EXTENSIONS OF THE HEISENBERG-WEYL INEQUALITY

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ABSTRACT. In this paper a number of generalizations of the classical Heisenberg-Weyl uncertainty inequality are given. We prove the n-dimensional Hirschman entropy inequality (Theorem 2.1) from the optimal form of the Hausdorff-Young theorem and deduce a higher dimensional uncertainty inequality (Theorem 2.2). From a general weighted form of the Hausdorff-Young theorem, a one-dimensional weighted entropy inequality is proved and some weighted forms of the Heisenberg-Weyl inequalities are given.

KEY WORDS AND PHRASES. Uncertainty Inequality, Fourier Transform, Variance, Entropy, Hausdorff-Young Inequality, Weighted Norm Inequalities.

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1. INTRODUCTION.

Let be the Fourier transform of defined by

\[ \hat{f}(x) = \int e^{-2\pi i xy}f(y)dy, \quad x \in \mathbb{R}. \]

If \( f \in L^2(\mathbb{R}) \) with \( L^2 \)-norm \( ||f||_2 = 1 \), then by Plancherel’s theorem \( ||\hat{f}||_2 = 1 \), so that \( |f(x)|^2 \) and \( |\hat{f}(y)|^2 \) are probability frequency functions. The variance of a probability frequency function \( g \) is defined by

\[ \text{Var}[g] = \int (x-m)^2 g(x)dx \quad \text{where} \quad m = \int xg(x)dx \]

is the mean. With these notations, the Heisenberg uncertainty principle of quantum mechanics can be stated in terms of the Fourier transform by the inequality

\[ \text{Var}[|f|^2] \text{Var}[|\hat{f}|^2] \geq (16\pi^2)^{-1}. \quad (1.1) \]

In the sequel, we assume without loss of generality that the mean \( m = 0 \). If \( g \) is a probability frequency function, then the entropy of \( g \) is defined by

\[ E[g] = \int_R g(x) \log g(x)dx. \]

With \( f \) as above, Hirschman [1] proved that

\[ E[|f|^2] + E[|\hat{f}|^2] \leq E_H \quad (1.2) \]

with \( E_H = 0 \), and suggested that (1.2) holds with \( E_H = \log 2 - 1 \). If \( E_H \) has that form, then by an inequality of Shannon and Weaver [2] it follows that (1.2) implies (1.1).

Using the Babenko-Beckner optimal form of the Hausdorff-Young inequality ([3])

\[ ||\hat{f}||_p \leq A(p)||f||_p, \quad 1 < p < 2, \quad A(p) = [p/p(p-1/p')]^{1/2}, \quad (1.3) \]
in Hirschman's proof of (1.2), then as Beckner [] noted, (1.2) holds with $E_{H} = \log 2 - 1$.

A modest extension of (1.1) is obtained as follows: Let $f$ on $\mathbb{R}$ be differentiable, such that $f(0) = 0$. Then H"older's and Hardy's inequality [4, Theorem 3.27] yield with $1 \leq p < 2$

$$\int_{0}^{\infty} |f(x)|^2 dx \leq \left( \int_{0}^{\infty} x |f(x)|^p dx \right)^{1/p} \left( \int_{0}^{\infty} |f'(x)|^p dx \right)^{1/p}.$$

Applying this estimate also to $f(-x)$, then

$$\|f\|_{L^{2}}^2 = \int_{0}^{\infty} |f(x)|^2 dx + \int_{0}^{\infty} |f(-x)|^2 dx$$

$$\leq p \left( \int_{0}^{\infty} x |f(x)|^p dx \right)^{1/p} \left( \int_{0}^{\infty} |f'(x)|^p dx \right)^{1/p'}$$

$$+ \left( \int_{0}^{\infty} x |f(-x)|^p dx \right)^{1/p} \left( \int_{0}^{\infty} |f'(-x)|^p dx \right)^{1/p'}$$

where the last inequality follows from H"older's inequality. Now by (1.3) and the fact that $\hat{f}'(y) = 2\pi i y \hat{f}(y)$ we obtain

**THEOREM 1.1.** If $f \in S (\mathbb{R})$ and $f(0) = 0$, then for $1 < p < 2$

$$\|f\|_{L^{2}}^2 \leq 2\pi p \ A(p) \ |x| \|f\|_{L^{p}} \ |y\| \|\hat{f}\|_{L^{p'}}.$$

Note that the constant in (1.4) is slightly better than that in [4, §1.4] but unlikely best possible.

