A QUASITOPOS CONTAINING CONV AND MET AS FULL SUBCATEGORIES

E. LOWEN
University of Brussels, V.U.B.
Departement Wiskunde
Pleinlaan 2
1050 Brussels, BELGIUM

R. LOWEN
University of Antwerp, R.U.C.A.
Wiskundige Analyse
Groenenborgerlaan 171
2020 Antwerp, BELGIUM

(Received May 12, 1987 and in revised form November 12, 1987)

ABSTRACT. We show that convergence spaces with continuous maps and metric spaces with contractions, can be viewed as entities of the same kind. Both can be characterized by a "limit function" \( \lambda \) which with each filter \( \mathcal{F} \) associates a map \( \lambda \mathcal{F} \) from the underlying set to the extended positive real line. Continuous maps and contractions can both be characterized as limit function preserving maps.

The properties common to both the convergence and metric case serve as a basis for the definition of the category, CAP. We show that CAP is a quasitopos and that, apart from the categories CONV, of convergence spaces, and MET, of metric spaces, it also contains the category AP of approach spaces as nicely embedded subcategories.

KEY WORDS AND PHRASES. Limit, distance, convergence space, metric space, approach space, cartesian closed, hereditary.

CLASSIFICATION. 54B30, 54A20, 54E35.

1. INTRODUCTION.

In [17] the categories TOP of topological spaces and continuous maps and pq-MET of extended pseudo-quasi-metric spaces and non-expansive maps were embedded in a common supercategory. The idea behind this embedding being that topological spaces and metric spaces can be viewed as objects of the same type, in the sense that they both can be described by a "distance between points and sets". Starting with a pq-MET space \((X,d)\) this distance is the usual one given by \( \delta(x,A) := \inf_{a \in A} d(x,a) \). Starting with a topological space \((X,\mathcal{C})\) a distance can be defined by \( \delta(x,A) := 0 \) if \( x \in \overline{A} \) and \( \delta(x,A) := \infty \) if \( x \notin \overline{A} \). A notion of distance has been axiomatized in [17] in such a way as to generalize both the metric and topological cases and resulted in the definition of the category AP of approach spaces and contractions.

There are several advantages to this.

In the first place that of unification, e.g. the notions of compactness (in TOP) and of total boundedness (in pq-MET) which turn out to be special cases of a measure
of compactness in AP [18] and which in turn makes a concept introduced by C. Kuratowski in [16] a canonical categorical notion.

A similar situation presents itself for the notions of connectedness (in TOP) and Cantor's "kettenzusammenhang" (in MET) [2], [19].

In the second place there are several classes of important topological spaces, e.g. spaces of measures with the weak topology and spaces of random variables with the topology of convergence in measure which can more naturally be equipped with AP-structures such that the topological structures are their TOP-coreflections [17].

In order to study these concepts and spaces it however soon became clear that we would need a theory of convergence in AP. We develop such a concept of convergence by means of assigning "limit functions" to filters, and moreover we show that AP can be completely characterized by four axioms about limit functions; two fundamental axioms - one on limit functions of principal ultrafilters and another on limit functions of comparable filters - a third axiom of a pretopological nature on limit functions of intersections of filters and a fourth one on limit functions of (Kowalsky-) diagonal filters [15].

Using this convergence-description of AP we obtain a very elegant characterization of initial structures in AP. AP is a topological construct in the sense of [1], [10], [11]. However from a categorical point of view some desirable properties are missing. For topologists and analysts, cartesian closedness is one such property [7], [9], [20], [21].

The existence of nice function space objects is indeed an important advantage in homotopy, topological algebra, and infinite dimensional differential calculus. The topological construct becomes extremely nice to work in when apart from being cartesian closed it also is hereditary, i.e. a quasitopos [5], [22], [23]. AP is neither cartesian closed nor hereditary. The situation is similar to the classical ones. Neither TOP nor pq-MET is a quasitopos. By weakening the axioms "bigger" categories with nicer properties result. For example CONV is a quasitopos containing TOP, and pqs-MET is a quasitopos containing pq-MET. By dropping the diagonal axiom and weakening the pretopological axiom on limit functions we introduce the supercategory CAP of convergence approach spaces. CAP is a quasitopos and moreover it contains both quasitopoi CONV and pqs-MET as nicely embedded subcategories. From this embedding it then follows that convergence spaces and extended pseudo-quasi-semi metric spaces can be viewed as entities of the same type, both being characterized by means of limit functions of filters. Moreover also AP is nicely embedded in this supercategory.

2. PRELIMINARIES.

If \( X \) is a set then the set of all filters on \( X \) shall be denoted \( \mathcal{F}(X) \).

If \( \mathcal{F} \in \mathcal{F}(X) \), then the set of all ultrafilters finer than \( \mathcal{F} \) shall be denoted \( \mathcal{U}(\mathcal{F}) \).

If \( \mathcal{F} = \{ X \} \) then we write shortly \( \mathcal{U}(X) \).

For any collection \( A \) of subsets of \( X \) we denote \( \text{stack}_{X} A := \{ B \subseteq X \mid A \in A : A \subseteq B \} \). If \( A \) consists of a single element \( A \) we put shortly \( \text{stack}_{X} A \) and if moreover \( A \) consists of a single point \( a \) then we put \( \text{stack}_{X} a \). If no confusion can occur, we drop the subscript and simply write \( \text{stack} A \) a.s.o.

If \( (S_j)_{j \in J} \) is a family of sets, then elements of their product \( \prod S_j \) shall sometimes be denoted in a functional notation, e.g. \( S \) where for all \( j \in J : s(j) \in S_j \).
stands for \([0,\infty]\) and all suprema and infima are taken in \(\mathbb{R}_+\). The following result shall be useful.

**Proposition 2.1.** If \(\mathcal{F} \in \mathcal{F}(X)\) and \(U \subseteq \mathcal{U}\) then there exists a finite set \(U_0 \in \mathcal{U}(\mathcal{F})\) such that \(U_0 \in \mathcal{U}(\mathcal{F})\).

**Proof.** Suppose not, in that case the family

\[
\mathcal{F} \cup \{X \setminus o(U) \mid U \in \mathcal{U}\}
\]

has the finite intersection property and thus is contained in some \(U_0 \in \mathcal{U}(\mathcal{F})\). Then however \(X \setminus o(U_0) \in U_0\) which is a contradiction.}

An **extended pseudo-quasi-metric space** (shortly \(\approx\)-**pq-metric space**) is a pair \((X,d)\) where

\[
d : X \times X \to \mathbb{R}_+
\]

fulfills (i) \(\forall x \in X : d(x,x) = 0\) and (ii) \(\forall x,y,z \in X : d(x,y) \leq d(x,z) + d(z,y)\).

The map \(d\) is called an **extended pseudo-quasi-metric** (shortly \(\approx\)-**pq-metric**). If \(d\) is moreover symmetric, everywhere "quasi-" ("q-") is dropped.

Given two \(\approx\)-**pq-metric spaces** \((X,d)\) and \((X',d')\) a function \(f : X \to X'\) is called **non-expansive** if \(d' \circ (f \circ f) \leq d\).

