), the above inequality reduces to

\[
1 - \alpha - \left[ \sum_{n=2}^{\infty} n(nk + n - k - \alpha) |a_n| + \sum_{n=1}^{\infty} n(nk + n + k + \alpha) |b_n| \right] r^{n-1} \geq 0. \tag{2.9}
\]

If condition (2.7) does not hold then the numerator in (2.9) is negative for \( r \) sufficiently close to 1. Thus there exists \( z_0 = r_0 \) in \((0,1)\) for which the quotient (2.9) is negative. This contradicts the required condition for \( f \in \overline{\mathcal{H}}(k, \alpha) \) and so the proof is complete. \( \square \)

3. **Inclusion relations.** As mentioned earlier in the proof of Theorem 2.1, the functions in \( \overline{\mathcal{H}}(k, \alpha) \) are convex harmonic in \( \Delta \). In the following example we show that this inclusion is proper.

**Example 3.1.** Consider the harmonic functions

\[
f_n(z) = z - \frac{1}{2}z^2 - \frac{1}{2n^2}z^n, \quad z \in \Delta, \ n = 2, 3, \ldots.
\tag{3.1}
\]

For \( a_n = 0 \) and \( b_n = -1/2n^2 \), we observe that

\[
\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| = \frac{1}{2} + n^2 \left( \frac{1}{2n^2} \right) = \frac{1}{2} + \frac{1}{2} = 1. \tag{3.2}
\]

Therefore, by Theorem 1.1, \( f_n \in \overline{\mathcal{H}}(0) \).

On the other hand,

\[
\frac{2k + 1 + \alpha}{1 - \alpha} - \frac{1}{2} + \frac{n(nk + n + k + \alpha)}{1 - \alpha} \left( \frac{1}{2n} \right) = \frac{2k + 1 + \alpha}{2(1 - \alpha)} + \frac{nk + n + k + \alpha}{2n(1 - \alpha)} > 1. \tag{3.3}
\]

Thus, by Theorem 2.2, \( f \notin \overline{\mathcal{H}}(k, \alpha) \).

More generally, we can prove the following theorem.

**Theorem 3.2.** Let \( 0 \leq k < \infty \), \( 0 \leq \alpha < 1 \), and \( 0 \leq \beta < 1 \). If \( k > \beta/(1 - \beta) \) then the proper inclusion relation \( \overline{\mathcal{H}}(k, \alpha) \subset \overline{\mathcal{H}}(\beta) \).
PROOF. Let \( f \in \overline{\mathcal{C} \in \mathcal{V}}(k, \alpha) \), then, by Theorem 2.2,
\[
\sum_{n=1}^{\infty} \frac{n(nk+n-k-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(nk+n+k+\alpha)}{1-\alpha} |b_n| \leq 1. \tag{3.4}
\]
Since \( (n-\beta)/(1-\beta) < (nk+n-k-\alpha)/(1-\alpha) \) and \( (n+\beta)/(1-\beta) < (nk+n+k+\alpha)/(1-\alpha) \), by Theorem 1.1, we conclude that \( f \in \overline{\mathcal{C} \in \mathcal{V}}(\beta) \).
To show that the inclusion is proper, consider the harmonic functions
\[
f_n(z) = z - \frac{1-\beta}{2(1+\beta)} z^n, \quad z \in \Delta, \quad n = 2, 3, \ldots \tag{3.5}
\]
By Theorem 1.1, \( f_n \in \overline{\mathcal{C} \in \mathcal{V}}(\beta) \), because
\[
\sum_{n=1}^{\infty} \frac{n(n-\beta)}{1-\beta} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\beta)}{1-\beta} |b_n| = \frac{1+\beta}{1-\beta} (1-\beta) \frac{n(n+\beta)}{2n(n+\beta)} = 1. \tag{3.6}
\]
On the contrary, by Theorem 2.2, \( f_n \notin \overline{\mathcal{C} \in \mathcal{V}}(k, \alpha) \), because
\[
\sum_{n=1}^{\infty} \frac{n(nk+n+k+\alpha)}{1-\alpha} |b_n| = \frac{1+\alpha+2k}{1-\alpha} \frac{1-\beta}{2(1+\beta)} + \frac{n(n+\alpha+(n+1)k)}{1-\alpha} \frac{1-\beta}{2n(n+\beta)}
\]
\[
= \frac{1-\beta}{2(1-\alpha)} \left( \frac{1+\alpha+2k}{1+\beta} + \frac{n+\alpha+(n+1)k}{n+\beta} \right)
\]
\[
> \frac{1-\beta}{2(1-\alpha)} \left( \frac{1+\alpha+2\beta/(1-\beta)}{1+\beta} + \frac{n+\alpha+(n+1)\beta/(1-\beta)}{n+\beta} \right)
\]
\[
= \frac{1}{2(1-\alpha)} \left( \frac{2 + (1-\beta)(n+1+2\beta)}{(1+\beta)(n+\beta)} \right) \geq 1. \tag{3.7}
\]