The purpose of this paper is to give extensions of the Heisenberg-Weyl inequality (1.1). In the next section a new proof of the entropy inequality (1.2) for functions on $\mathbb{R}^n$ is given and an $n$-dimensional Heisenberg-Weyl inequality is deduced. The $n$-dimensional generalization of inequality (1.4) is also given in the next section. The two inequalities are quite different, even in the case $p = 2$, but depend strongly on the sharp Hausdorff-Young inequality. In the third section a weighted form of the Heisenberg-Weyl inequality in one dimension is obtained from a weighted form of the Hausdorff-Young inequality (§5[6][7][8]). Unlike the constant $A(p)$ in (1.3) the constant of the weighted Hausdorff-Young inequality (3.3) of (Theorem 3.1) is far from sharp. If the constant is not too large, then a weighted form of Hirschman's entropy inequality can also be given, from which another uncertainty inequality is deduced.

Throughout, $p' = p/(p-1)$, with $p' = \infty$ if $p = 1$, is the conjugate index of $p$, and similarly for other letters. $S(\mathbb{R}^n)$ is the Schwartz class of slowly increasing functions on $\mathbb{R}^n$. We say $g$ is in the weighted $L^r_w$-space with weight $w$, if $wg \in L^r$ and norm $\|g\|_{L^r_w} = \|wg\|_{L^r}$. If $x \in \mathbb{R}^n$, then $x = (x_1, x_2, \ldots, x_n)$ and $dx = dx_1 \ldots dx_n$ the $n$-dimensional Lebesgue measure. $f_i(x)$, $x \in \mathbb{R}^n$ denotes the partial derivative of $f$ with respect to the $i^{th}$ component and $f_{ij} = (f_i)_j$. The letter $C$ denotes a constant which may be different at different occurrences, but is independent of $f$.

2. THE HIRSCHMAN INEQUALITY.

The Fourier transform of $f$ on $\mathbb{R}^n$ is given by

$$\hat{f}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} f(y) dy, \quad x \in \mathbb{R}^n, \quad x \cdot y = x_1 y_1 + \ldots + x_n y_n;$$

and the entropy of a function on $\mathbb{R}^n$ is defined as before with $\mathbb{R}$ replaced by $\mathbb{R}^n$. We shall need the following well known result (c.f. [9; §13.32 ii]):
If $\mu = 1$, then
$$\lim_{p \to 0^+} \left( \int |f|^p \mu \right)^{1/p} = \exp \int \log |f| \mu.$$ (2.1)

Using this fact we obtain easily the $n$-dimensional form of Hirschman's inequality (1.2)

**Theorem 2.1.** If $f \in L^2(\mathbb{R}^n)$ such that $||f||_2 = ||\hat{f}||_2 = 1$, then
$$E[|f|^2] + E[|\hat{f}|^2] \leq n \log 2 - 1,$$ (2.2)

whenever the left side has meaning.

**Proof.** Let $f \in L^2(\mathbb{R}^n)$, then $f \in L^p(\mathbb{R}^n)$, $1 < p < 2$, and by the $n$-dimensional form of the sharp Hausdorff-Young inequality [3] (that is, (1.3) with $A(p)$ replaced by $[A(p)]_n$) we obtain with $p = 2 - r$, $r > 0$ and $p' = 2 - r'$, $r' > 0$

$$\left( \int_{\mathbb{R}^n} |\hat{f}(y)|^{2-r'} dy \right)^{-r'/r} \leq \left( \frac{(2-r')^{-1}}{2-r} \right)^n \left( \int_{\mathbb{R}^n} |f(x)|^{2-r} dx \right)^{1/r}.$$