The category with objects \(\approx\)-**pq-metric spaces** and morphisms non-expansive maps is denoted \(\text{pq-MET}^\approx\). See also [13], [14].

An **approach space** is a pair \((X,\delta)\) where

\[
\delta : X \times 2^X \to \mathbb{R}_+
\]

fulfills

\[
\begin{align*}
(D1) & \quad \forall A \in 2^X, X \subseteq A \colon \delta(x,A) = 0 \\
(D2) & \quad \forall x \in X : \delta(x,\emptyset) = \infty \\
(D3) & \quad \forall A,B \in 2^X, x \in X : \delta(x,A \cup B) = \delta(x,A) \wedge \delta(x,B) \\
(D4) & \quad \forall A \in 2^X, x \in X, \forall \varepsilon \in \mathbb{R}_+ : \\
& \quad \delta(x,A) \leq \delta(x,A(\varepsilon)) + \varepsilon
\end{align*}
\]

where \(A(\varepsilon) := \{y \in X \mid \delta(y,A) < \varepsilon\}\).

The map \(\delta\) is called a **distance**.

Given \(A \in 2^X\) we denote

\[
\delta_A : X \to \mathbb{R}_+ \\
: x \mapsto \delta(x,A).
\]

Given two approach spaces \((X,\delta)\) and \((X',\delta')\) a function \(f : X \to X'\) is called a **contraction** if for all \(x \in X\) and \(A \in 2^X\):

\[
\delta'(f(x),f(A)) \leq \delta(x,A)
\]

or equivalently, if for all \(A \in 2^X\):

\[
\delta'_{f(A)} \circ f \leq \delta.
\]
In this section we shall give alternative characterizations of both approach spaces and contractions.

Let $X \in \{\text{SEIT}\}$. We recall the Kowalsky diagonal operator $\mathcal{O}$ [15] defined as follows. For any index set $J$, any collection of filters $(\mathcal{F}_j)_{j \in J}$ on $X$, and filter $\mathcal{F}$ on $J$:

$$\mathcal{O}((\mathcal{F}_j)_{j \in J}, \mathcal{F}) := \bigvee_{F \in \mathcal{F}_j, j \in J} \mathcal{F}_j.$$  

In the case the collection of filters is a selection on $X$ in the sense that we have a map

$$\mathcal{S} : X \to \mathcal{F}(X)$$

$$x \mapsto \mathcal{S}(x)$$

then we put shortly $\mathcal{O}(\mathcal{S}, \mathcal{F})$ for $\mathcal{O}((\mathcal{S}(y))_{y \in X}, \mathcal{F})$.

In the sequel we require the following results. Easy proofs are left to the reader.

**PROPOSITION 3.1.** Let $(\mathcal{F}_j)_{j \in J}$ be a collection of filters on $X$ and $\mathcal{F}$ a filter on $J$, then the following properties hold:

1. $\mathcal{O}((\mathcal{F}_j)_{j \in J}, \mathcal{F}) = \bigvee_{F \in \mathcal{F}_j, j \in J} \mathcal{F}_j.$
2. If $(\mathcal{G}_j)_{j \in J}$ is a family of filters on $J$ and $\mathcal{F} = \bigcap_{j \in J} G_j$ then $\mathcal{O}((\mathcal{F}_j)_{j \in J}, \mathcal{F}) = \bigcap_{j \in J} \mathcal{O}((\mathcal{F}_j)_{j \in J}, \mathcal{G}_j).$
3. $\mathcal{O}((\mathcal{F}_j)_{j \in J}, \mathcal{F}) = \bigcap_{F \in \mathcal{F}_j, j \in J} (\mathcal{U}_j)_{j \in J}.$
4. If each $\mathcal{F}_j, j \in J$ is ultra and $\mathcal{F}$ is ultra then $\mathcal{O}((\mathcal{F}_j)_{j \in J}, \mathcal{F})$ is ultra.

**THEOREM 3.1.** If $(X, \mathcal{O}) \in |\text{AP}|$ then the map

$$\lambda : \mathcal{F}(X) \to \mathbb{R}^\mathbb{K}_+$$

$$\mathcal{F} \mapsto \sup_{\mathcal{U} \in \mathcal{U}(\mathcal{F})} \sup_{U \in \mathcal{U}} \delta(U)$$

fulfills

(CAL1) For any $x \in X : \lambda(\text{stack } x)(x) = 0$.
(CAL2) $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \lambda(\mathcal{G}) \leq \lambda(\mathcal{F})$.
(PRAL) For any family $(\mathcal{F}_j)_{j \in J}$ of filters on $X :$

$$\lambda(\bigcap_{j \in J} \mathcal{F}_j) = \sup_{j \in J} \lambda(\mathcal{F}_j).$$

(AL) For any $\mathcal{F} \in \mathcal{F}(X)$ and any selection of filters $(\mathcal{S}(y))_{y \in X}$ on $X :$

$$\lambda(\mathcal{O}(\mathcal{S}, \mathcal{F})) \leq \lambda(\mathcal{F}) + \sup_{y \in X} \lambda(\mathcal{S}(y))(y).$$

Moreover, for any $x \in X$ and $A \subseteq X :$

$$\delta(x, A) = \inf_{\mathcal{U} \in \mathcal{U}(\text{stack } A)} \lambda(\mathcal{U}(x)).$$

**PROOF.** (CAL1) follows from (D1) whereas (CAL2) follows from the fact that $\mathcal{F} \subseteq \mathcal{G}$ implies $\mathcal{U}(\mathcal{G}) \subseteq \mathcal{U}(\mathcal{F})$. 
To prove (PRAL), let \((\mathcal{F}_j)_{j \in J} \subseteq \mathcal{F}(X)\). One inequality follows from (CAL2), to show the other one observe that for all \(U \in \bigcap_{j \in J} \mathcal{F}_j\) and \(U \in \mathcal{U}\) there exists \(j \in J\) and \(\mathcal{W} \in \mathcal{U}(\mathcal{F}_j)\) such that \(U \in \mathcal{W}\). Consequently we have

\[
\lambda(\bigcap_{j \in J} \mathcal{F}_j) = \sup_{U \in \mathcal{U}(\bigcap_{j \in J} \mathcal{F}_j)} \sup_{U \in \mathcal{U}} \sup_{j \in J} \sup_{\mathcal{W} \in \mathcal{U}(\mathcal{F}_j)} \sup_{U \in \mathcal{W}} \delta.
\]

To prove (AL) let us first suppose that \(\mathcal{F} \subseteq \mathcal{U}(X)\) and that for all \(y \in X:\)
\(\mathcal{S}(y) \subseteq \mathcal{U}(X)\) too. Now put \(\varepsilon := \sup_{y \in X} \lambda(\mathcal{S}(y))(y)\). Then for any \(D \in \mathcal{O}(\mathcal{S}, \mathcal{F})\), by Proposition 3.1.1, there exists \(F \in \mathcal{F}\) such that for all \(y \in F : D \subseteq \mathcal{S}(y)\), and consequently

\[
\delta(y, D) \leq \lambda(\mathcal{S}(y))(y) \leq \varepsilon.
\]

Thus \(D(\varepsilon) \in \mathcal{F}\) and it follows from (D4) that

\[
\delta \leq \delta(\varepsilon) + \varepsilon
\]