4. Extreme points and distortion bounds. Using definition (1.7), and according to the arguments given in [3], we obtain the following extreme points of the closed convex hulls of \( \overline{\mathcal{C} \in \mathcal{V}}(k, \alpha) \) denoted by \( \overline{\mathcal{C} \in \mathcal{V}}(k, \alpha) \).

**Theorem 4.1.** Let \( f \) be the form of (1.4). Then \( f \in \overline{\mathcal{C} \in \mathcal{V}}(k, \alpha) \) if and only if \( f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n) \) where \( h_1(z) = z, \ h_n(z) = z - ((1-\alpha)/n(n+nk-k-\alpha))z^n(n = 2, 3, \ldots), \ g_n(z) = z - ((1-\alpha)/n(n+nk+k+\alpha))z^n(n = 1, 2, 3, \ldots), \sum_{n=1}^{\infty} (X_n + Y_n) = 1, \ X_n \geq 0 \) and \( Y_n \geq 0 \). In particular, the extreme points of \( \overline{\mathcal{C} \in \mathcal{V}}(k, \alpha) \) are \{\( h_n \} \) and \{\( g_n \} \).

Similarly, follows the distortion bounds for functions in \( \overline{\mathcal{C} \in \mathcal{V}}(k, \alpha) \).

**Theorem 4.2.** If \( f \in \overline{\mathcal{C} \in \mathcal{V}}(k, \alpha) \) then
\[
|f(z)| \leq (1 + |b_1|) r + \frac{1}{2} \left( \frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha} |b_1| \right) r^2, \quad |z| = r < 1,
\]
\[
|f(z)| \geq (1 - |b_1|) r - \frac{1}{2} \left( \frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha} |b_1| \right) r^2, \quad |z| = r < 1. \tag{4.1}
\]
If we let $r \to 1$ in the left-hand inequality of Theorem 4.2 and collect the like terms, we obtain the following theorem.

**Theorem 4.3.** If $f \in \mathcal{H}(k, \alpha)$ then

$$w : |w| < \left(3 + 2k - \alpha\right)/\left(2 + k - \alpha\right) \subset f(\Delta).$$

5. **Convolutions and convex combinations.** For harmonic functions $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ and $F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n$, we define the convolution of $f$ and $F$ as

$$(f \ast F)(z) = f(z) \ast F(z) = z - \sum_{n=2}^{\infty} |a_n| |A_n| z^n - \sum_{n=1}^{\infty} |b_n| |B_n| z^n. \quad (5.1)$$

In the following theorem we examine the convolution properties of the class $\mathcal{H}(k, \alpha)$.

**Theorem 5.1.** For $0 \leq \alpha \leq \beta < 1$, let $f \in \mathcal{H}(k, \beta)$ and $F \in \mathcal{H}(k, \alpha)$ then

$$f \ast F \in \mathcal{H}(k, \beta) \subset \mathcal{H}(k, \alpha). \quad (5.2)$$

**Proof.** Express the convolution of $f$ and $F$ as that given by (5.1) and note that $|A_n| \leq 1$ and $|B_n| \leq 1$. Now the theorem follows upon the application of the required condition (2.7).

The convex combination properties of the class $\mathcal{H}(k, \alpha)$ is given in the following theorem.

**Theorem 5.2.** The class $\mathcal{H}(k, \alpha)$ is closed under convex combinations.

**Proof.** For $i = 1, 2, \ldots$, suppose that $f_i \in \mathcal{H}(k, \alpha)$ where $f_i$ is given by $f_i(z) = z - \sum_{n=2}^{\infty} |a_{i,n}| z^n - \sum_{n=1}^{\infty} |b_{i,n}| z^n$. For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combinations of $f_i$ may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i,n}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i,n}| \right) z^n. \quad (5.3)$$

Now, the theorem follows by (2.7) upon noting that $\sum_{i=1}^{\infty} t_i = 1$. \hfill \Box

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**References**


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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

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