Now let $d\hat{f} = |\hat{f}(y)|^{2-r} dy$ and $d\hat{f} = |f(x)|^{2-r} dx$, then $\int_{\mathbb{R}^n} d\hat{f} = \int_{\mathbb{R}^n} d\hat{f} = 1$, so that the inequality becomes

$$\left( \int_{\mathbb{R}^n} |\hat{f}(y)|^{2-r'} d\hat{f} \right)^{-r'/r} \left( \int_{\mathbb{R}^n} (1/|f(x)|^{2-r}) d\hat{f} \right)^{1/r} \leq \frac{(2-r')^{-1}(2-r)}{(2-r')}.$$

But as $r \to +$, $-r' \to +$, so that by (2.1)

$$\exp \left( \int_{\mathbb{R}^n} \log |\hat{f}(y)| d\hat{f} \right) / \exp \left( \int_{\mathbb{R}^n} \log |f(x)| d\hat{f} \right) = \exp \left( \int_{\mathbb{R}^n} |\hat{f}(y)|^{2-r} dy \right)^{2-r} \int_{\mathbb{R}^n} |f(x)|^{2-r} dx = 2^{n/2}e^{-n/2}.$$

Taking logarithms on both sides we get

$$\int_{\mathbb{R}^n} |\hat{f}(y)|^{2-r} dy \leq \int_{\mathbb{R}^n} |f(x)|^{2-r} dx \leq \frac{n}{2} \log 2 - 1$$

and this implies (2.2) in the case $f \in (L^1 \cap L^2)(\mathbb{R}^n)$.

If $f \in L^2$ the result is obtained as in [1] only now one takes for $\omega_T$, $\omega_R(x) = e^{-\pi |x|^2}$ and for $\omega_T$, $\omega_R(x) = e^{-n/2e-\pi |y|^2}/\pi$. We omit the details.

If $g \in L^2(\mathbb{R})$ is a probability frequency function, then the relation between entropy and variance is expressed by $E[|g|^2] = -1 - 1/2 \log (2\pi e |g|^2)(2; p. 55-56)$. The $n$-dimensional form of this inequality is given in the following lemma:

**Lemma 2.1.** ([2; p. 56-57]). Let $g \in L^2(\mathbb{R}^n)$ with $||g||_2 = 1$. If $B = (b_{ij})$ is the matrix with entries

$$b_{ij} = \int_{\mathbb{R}^n} x_i x_j |g(x)|^2 dx,$$

then

$$E[|g|^2] > \frac{n}{2} \log (2\pi |b_{ij}|^{1/n}) - n/2$$

where $|b_{ij}| = \det B$.

Using the lemma and Theorem 2.1, we easily establish an $n$-dimensional extension of the Heisenberg-Weyl inequality.

**Theorem 2.2.** Let $f \in L^2(\mathbb{R}^n)$ with $||f||_2 = ||\hat{f}||_2 = 1$ and

$$b_{ij} = \int_{\mathbb{R}^n} x_i x_j |f(x)|^2 dx, \quad \tilde{b}_{ij} = \int_{\mathbb{R}^n} x_i x_j |\hat{f}(y)|^2 dy,$$
1, j = 1, 2, ..., n; be the entries of the matrices $B$ and $\hat{B}$ respectively, then

$$(\det B)(\det \hat{B}) \geq (16 \pi^2)^{-n}.$$  

PROOF. By (2.2) and Lemma 2.1,

$$n\log 2 - 1 \geq E[|f|^2] + E[|\hat{f}|^2]$$

so that

$$\log 2 \geq -\frac{1}{2} \log(4\pi^2 |b_{ij}|^{1/n} |\hat{b}_{ij}|^{1/n}).$$

But then

$$4 \geq 1/(\det B)^{1/n}(\det \hat{B})^{1/n}\pi^2,$$

which implies the result.