By the arbitrariness of \(D \in \mathcal{O}(\mathcal{S}, \mathcal{F})\) and Proposition 3.1.4 it follows herefrom that

\[
\lambda(\mathcal{F}) = \sup_{D \in \mathcal{O}(\mathcal{S}, \mathcal{F})} \delta(\varepsilon) \leq \lambda(\mathcal{F}) + \varepsilon.
\]

Second, let us now suppose \(\mathcal{F}\) and all \(\mathcal{S}(y), y \in X\) are arbitrary filters on \(X\), let again \(\varepsilon := \sup_{y \in X} \lambda(\mathcal{S}(y))(y)\) and for each \(\mathcal{R} \in \bigcap_{y \in X} \mathcal{U}(\mathcal{S}(y))\) let \(\varepsilon(\mathcal{R}) := \sup_{y \in X} \lambda(\mathcal{R}(y))(y)\). Then by straightforward verification we have

\[
\varepsilon = \sup_{\mathcal{R} \in \bigcap_{y \in X} \mathcal{U}(\mathcal{S}(y))} \varepsilon(\mathcal{R}).
\]  

By the foregoing result we know that for any \(\mathcal{R} \in \bigcap_{y \in X} \mathcal{U}(\mathcal{S}(y))\) and \(U \in \mathcal{U}(\mathcal{F})\):

\[
\lambda(\mathcal{O}(\mathcal{R}, U)) \leq \lambda(U) + \varepsilon(\mathcal{R}).
\]  

Further, by Proposition 3.1.2 and 3 we also have

\[
\mathcal{O}(\mathcal{S}, \mathcal{F}) = \bigcap_{y \in X} \mathcal{U}(\mathcal{S}(y)) \bigcap_{\mathcal{U} \in \mathcal{U}(\mathcal{F})} \mathcal{O}(\mathcal{R}, \mathcal{U})
\]

Combining (3.1), (3.2), (3.3) and upon applying (PRAL) it follows that

\[
\lambda(\mathcal{O}(\mathcal{S}, \mathcal{F})) \leq \lambda(\mathcal{F}) + \varepsilon.
\]

To prove the final claim of the theorem, first from the fact that for any
\[ \forall \mathcal{U} \in \mathcal{U}(\text{stack } A) \text{ we have } A \in \mathcal{U} \text{ one inequality is clear. Second, by definition of } \lambda \text{ and applying complete distributivity we obtain} \]

\[
\inf_{\mathcal{U} \in \mathcal{U}(\text{stack } A)} \lambda(\mathcal{U}) = \sup_{\theta \in \cap \mathcal{U}} \inf_{\mathcal{U} \in \mathcal{U}(\text{stack } A)} \delta(\mathcal{U})
\]

By Proposition 2.1, for each \( \theta \in \cap \mathcal{U} \) we can find \( \mathcal{U}_\theta \subset \mathcal{U}(\text{stack } A) \) finite such that \( A \subset \bigcup_{\mathcal{U} \in \mathcal{U}_\theta} \theta(\mathcal{U}) \). Consequently by (D3), we obtain

\[
\inf_{\mathcal{U} \in \mathcal{U}(\text{stack } A)} \lambda(\mathcal{U}) \leq \sup_{\theta \in \cap \mathcal{U} \mathcal{U} \in \mathcal{U}_\theta} \delta(\mathcal{U} \cap \theta(\mathcal{U})) \leq \delta_A
\]

which proves the remaining inequality. 

**THEOREM 3.2.** If \( X \subseteq \text{SET} \) and \( \lambda : \mathcal{P}(X) \to \mathbb{R}_+^X \) is a map fulfilling (CAL1), (CAL2), (PRAL) and (AL) then the map

\[
\delta : X \times \mathcal{P}(X) \to \mathbb{R}_+
\]

\[
(\mathcal{F}, A) \mapsto \inf_{\mathcal{U} \in \mathcal{U}(\text{stack } A)} \lambda(\mathcal{U})
\]

is a distance on \( X \).

Moreover, for any \( \mathcal{F} \in \mathcal{F}(X) \):

\[
\lambda(\mathcal{F}) = \sup_{\mathcal{U} \in \mathcal{U}(\mathcal{F})} \sup_{\mathcal{U} \in \mathcal{U}} \delta_U.
\]

**PROOF.** (D1) follows from (CAL1), (D2) follows from the fact that the infimum over an empty set is infinite, and (D3) follows from the fact that for any \( A, B \subseteq X : \mathcal{U}(\text{stack } \mathcal{A} \cup \mathcal{B}) = \mathcal{U}(\text{stack } A) \cup \mathcal{U}(\text{stack } B) \). Before tackling (D4), we prove the final claim of the theorem. Let the map \( \lambda' \) be defined as

\[
\lambda' : \mathcal{F}(X) \to \mathbb{R}_+^X
\]

\[
\mathcal{F} \mapsto \sup_{\mathcal{U} \in \mathcal{U}(\mathcal{F})} \sup_{\mathcal{U} \in \mathcal{U}} \delta_U.
\]

Now, let \( \mathcal{U} \in \mathcal{U}(X) \). Then first we have

\[
\lambda'(\mathcal{U}) = \sup_{\mathcal{U} \in \mathcal{U}} \delta_U = \sup_{\mathcal{U} \in \mathcal{U}} \inf_{\mathcal{W} \in \mathcal{U}(\text{stack } U)} \lambda(\mathcal{W}) \leq \lambda(\mathcal{U}). \tag{3.4}
\]

Second, by complete distributivity and (PRAL) it follows that

\[
\lambda'(\mathcal{U}) = \sup_{\mathcal{W} \in \mathcal{U}(\text{stack } U)} \inf_{\mathcal{W} \in \mathcal{U}(\text{stack } U)} \lambda(\mathcal{W}) \leq \lambda(\cap \theta(\mathcal{U})). \tag{3.5}
\]

\[
\delta \in \lambda(\mathcal{U}).
\]
Now for any $\theta \in \cap \cup \{U(\text{stack } U) \text{ and any } U \in \mathcal{U}\}$ we have $U \in \theta(U)$ and thus we have
\[ \lambda(\mathcal{U}) \leq \lambda(\cap \{U(\theta(U)) \text{ and it follows from (CAL2) that} \}
\]
\[ \lambda(\mathcal{U}) \leq \lambda(\cap \{U(\theta(U)) \text{ and it follows from (CAL2) that} \}
\]
By the arbitrariness of $\theta$ it follows in combination with (3.5) that
\[ \lambda(\mathcal{U}) \leq \lambda'(\mathcal{U}). \]

Together with (3.4) this shows that $\lambda$ and $\lambda'$ coincide on ultrafilters.