Clearly, if $n = 1$ we obtain at once (1.1). If $n = 2$ then

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \int_{\mathbb{R}^2} x_1^2 |f|^2 \, dx, & \int_{\mathbb{R}^2} x_1 x_2 |f|^2 \, dx \\ \int_{\mathbb{R}^2} x_2^2 |f|^2 \, dx, & \int_{\mathbb{R}^2} x_2^2 |f|^2 \, dx \end{pmatrix},$$

with a similar expression for $\hat{B}$. Applying Theorem 2.2 we obtain

$$(\det B)(\det \hat{B}) = [(\int_{\mathbb{R}^2} x_1^2 |f|^2 \, dx)(\int_{\mathbb{R}^2} x_2^2 |f|^2 \, dx) - (\int_{\mathbb{R}^2} x_1 x_2 |f|^2 \, dx)^2]$$

$$\cdot [(\int_{\mathbb{R}^2} y_1^2 |\hat{f}|^2 \, dy)(\int_{\mathbb{R}^2} y_2^2 |\hat{f}|^2 \, dy) - (\int_{\mathbb{R}^2} y_1 y_2 |\hat{f}|^2 \, dy)^2] \geq (16 \pi^2)^{-2}.$$  

If we denote the bracketed terms above by $D[|f|^2]$ and $D[|\hat{f}|^2]$, the discrepancy of Schwarz's inequality, or the difference between variance and covariance of $|f|^2$ and $|\hat{f}|^2$, then the two dimensional Heisenberg-Weyl inequality shows that the discrepancies of $|\hat{f}|^2$ and $|\hat{f}|^2$ cannot both be small; $D[|f|^2] D[|\hat{f}|^2] \geq (16 \pi^2)^{-2}$.

A different generalization of (1.1) may be obtained along the lines of Theorem 1.1.

THEOREM 2.3. Let $f, g \in S(\mathbb{R}^n)$, such that $f(x_1, x_2, ..., x_n) = 0$, whenever $x_i = 0$ for some $i$. If $1 < p \leq 2$ and $A(p)$ is the constant of (1.3), then

$$\left\| \frac{f}{|x|^p} \right\|_2 \leq \frac{2^{p-1} A(p)}{\pi^{n/2}} \left\| x_1 \cdots x_n f \right\|_p \left\| y_1 \cdots y_n \hat{f} \right\|_p.$$  

PROOF. We only give the proof for $n = 2$ since the general case follows in exactly the same way. Let $f_{21}(x,y) = g(x,y)$, then

$$f(x,y) = \int_0^y \int_0^x g(s,t) \, ds \, dt$$

and by Hölder's and the two dimensional Hardy inequality, with $\mathbb{R}^2_+ = (0,\infty) \times (0,\infty)$,

$$\frac{1}{p} \int_{\mathbb{R}^2_+} |f(x,y)|^2 \, dx \, dy < \left\| \int_{\mathbb{R}^2_+} x y \, f(x,y) \, P \, dx \, dy \right\|_1 / p \left\| \int_{\mathbb{R}^2_+} x y \, f(x,y) \, P \, dx \, dy \right\| / p,$$

$$< p^2 \left\| \int_{\mathbb{R}^2_+} |x y f(x,y)| \, P \, dx \, dy \right\|_1 / p \left\| \int_{\mathbb{R}^2_+} |x y f(x,y)| \, P \, dx \, dy \right\| / p.$$