By definition of $\lambda'$ and the fact that $\lambda$ fulfills (PRAL) it then follows that $\lambda' = \lambda$. In
order to prove now (D4) let $A \subseteq X, \epsilon \in \mathbb{R}_+$ and choose any $\mathcal{W} \in \mathcal{U}(A(e)).$ Now suppose that
for some $y \in A(e)$ and for all $U \in \mathcal{U}(\text{stack } A) :$
\[ \epsilon < \lambda(\mathcal{U})(y) \]
\[ = \lambda'(\mathcal{U})(y) = \sup_{U \in \mathcal{U}} \delta(y, U). \]

This implies that for all $U \in \mathcal{U}(\text{stack } A)$ there exists $U_0 \in \mathcal{U}$ such that
\[ \epsilon < \delta(y, U_0). \]

By Proposition 2.1 we can then find $U_1, \ldots, U_n \in \mathcal{U}(\text{stack } A)$ such that $A \subseteq \bigcup_{i=1}^n U_i$
and then it follows from (D3) that
\[ \epsilon < \inf_{i=1}^n \delta(y, U_i) \]
\[ = \delta(y, \bigcup_{i=1}^n U_i) \]
\[ \leq \delta(y, A) \]
which is in contradiction to the choice of $y$. Thus for all $y \in A(e)$ we can find
\[ \mathcal{S}(y) \in \mathcal{U}(\text{stack } A) \text{ such that} \]
\[ \lambda(\mathcal{S}(y))(y) \leq \epsilon. \]

For $y \notin A(e)$ put $\mathcal{S}(y) := \text{stack } y$ and then put
\[ \epsilon' := \sup_{y \in X} \lambda(\mathcal{S}(y))(y). \]

Now consider the diagonal filter $\mathcal{D}(\mathcal{S}, \mathcal{W})$ then $A \in \cap \{y \in A(e) \mathcal{S}(y) \text{ and thus too} \}
A \in \mathcal{D}(\mathcal{S}, \mathcal{W})$. From Proposition 3.1.4° it then also follows that $\mathcal{D}(\mathcal{S}, \mathcal{W}) \in \mathcal{U}(\text{stack } A)$
and from the definition of $\delta$ and (AL) it then follows further that for any $x \in X$ :
\[ \delta(x, A) \leq \lambda(\mathcal{D}(\mathcal{S}, \mathcal{W}))(x) \]
\[ \leq \lambda(\mathcal{W})(x) + \epsilon' \]
\[ \leq \lambda(\mathcal{W})(x) + \epsilon. \]

From the arbitrariness of $\mathcal{W} \in \mathcal{U}(\text{stack } A(e))$ and the definition of $\delta$ it then finally
follows that
\[ \delta(x, A) \leq \delta(x, A(e)) + \epsilon. \]

The combined results of Theorems 3.1 and 3.2 give yet another way to describe the objects of the category $\mathcal{A}_{P}$.

In what follows objects of $\mathcal{A}_{P}$ shall then often also be denoted $(X, \lambda)$ where $\lambda$ then
is a map on $X$ fulfills (CAL1), (CAL2), (PRAL) and (AL). We shall characterize the morphisms of $\mathcal{A}_{P}$ using this new description of objects.
THEOREM 3.3. If $(X, \lambda), (X', \lambda') \in |\text{AP}|$ and $f : X \to X'$ is a function then the following are equivalent:

1° $f$ is a contraction
2° $\forall \mathcal{F} \in F(X) : \lambda'(\text{stack } f(\mathcal{F})) \circ f \preceq \lambda(\mathcal{F})$
3° $\forall \mathcal{F} \in U(X) : \lambda'(\text{stack } f(\mathcal{F})) \circ f \preceq \lambda(\mathcal{F})$.

PROOF. 1° $\Rightarrow$ 2°: from Theorem 3.1 we obtain for any $\mathcal{F} \in F(X)$:

$$\lambda'(\text{stack } f(\mathcal{F})) \circ f = \sup_{\mathcal{W} \in U(\text{stack } f(\mathcal{F}))} \sup_{\mathcal{W} \in W} \delta_\mathcal{W} \circ f \preceq \sup_{\mathcal{U} \in U(\mathcal{F})} \sup_{\mathcal{U} \in U} \delta_\mathcal{U} \circ f = \lambda(\mathcal{F}).$$

2° $\Rightarrow$ 3°: clear.

3° $\Rightarrow$ 1°: from Theorem 3.2 we obtain for any $A \subset X$:

$$\delta_\mathcal{F}(A) \circ f = \inf_{\mathcal{W} \in U(\text{stack } f(A))} \lambda'(\mathcal{W}) \circ f = \inf_{\mathcal{U} \in U(\text{stack } A)} \lambda'(\text{stack } f(\mathcal{U})) \circ f = \inf_{\mathcal{U} \in U(\text{stack } A)} \lambda(\mathcal{U}) = \delta_A'.$$

4. THE QUASITOPOS CAP

DEFINITION 4.1. Given $X \in |\text{SET}|$ a map

$$\lambda : F(X) \to \mathbb{R}_+^X$$

is called a convergence-approach limit if it fulfils (CAL1), (CAL2) and the following weakening of (PRAL):

(CAL3) For all $\mathcal{F}, \mathcal{G} \in F(X) : \lambda(\mathcal{F} \cap \mathcal{G}) = \lambda(\mathcal{F}) \lor \lambda(\mathcal{G})$. The pair $(X, \lambda)$ is called a convergence-approach space.

DEFINITION 4.2. Given convergence-approach spaces $(X, \lambda)$ and $(X', \lambda')$ a function $f : X \to X'$ is called a contraction if it fulfills:

(C) For all $\mathcal{F} \in F(X) : \lambda'(\text{stack } f(\mathcal{F})) \circ f \preceq \lambda(\mathcal{F})$.

In the sequel, a convergence-approach limit and a convergence-approach space will be denoted shortly a CAP-limit and a CAP-space respectively.

We recall that a category of structured sets which is fibre-small and has the property that all constant maps between objects are morphisms is called a construct [1], [11].
If we denote \( \text{CAP} \) the category with objects all \( \text{CAP} \)-spaces and morphisms all contractions, then we obtain the following result, the verification of which is quite trivial.

**Proposition 4.1.** \( \text{CAP} \) is a construct. □

A construct is called topological [11] if it is finally (or equivalently initially) complete.

**Theorem 4.1.** \( \text{CAP} \) is a topological construct.

**Proof.** In order to show that \( \text{CAP} \) is initially complete consider the source

\[
(X \xrightarrow{f_j} (X_j, \lambda_j))_{j \in J}
\]

where all items have their obvious meaning.

Let \( \lambda \) be defined by

\[
\lambda : \mathcal{F}(X) \rightarrow \mathbb{R}_+^X
\]

\[
\mathcal{F} \rightarrow \sup_{j \in J} \lambda_j(\text{stack } f_j(\mathcal{F})) \circ f_j.
\]

To show that \( \lambda \) is a \( \text{CAP} \)-limit on \( X \) is quite simple. (CAL1) and (CAL2) are trivial and (CAL3) follows from the observation that for any \( j \in J \) and any \( \mathcal{F}, \mathcal{G} \in \mathcal{F}(X) \), we have

\[
\text{stack } f_j(\mathcal{F} \cap \mathcal{G}) = \text{stack } f_j(\mathcal{F}) \cap f_j(\mathcal{G}).
\]

To show that \( \lambda \) is initial, let \( (X', \lambda') \in |\text{CAP}| \) and let \( g : X' \rightarrow X \) be a function such that for all \( j \in J : f_j \circ g \) is a contraction. Then for any \( \mathcal{F} \in \mathcal{F}(X') \) we have

\[
\lambda(\text{stack } g(\mathcal{F})) \circ g = \sup_{j \in J} \lambda_j(\text{stack } f_j(\text{stack } g(\mathcal{F}))) \circ f_j \circ g
\]

\[
= \sup_{j \in J} \lambda_j(\text{stack } f_j(\text{stack } g(\mathcal{F}))) \circ (f_j \circ g)
\]

\[
\leq \lambda'(\mathcal{F}).
\]

Consequently \( g \) too is a contraction and we are done. □

Before proceeding we now need some further notational conventions and definitions.

If \( X, Y \in |\text{CAP}| \) then \( \text{HOM}_{\text{CAP}}(X,Y) \) stands for the set of all morphisms i.e. contractions from \( X \) to \( Y \). If no confusion can occur concerning the category under study we often omit the subscript and simply write \( \text{HOM}(X,Y) \).

Given \( \psi \in \mathcal{F}(\text{HOM}(X,Y)) \) and \( \mathcal{F} \in \mathcal{F}(X) \) we define

\[
\psi(\mathcal{F}) := \{ \psi(F) | \psi \in \psi, F \in \mathcal{F} \}
\]

where for all \( \psi \in \psi \) and \( F \in \mathcal{F} : \psi(F) := \{ g(y) | g \in \psi, y \in F \} \).

Clearly, \( \text{stack } \psi(\mathcal{F}) \in \mathcal{F}(Y) \).

Next for any \( f \in \text{HOM}(X,Y) \) if \( \lambda_X \) and \( \lambda_Y \) are the \( \text{CAP} \)-limits on \( X \) and \( Y \) respectively, we define

\[
(\psi, f) := \{ \alpha \in \mathbb{R}_+ | \forall \mathcal{F} \in \mathcal{F}(X) : \lambda_Y(\text{stack } \psi(\mathcal{F})) \circ f \leq \lambda_X(\mathcal{F}) \land \alpha \}.
\]
Quite obviously, \( L(\psi, f) \) is a subinterval of \( \overline{\mathbb{R}}_+ \) and \( \alpha \in L(\psi, f) \). Consequently the map

\[
\lambda : \mathbb{F}(\text{HOM}(X,Y)) \rightarrow \overline{\mathbb{R}}_+
\]

\[\psi \rightarrow \inf L(\psi, \cdot)\]

is well-defined.

**Proposition 4.2.** \( \lambda \) is a CAP-limit on \( \text{HOM}(X,Y) \).

**Proof.** We leave the details to the reader. (CAL1) follows from the fact that for any \( f \in \text{HOM}(X,Y) \) and any \( \mathcal{F} \in \mathbb{F}(X) : \text{stack}(\text{stack } f \mathcal{F}) = \text{stack } f(\mathcal{F}) \). (CAL2) and (CAL3) follow from the facts that for any \( f \in \text{HOM}(X,Y) \) and any \( \psi, \phi \in \mathbb{F}(\text{HOM}(X,Y)) \) respectively, if \( \psi \subset \phi \) then \( L(\psi, f) \subset L(\phi, f) \) and if \( \psi \) and \( \phi \) are arbitrary then \( L(\psi \cap \phi, f) = L(\psi, f) \cap L(\phi, f) \).

If \( G \) is a topological construct, then \( G \) is called **cartesian closed** if for all objects \( A, B \in |G| \), the set \( \text{HOM}_G(A, B) \) can be endowed with a \( G \)-structure such that the evaluation map

\[
ev : A \times \text{HOM}_G(A, B) \rightarrow B
\]

defined by \( \ev(a, f) := f(a) \) is co-universal with respect to the endofunctor \( A \times - \). For more information on cartesian closedness, we refer to [7], [9], [20], [21].

**Theorem 4.2.** CAP is cartesian closed.

**Proof.** The assertion we have to prove breaks up in two parts:

1. For any two objects \( (X, \lambda_X) \) and \( (Y, \lambda_Y) \) in CAP and \( \lambda \) as defined in Proposition 4.2, the evaluation

\[
ev : (X, \lambda_X) \times (\text{HOM}(X,Y), \lambda) \rightarrow (X, \lambda_Y)
\]

is a contraction.

2. For any three objects \( (X, \lambda_X) \), \( (Y, \lambda_Y) \) and \( (Z, \lambda_Z) \) in CAP and a contraction

\[
f : (X \times Z, \lambda_X \times \lambda_Z) \rightarrow (Y, \lambda_Y)
\]

the transpose

\[
f^* : (Z, \lambda_Z) \rightarrow (\text{HOM}(X,Y), \lambda)
\]

defined by \( f^*(z)(x) := f(x, z) \) is a contraction.

In order to verify (1) let \( \mathcal{G} \in \mathbb{F}(X \times \text{HOM}(X,Y)) \) and put \( \mathcal{F} := \text{pr}_1(\mathcal{G}) \) and \( \psi := \text{pr}_2(\mathcal{G}) \).

where

\[
X \times \text{HOM}(X, Y) \xrightarrow{\text{pr}_1} X
\]

\[
\xrightarrow{\text{pr}_2} \text{HOM}(X, Y)
\]

are the canonical projections.

Now fix \( (x, f) \in X \times \text{HOM}(X, Y) \) then from the definition of \( \lambda \) and the construction of initial structures in CAP, it follows that

\[
(\lambda_X \times \lambda)(\mathcal{G})(x, f) = \lambda_X(\mathcal{F})(x) \vee \lambda(\psi)(f)
\]

\[
= \inf \{ \lambda_X(\mathcal{F})(x) \vee \alpha | \alpha \in L(\psi, f) \}.
\]

From the definition of \( L(\psi, f) \) it follows that for any \( \alpha \in L(\psi, f) : \)

\[
\lambda_X(\text{stack } \psi(\mathcal{F}))(f(x)) \leq \lambda_X(\mathcal{F})(x) \vee \alpha.
\]
From (4.1), (4.2) and the fact that ev(\(\mathcal{F} \times \mathcal{G}\)) = \(\mathcal{V}(\mathcal{F})\) it follows that

\[ \lambda_Y(\text{stack } ev(\mathcal{G}))(f(x)) \]
\[ \leq \lambda_Y(\text{stack } ev(\mathcal{F} \times \mathcal{G}))(f(x)) \]
\[ = \lambda_Y(\text{stack } \mathcal{V}(\mathcal{F}))(f(x)) \]
\[ \leq \lambda_X(\mathcal{F})(x) \lor \lambda(\mathcal{V})(f) \]
\[ = (\lambda_X \lor \lambda)(\mathcal{G})(x,f). \]

This proves (1).