On applying this estimate four times we obtain with $d\nu = dx \, dy$

$$\left\| \frac{f}{|x|^p} \right\|_2^2 = \int_{\mathbb{R}^2_+} \left( |f(x,y)|^2 + |f(x,-y)|^2 + |f(-x,y)|^2 + |f(-x,-y)|^2 \right) d\nu.$$
\[
\begin{align*}
&\leq p^2 \left( \int_{\mathbb{R}^2} |xy f(x,y)| P du \right)^{1/p} \left( \int_{\mathbb{R}^2} |f_{21}(x,y)| P' du \right)^{1/p'} \\
&+ \left( \int_{\mathbb{R}^2} |xy f(x,-y)| P du \right)^{1/p} \left( \int_{\mathbb{R}^2} |f_{21}(x,-y)| P' du \right)^{1/p'} \\
&+ \left( \int_{\mathbb{R}^2} |xy f(-x,-y)| P du \right)^{1/p} \left( \int_{\mathbb{R}^2} |f_{21}(-x,-y)| P' du \right)^{1/p'} \\
&+ \left( \int_{\mathbb{R}^2} |xy f(-x,y)| P du \right)^{1/p} \left( \int_{\mathbb{R}^2} |f_{21}(-x,y)| P' du \right)^{1/p'} \\
&\leq p^2 \left( \int_{\mathbb{R}^2} \left| f(x,y) \right|^p + \left| f(x,-y) \right|^p + \left| f(-x,-y) \right|^p + \left| f(-x,y) \right|^p \right) du \right)^{1/p} \\
&\quad \times \left( \int_{\mathbb{R}^2} \left| f_{21}(x,y) \right|^p + \left| f_{21}(x,-y) \right|^p + \left| f_{21}(-x,-y) \right|^p + \left| f_{21}(-x,y) \right|^p \right) du \right)^{1/p'} \\
&= p^2 \left( \int_{\mathbb{R}^2} \left| xy f(x,y) \right|^p du \right)^{1/p} \left( \int_{\mathbb{R}^2} \left| f_{21}(x,y) \right| P' du \right)^{1/p'},
\end{align*}
\]

where the last inequality follows from Hölder's inequality. But by the sharp form of the Hausdorff-Young inequality with \( n = 2 \) we obtain \[ \| f \|^{2/p} \leq \| A(p) \|^{2/p} \| xy f \|_p \| \hat{f}_{21} \|_p', \]

Since \( (\hat{f}_{-1})(s,t) = 4\pi^2 st \hat{f}(s,t) \) the result follows.

3. WEIGHTED HIRSCHMAN ENTROPY INEQUALITY AND WEIGHTED HEISENBERG-WEYL INEQUALITY.

The results of the last section show that the Heisenberg-Weyl inequality is a consequence of the Hausdorff-Young theorem. Recently a number of weighted Hausdorff-Young inequalities have been obtained [5], [6], [7] and [8]. We shall use these results in this section to obtain a weighted Hirschman entropy inequality as well as weighted form of the Heisenberg-Weyl inequality. Here we consider weighted extensions in \( \mathbb{R}^1 \) only.

Recall that if \( g \) is a Lebesgue measurable function on \( \mathbb{R} \), then the equi-measurable decreasing rearrangement of \( g \) is defined by \( g^*(t) = \inf \{ y > 0 : \int_{|x| > y} |g(x)| \, dx \leq t \} \), where \( y > 0 \) and \( |E| \) denotes Lebesgue measure of the set \( E \). Clearly, if \( g \) is an even function on \( \mathbb{R} \), decreasing on \( (0, \infty) \), then for \( t > 0 \), \( g^*(t) = g(t/2) \). We shall use this fact below.

**Definition 3.1.** Let \( u \) and \( v \) be locally integrable functions of \( \mathbb{R} \). We write \((u,v) \in F_{p,q}^*\), \( 1 \leq p \leq q < \infty \), if

\[
\sup \left( \int_0^\infty [u^*(t)]^q \, dt \right)^{1/q} \left( \int_0^\infty [(1/v)^*(t)]^p \, dt \right)^{1/p'} < \infty,
\]

where in the case \( p = 1 \) the second integral is replaced by the essential supremum of \((1/v)^*(t)\) over \((0,1/s)\).

If \( u \) and \( 1/v \) are even and decreasing on \((0, \infty)\) then \((3.1)\) is equivalent to

\[
\sup \left( \int_0^{s/2} [u(x)]^q \, dx \right)^{1/q} \left( \int_0^{s/2} [(1/v)^*(x)]^p \, dx \right)^{1/p'} < \infty
\]

and in this case we write \((u,v) \in F_{p,q}^*\).

The weighted Hausdorff-Young inequality is given in the following theorem:

**Theorem 3.1.** ([5; Theorem 1.1]). Suppose \((u,v) \in F_{p,q}^*\), \( 1 \leq p \leq q < \infty \) and \( f \in L_p^\mathbb{R} \).

(i) If \( \lim n \to \infty \| f_n - f \|_{p,v} = 0 \) for a sequence of simple functions, then \( \{f_n\} \) converges in \( L_q^u \) to a function \( \hat{f} \in L_q^u \). \( \hat{f} \) is independent of the sequence \( \{f_n\} \) and is called
the Fourier transform of \( f \).

(ii) there is a constant \( B > 0 \) such that for all \( f \in L^p_v \)

\[
||\hat{f}||_{q,u} < B ||f||_{p,v}. \tag{3.3}
\]

(iii) If \( g \in L^q_{1/u}, q > 1 \), then Parseval's formula

\[
\int \hat{f}(y)g(y)dy = \int f(t)\hat{g}(t)dt
\]

holds.

We note ([5], [6], [8]) that Theorem 3.1 is sharp in the sense that if \( u \) and \( v \) are even and satisfy (3.3), then \((u, v)\) satisfies (3.2). The constant \( B \) in (3.3) is not sharp, however it is of the form \( B = k.C \) where \( k = k(p,q) \) is independent of \( u \) and \( v \) and \( C \) is the supremum of (3.1), and in the case \( u, 1/v \) decreasing and even the supremum (3.2).

A special case of Theorem 3.1 is the following:

**COROLLARY 3.1.** Suppose \( f \in L^1_{1/2/p}, (u^{1-2/p'}, v^{1-2/p'}) \in F^*_{p,p''}, 1 < p < 2 \), where \( u \) and \( v \) are even, decreasing as \( 0, \infty \) then

\[
(\int u(y)^{p'-2}f(y)^{p'} \leq k.C (\int v(x)^{p-2}f(x)^{p} dx)^{1/p}
\]

where \( C_p = \sup_{s>0} (\int u(x)^{p-2}dx)^{1/p} (\int v(x)^{p'}^{1/2s}dx)^{1/p'} \).

Utilizing the last result we now give a weighted form of Hirschman's entropy inequality.

**PROPOSITION 3.1.** Suppose \( f \in L^2 \cap L^1_{1/v}, \) where \( u \) and \( v \) satisfy the conditions of Corollary 3.1. If \( ||f||_2 = 1 \) and (3.4) holds with \( 0 < k < 2 \) and \( C_p \) remains bounded as \( p \to 2 \), then

\[
\int f(y)^2 \log |u(y)f(y)|^2 dy + \int f(x)^2 \log |v(x)f(x)|^2 dx
\]

\[
\leq 2 \log k + 8 \sup_{s>0} (\int u(x)v(y) \log |u(x)v(y)| dy).
\]

**PROOF.** Since \( f \in L^p_v 1-2/p, 1 < p < 2 \), we apply Corollary 3.1 with \( p = 2-r, r > 0, p' = 2-r', r' < 0 \) and \( du = \int f(x)^2 dx, dv(x) = |f(x)|^2 dx \). Then (3.4) has the form

\[
(\int u(y)^{p'}|\hat{f}(y)|^{-r'du})^{1/(2-r')} \leq k \sup_{s>0} (\int u(x)^{-r'}dx)^{1/(2s)} \sup_{s>0} (\int v(x)^{-r'}dx)^{1/(2s)}
\]

or, on raising the inequality at the power \((2-r')(-1/r')\), equivalently

\[
(\int u(y)^{p'}|\hat{f}(y)|^{-r'}\hat{f}(y)^{-1/r')} \leq k (\int v(x)^{-r'}f(x)^{1/r'})^{1/r}
\]

where

\[
M_r = \sup_{s>0} (\int u(x)v(y)^{-r'}4dx)\int v(x)v(y)^{-r'}4dx
\]