In order to verify (2) notice that for any \(\mathcal{G} \in \mathcal{F}(Z)\), \(\mathcal{F} \in \mathcal{F}(X)\), \(z \in Z\) and \(x \in X\), since \(f \in \text{HOM}(X \times Z, Y)\), we have

\[ \lambda_Y(\text{stack } f^*(\mathcal{G}))(f^*(z))(x) \]
\[ = \lambda_Y(\text{stack } f(\mathcal{F} \times \mathcal{G}))(f(x,z)) \]
\[ \leq \lambda_X(\mathcal{F})(x) \lor \lambda_Z(\mathcal{G})(z). \]

Consequently \(\lambda_Z(\mathcal{G})(z) \leq \lambda(\mathcal{F}, f^*(\mathcal{G}), f^*(z))\) which implies that

\[ \lambda(f^*(\mathcal{G}))(f^*(z)) \leq \lambda_Z(\mathcal{G})(z). \]

The arbitrariness of \(\mathcal{G}\) and \(z\) shows that \(f^*\) is indeed again a contraction. This ends the proof of the theorem.

A topological construct is called hereditary provided final epo-sinks are hereditary, or equivalently as was shown in [11], if partial morphisms are representable.

If \(\mathcal{G}\) is a construct and \(A, B \in |\mathcal{G}|\), then a partial morphism from \(A\) to \(B\) is a morphism \(f \in \text{HOM}_\mathcal{G}(C, B)\) where \(C\) is a subobject of \(A\).

If \(\mathcal{G}\) has subobjects then partial morphisms are representable if every object \(B \in |\mathcal{G}|\) can be embedded via the addition of a single point \(\omega_B\), into an object \(B^* \in |\mathcal{G}|\) such that for every partial morphism \(f : C \rightarrow B\) from \(A\) to \(B\) the map

\[ f^* : A \rightarrow B^* \]
\[ a \rightarrow f(a) \quad \text{if } a \in C \]
\[ \omega_B \quad \text{if } a \notin C \]

is a morphism in \(\mathcal{G}\). We shall use this characterization to prove our next result.

THEOREM 4.3. CAP is hereditary.

PROOF. Let \((X, \lambda_X), (Y, \lambda) \in |\text{CAP}|\) and let \(Z \subset X\). The subobject determined by \(Z\) we shall denote \((Z, \lambda_Z)\) where then for any \(\mathcal{F} \in \mathcal{F}(Z)\):

\[ \lambda_Z(\mathcal{F}) = \lambda_X(\text{stack}_X \mathcal{F}). \]

Let \(f : (Z, \lambda_Z) \rightarrow (Y, \lambda)\) be a partial morphism from \((X, \lambda_X)\) to \((Y, \lambda)\). Let \(Y^* := Y \cup \{\omega_Y\}\)
where \(\omega_Y \notin Y\) and define

\[ \lambda^* : \mathcal{F}(Y^*) \rightarrow \overline{B}_+^\mathcal{F}(Y^*) \]

as follows. If \(\mathcal{F} \in \mathcal{F}(Y^*) \setminus \{\text{stack}_Y \omega_Y\}\) then
\[
\lambda^*(\mathcal{F})(y) := \begin{cases} 
\lambda(\mathcal{F}|_Y)(y) & \text{if } y \in Y \\
0 & \text{if } y = \infty_Y 
\end{cases}
\]
and
\[
\lambda^*(\text{stack}_Y) = 0.
\]

It is rather dreary but straightforward to verify that \((Y^*, \lambda^*) \subseteq \cap \) and that \((Y, \lambda)\) is embedded in \((Y^*, \lambda^*)\) by inclusion, so we omit this. Now we define
\[
f^* : (X, \lambda_X) \rightarrow (Y^*, \lambda^*)
\]
by \(f^*(x) = f(x)\) if \(x \in Z\) and \(f^*(x) = \infty_Y\) if \(x \in X \setminus Z\).

To show that \(f^*\) is a contraction, let \(\mathcal{F} \in \mathcal{F}(X)\) and \(x \in X\). If \(\mathcal{F}\) has a trace on \(Z\) then it is clear that \(\text{stack}_Y f^*(\mathcal{F})\) has a trace on \(Y\) equal to \(\text{stack}_Y f(\mathcal{F}|_Z)\). If then \(x \in Z\) it follows that \(f^*(x) = f(x) \in Y\) and by definition of \(\lambda^*\) and the fact that \(f \in \text{HOM}(Z, Y)\) we then obtain
\[
\lambda^*(\text{stack}_Y f^*(\mathcal{F}))(f^*(x)) = \lambda(\text{stack}_Y f(\mathcal{F}|_Z))(f(x)) \leq \lambda_Z(\mathcal{F}|_Z)(x) = \lambda_X(\text{stack}_X f(\mathcal{F}|_Z))(x) \leq \lambda_X(\mathcal{F})(x).
\]

If \(x \in X \setminus Z\) the same inequality results at once from the definition of \(\lambda^*\) and from \(f^*(x) = \infty_Y\). If \(\mathcal{F}\) does not have a trace on \(Z\) then again the same inequality holds for any \(x \in X\) by definition of \(\lambda^*\) and the fact that \(\text{stack}_Y f^*(\mathcal{F}) = \text{stack}_Y \infty_Y = Y\). By Theorem 1 [11] this proves the theorem. 

Since by definition a quasitopos is a hereditary cartesian closed topological construct [10] our main result now is an immediate consequence of the foregoing theorems.

**THEOREM 4.4.** \(\cap \) is a quasitopos. 

5. THE HEREDITARY TOPOLOGICAL CONSTRUCT PRAP

**DEFINITION 5.1.** Given \(X \in \text{SET}\) a map
\[
\lambda : \mathcal{F}(X) \rightarrow \mathcal{F}_X
\]
is called a pre-approach limit (or PRAP-limit for short), if it fulfills (CAL1), (CAL2) and (PRAL). The pair \((X, \lambda)\) is then called a pre-approach space (or PRAP-space for short).

Clearly each pre-approach space is a convergence-approach space. The full subcategory of \(\cap \) with objects all pre-approach spaces shall be denoted \(\text{PRAP}\). From Proposition 4.1 we at once obtain the next result.

**PROPOSITION 5.1.** \(\text{PRAP}\) is a construct.

In Theorems 3.1 and 3.2, we proved that giving a distance on a set \(X\) is equivalent to giving an approach limit on \(X\). A simple inspection of the proofs of these two theorems reveals that (D1), (D2) and (D3) are equivalent to (CAL1), (CAL2) and (PRAL). Consequently, if we call a map
\[
\delta : X \times \mathcal{F}_X \rightarrow \mathcal{F}_X
\]
fulfilling (D1), (D2) and (D3) a pre-distance, then without further proof we can state
the following two results.

**Theorem 5.1.** If $X \in \text{SET}$ and $\delta$ is a pre-distance on $X$, then the map
\[ \lambda : F(X) \to \mathbb{R}_+^X \]
\[ \mathcal{F} \longmapsto \sup_{U \in \mathcal{U}(\mathcal{F})} \sup_{V \in \mathcal{U}} \delta_{U} \]
is a pre-approach limit on $X$.
Moreover, for any $x \in X$ and $A \subset X$:
\[ \delta(x, A) = \inf_{U \in \mathcal{U}(\text{stack } A)} \lambda(U)(x). \]

**Theorem 5.2.** If $(X, \lambda) \in \text{PRAP}$ then the map
\[ \delta : X \times 2^X \to \mathbb{R}_+^X \]
\[ (x, A) \longmapsto \inf_{U \in \mathcal{U}(\text{stack } A)} \lambda(U)(x) \]
is a pre-distance on $X$.
Moreover, for any $\mathcal{F} \in F(X)$:
\[ \lambda(\mathcal{F}) = \sup_{U \in \mathcal{U}(\mathcal{F})} \sup_{V \in \mathcal{U}} \delta_{U}. \]

As was the case for approach spaces, the structure on a pre-approach space shall be
determined either by a pre-approach limit or by a pre-distance, whichever is more conve-
nient.