Given \( \varepsilon > 0 \) there is an \( s_o > 0 \) such that

\[
M_r \leq (\int u(x)v(y)^{-r'}rdx)^{-1/r'} + \varepsilon
\]

so that
where we used the fact that $k/2 < 1$. Now as $r \to +\infty$, $r' \to -\infty$, then on applying (2.1) to both sides of (3.5) we obtain

$$\exp(\int R \log |u(y)\hat{f}(y)|dy)/\exp(\int R \log |v(x)f(x)|dx) \leq k(\int s/2 \log v(y)dy + e).$$

But $e > 0$ is arbitrary so that on taking logarithms we have

$$\frac{1}{2} \int -\infty \hat{f}(y)^2 \log |u(y)\hat{f}(y)|^2 dy + \frac{1}{2} \int -\infty \hat{f}(x)^2 \log |v(x)f(x)|^2 dx$$

$$\leq \log k + 4 \sup_{s > 0} \int s/2 \log |u(x)|^2 dx \int s/2 \log |v(y)|^2 dy$$

which yields the result.

Note that if $u = v \equiv 1$ and if $k \leq \sqrt{2/\varepsilon}$ we obtain (2.2) with $n = 1$.

We can write the conclusion of Proposition 3.1 in the form

$$\mathbb{E}[|f|^2] + \mathbb{E}[|\hat{f}|^2] \leq 2 \log k + 8 \sup_{s > 0} \int \frac{s/2}{2} \log |u(x)v(y)|^2 dx$$

But since $(1/2) \mathbb{E}[|f|^2] \geq -\frac{1}{2} - \frac{1}{2} \log (2\pi V[|f|^2])$ and also with $f$ replaced by $\hat{f}$ we obtain another uncertainty inequality

$$\mathbb{V}[|f|^2] + \mathbb{V}[|\hat{f}|^2] \geq \frac{\kappa^4}{4 \pi^2 e^2} \exp(-16 \sup_{s > 0} \int \frac{s/2}{2} \log |u| v^2 dy)$$

$$\times \exp(2 \int \mathbb{E}[|\hat{f}|^2 \log |u|^2 dy \exp(2 \int |f|^2 \log |v|^2 dx).$$

If $u = v \equiv 1$ and $k = \sqrt{2/\varepsilon}$ in this estimate we obtain (1.1).

**Theorem 3.2.** (Heisenberg-Weyl inequality). If $(1/u, v) \in \mathbb{F}_{p,q}$, $1 \leq p \leq q < \infty$ and $f \in S(\mathbb{R})$, then

$$||f||^2 \leq C(\int_\mathbb{R} |u(x)xf(x)| |q'dx|^{1/q} (\int_\mathbb{R} |v(y)y\hat{f}(y)| |Pdy|^{1/p}.$$  

**Proof.** Integration by parts and Hölder's inequality show that for $1 \leq q < \infty$

$$||f||^2 \leq 2 \int |x||f(x)| |f'(x)| dx$$

$$\leq 2(\int |ux(x)f(x)| q'dx)^{1/q'} (\int |y\hat{f}'(x)/u(x)| qdx)^{1/q}$$

$$\leq 2C(\int |ux(x)f(x)| q'dx)^{1/q'} (\int |v(y)\hat{f}(y)| |Pdx|)^{1/p},$$

where the last inequality follows from (3.3). Since $\hat{f}'(y) = 2\pi iy \hat{f}(y)$ the result follows.

Note that the case $p = 1$ also holds, provided the second integral in the $\mathbb{F}_{p,q}^*$ condition is interpreted as the essential supremum of $(1/v)^*$ over $(0, 1/s)$. 


The same result holds also if we take \((1/u, v) \in F_{p,q}\).
Observe also that the case \(u = v = 1\) and \(q = p'\), \(1 < p \leq 2\) reduces to (1.4), but
with a different constant.

Weighted inequalities of the form (3.6) were also obtained by Cowling and Price [3] but by quite different methods.

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As a multidisciplinary field, financial engineering is becoming increasingly important in today’s economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems).

This special issue will include (but not be limited to) the following topics:

- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

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