**Theorem 5.3.** PRAP is a bireflective subcategory of CAP.

**Proof.** Since PRAP contains all indiscrete CAP-objects, it will suffice to show that
PRAP is initially closed in CAP. Let $(X_j, \lambda_j)_{j \in J}$ be a collection of PRAP-spaces and con-
sider the source
\[ (X \xrightarrow{\lambda} (X_j, \lambda_j))_{j \in J}. \]
Let $\lambda$ be the initial CAP-limit on $X$ given by Theorem 4.1. To prove that $\lambda$ fulfills
(PRAL), let $(\mathcal{F}_k)_{k \in K}$ be a collection of filters on $X$ then
\[ \lambda(\bigcap_{k \in K} \mathcal{F}_k) = \sup_{j \in J} \lambda_j(\text{stack } f_j(\bigcap_{k \in K} \mathcal{F}_k)) o f_j \]
\[ = \sup_{j \in J} \lambda_j(\text{stack } f_j(\mathcal{F}_k)) o f_j \]
\[ = \sup_{j \in J} \sup_{k \in K} \lambda_j(\text{stack } f_j(\mathcal{F}_k)) o f_j \]
\[ = \sup_{k \in K} \lambda(\mathcal{F}_k) \]
and we are done.  

**Remark.** It is easily verified that the PRAP-reflection of a CAP-space $(X, \lambda)$ is given
by
\[ (X, \lambda) \xrightarrow{id_X} (X, \lambda_p) \]
where the pre-distance associated with $\lambda_p$ is given by
\[ \delta(x,A) := \inf_{U \in \mathcal{U}(\text{stack } A)} \lambda(U)(x). \]

**Corollary 5.1.** PRAP is a topological construct.  

We shall later give a simple reason why PRAP is not cartesian closed, it is however hereditary as we shall now prove.

**Theorem 5.4.** PRAP is an hereditary topological construct.

**Proof.** The proof goes exactly the same as that of Theorem 4.3., the only difference being that now one starts with $(X,\lambda_X),(Y,\lambda) \in |\text{PRAP}|$ and one has to show that $(Y^*,\lambda^*) \in |\text{PRAP}|$. We leave this to the reader.  

**6. Embedding Conv in CAP**

A convergence space [6], [15] is a pair $(X,q)$ where $X \in \text{SET}$ and $q \subset \mathcal{F}(X) \times X$ fulfills

(C1) for all $x \in X$ : $(\text{stack } x,x) \in q$.

(C2) For all $\mathcal{F},\mathcal{G} \in \mathcal{F}(X)$ and $x \in X$ : $(\mathcal{F},x) \in q$, $\mathcal{F} \subset \mathcal{G} \Rightarrow (\mathcal{G},x) \in q$.

(C3) For all $\mathcal{F},\mathcal{G} \in \mathcal{F}(X)$ and $x \in X$ : $(\mathcal{F},x) \in q$ and $(\mathcal{G},x) \in q \Rightarrow (\mathcal{F} \cap \mathcal{G},x) \in q$.

Given convergence spaces $(X,q),(X',q')$ a function $f : X \to X'$ is called continuous if for all $(\mathcal{F},x) \in q$ we have $(\text{stack } f(),f(x)) \in q'$.

The class with objects all convergence spaces and morphisms all continuous maps, is a quasitopos [10], denoted Conv.

The proof of the following result is quite straightforward and so we omit it.

**Theorem 6.1.** Conv is embedded as a full subcategory in CAP by the functor

\[ \text{CONV} \to \text{CAP} \]

\[ (X,q) \mapsto (X,\lambda_q) \]

where for all $\mathcal{F} \in \mathcal{F}(X)$, and $x \in X$ :

\[ \lambda_q(\mathcal{F})(x) := \begin{cases} 0 & \text{if } (\mathcal{F},x) \in q \\ \infty & \text{otherwise}. \end{cases} \]

We shall now show that this embedding actually is extremely nice, but first we mention the following useful characterization of Conv in CAP, similar to that of TOP in AP [17].

**Proposition 6.1.** A space $(X,\lambda) \in |\text{CAP}|$ is a convergence space, if and only if for all $\mathcal{F} \in \mathcal{F}(X)$ : $\lambda(\mathcal{F})(x) \subset (0,\infty)$.  

As the formulation of this proposition suggests we shall not differentiate between the notion of a convergence space and of a CAP-space fulfilling the condition of Proposition 6.1. This is after all entirely justified by Theorem 6.1.

**Theorem 6.2.** Conv is a bireflective subcategory of CAP.

**Proof.** Given $(X,\lambda) \in |\text{CAP}|$ define

\[ \lambda^* : \mathcal{F}(X) \to \mathbb{R}_+^X \]
by \( \lambda_\beta(\mathcal{F})(x) = 0 \) if \( \lambda(\mathcal{F})(x) < \infty \) and \( \lambda_\beta(\mathcal{F})(x) = \infty \) if \( \lambda(\mathcal{F})(x) = \infty \). \((X, \lambda_\beta)\) clearly is a convergence space and the bireflection of \((X, \lambda)\) is given by \( (X, \lambda) \xrightarrow{id_X} (X, \lambda_\beta) \).

**THEOREM 6.3.** \(\text{CONV}\) is a bicoreflective subcategory of \(\text{CAP}\).

**PROOF.** Given \((X, \lambda) \in \text{CAP}\) define
\[
\lambda^*: \mathcal{F}(X) \to \mathbb{R}_+
\]
by \( \lambda^*(\mathcal{F})(x) = 0 \) if \( \lambda(\mathcal{F})(x) = 0 \) and \( \lambda^*(\mathcal{F})(x) = \infty \) if \( \lambda(\mathcal{F})(x) > 0 \). Again it is clear, that \((X, \lambda^*)\) is a convergence space and that the bicoreflection of \((X, \lambda)\) is given by \( (X, \lambda^*) \xrightarrow{id_X} (X, \lambda) \).

---

**7. EMBEDDING PRETOP IN PRAP**

A pre-topological space \([4], [6]\) is a convergence space \((X, q)\) where instead of (C3) \(q\) fulfills the stronger condition:

\[(PR)\quad \text{For any collection } (\mathcal{F}_j, x_j)_{j \in J} \subseteq q \text{ we have } \bigcap_{j \in J} \mathcal{F}_j(x) \subseteq q.\]

The full subcategory of \(\text{CONV}\) with objects all pre-topological spaces is denoted \(\text{PRETOP}\). It is quite easy to see that precisely the same results hold for \(\text{PRETOP}\) w.r.t. \(\text{PRAP}\), as those proven in Section 6 for \(\text{CONV}\) w.r.t. \(\text{CAP}\). We therefore list them without further explanation.

**THEOREM 7.1.** \(\text{PRETOP}\) is embedded as a full subcategory in \(\text{PRAP}\) by the functor
\[
\text{PRETOP} \longrightarrow \text{PRAP}
\]
\[
(X, q) \mapsto (X, \lambda_q)
\]
where for all \(\mathcal{F} \in \mathcal{F}(X)\) and \(x \in X:\)
\[
\lambda_q(\mathcal{F})(x) := \begin{cases} 
0 & \text{if } (\mathcal{F}, x) \in q \\
\infty & \text{otherwise}.
\end{cases}
\]

**PROPOSITION 7.1.** A space \((X, \lambda) \in \text{PRAP}\) is a pre-topological space, if and only if for all \(\mathcal{F} \in \mathcal{F}(X) : \lambda(\mathcal{F})(X) \subseteq \{0, \infty\}\), or equivalently, if \(\delta\) is the pre-distance associated with \(\lambda\), if and only if \(\delta(X \times X) \subseteq \{0, \infty\}\).

**THEOREM 7.2.** \(\text{PRETOP}\) is a bireflective subcategory of \(\text{PRAP}\), the bireflection of any \(\text{PRAP}\)-space being the same as its \(\text{CONV}\)-bireflection.

**THEOREM 7.3.** \(\text{PRETOP}\) is a bicoreflective subcategory of \(\text{PRAP}\), the bicoreflection of any \(\text{PRAP}\)-space being the same as its \(\text{CONV}\)-bicoreflection.

---

Again, we shall not differentiate between pre-topological spaces and \(\text{PRAP}\)-spaces fulfilling the condition of Proposition 7.1.
8. EMBEDDING AP IN PRAP

From Section 3 it is quite clear that AP is embedded as a full subcategory of PRAP.

**THEOREM 8.1.** AP is a bireflective subcategory of CAP.

**PROOF.** Since AP contains all indiscrete CAP-objects, it will suffice to show that AP is initially closed in CAP. Let \((X_j, \lambda_j)_{j \in J}\) be a family of AP-spaces and consider the source

\[
\xymatrix{ (X) \ar[r]^f & (X_j, \lambda_j)_{j \in J} }.
\]

Let \(\lambda\) be the initial CAP-limit on \(X\). From Theorem 5.3 we already know that \(\lambda\) fulfills (PRAL). To show that it also fulfills (AL), let \(\mathcal{F} \in \mathcal{P}(X)\), let \((\mathcal{S}(y))_{y \in X}\) be a selection of filters on \(X\) and put \(\varepsilon := \sup \lambda(\mathcal{S}(y))(y)\). Now, for all \(j \in J\) define the following selection of filters on \(X_j\):

\[
\mathcal{Q}_j(z) := \left\{ \begin{array}{ll}
\bigcap_{y \in f_j^{-1}(z)} f_j(\mathcal{S}(y)) & \text{if } z \in f_j(X) \\
\text{stack } z & \text{if } z \notin f_j(X)
\end{array} \right.
\]

We leave to the reader the straightforward verification that for all \(j \in J\):

\[
\emptyset(\mathcal{Q}_j, \text{stack } f_j(\mathcal{F})) \subset \text{stack } f_j(\emptyset(\mathcal{S}, \mathcal{F})). \tag{6.1}
\]

Next for all \(j \in J\), put \(\varepsilon_j := \sup_{z \in f_j^{-1}(X)} \lambda_j(\mathcal{Q}_j(z))(z)\). Now if \(z \notin f_j(X)\) then

\[
\lambda_j(\mathcal{Q}_j(z))(z) = \lambda_j(\text{stack } z)(z) = 0 \preceq \varepsilon
\]

whereas, if \(z \in f_j(X)\) then

\[
\lambda_j(\mathcal{Q}_j(z))(z) = \lambda_j\left(\bigcap_{y \in f_j^{-1}(z)} f_j(\mathcal{S}(y))(z)\right) = \sup_{y \in f_j^{-1}(z)} \lambda_j(\text{stack } f_j(\mathcal{S}(y)))(f_j(y)) \leq \sup_{y \in X} \sup_{j \in J} \lambda_j(\text{stack } f_j(\mathcal{S}(y)))(f_j(y)) = \varepsilon.
\]

By the arbitrariness of \(j \in J\), this implies that

\[
\sup_{j \in J} \varepsilon_j \preceq \varepsilon. \tag{6.2}
\]

From (6.1) and (6.2) we then obtain

\[
\lambda(\emptyset(\mathcal{S}, \mathcal{F})) = \sup_{j \in J} \lambda_j(\text{stack } f_j(\emptyset(\mathcal{S}, \mathcal{F}))) \circ f_j \preceq \sup_{j \in J} \lambda_j(\mathcal{Q}_j, \text{stack } f_j(\mathcal{F})) \circ f_j
\]
COROLLARY 8.1. AP is a bireflective subcategory of PRAP. ■

9. EMBEDDING pqs-MET in PRAP

The most general kind of map measuring a distance between points of a set X is an extended pseudo quasi-semimetric (shortly -pqs-metric). An -pqs-metric

\[ d : X \times X \to \mathbb{R}_+ \]

need only fulfil \( d(x,x) = 0 \) for all \( x \in X \). The pair \((X,d)\) then is called an -pqs-metric space. Given -pqs-metric spaces \((X,d)\) and \((X',d')\) a function \( f : X \to X' \) is called non-expansive if \( d' \circ (f \circ f) \leq d \).

Let pqs-MET stand for the category with objects all -pqs-metric spaces and morphisms all non-expansive maps.

THEOREM 9.1. pqs-MET is embedded as a full subcategory in PRAP by the functor

\[ \text{pqs-MET} \to \text{CAP} \]

\[ (X,d) \mapsto (X,\lambda_d) \]

where for all \( \mathcal{F} \in \mathcal{P}(X) \) and \( x \in X \):

\[ \lambda_d(\mathcal{F})(x) := \inf_{F \in \mathcal{F}} \sup_{y \in F} d(x,y). \]

PROOF. That \( \lambda_d \) fulfils (CAL1) and (CAL2) is clear. That it also fulfils (PRAL) is seen as follows. Let \( (\mathcal{F}_j)_{j \in J} \subset \mathcal{P}(X) \) then for any \( x \in X \) we have

\[ \lambda_d(\bigcap_{j \in J} \mathcal{F}_j)(x) = \inf_{F \in \bigcap_{j \in J} \mathcal{F}_j} \sup_{y \in F} d(x,y) \]

\[ = \inf_{\theta \in \bigcap_{j \in J} \mathcal{F}_j} \sup_{y \in \bigcup_{j \in J} \theta(j)} d(x,y) \]

\[ = \inf_{\theta \in \bigcap_{j \in J} \mathcal{F}_j} \sup_{j \in J} \sup_{y \in \theta(j)} d(x,y) \]

\[ = \sup_{j \in J} \lambda_d(\mathcal{F}_j)(x). \]

If \((X,d), (X',d') \in \text{pqs-MET} \) and \( f : (X,d) \to (X',d') \) is non-expansive it is easily verified that \( f : (X,\lambda_d) \to (X',\lambda_{d'}) \) is a contraction. The converse is equally simple upon noticing that from the definition of \( \lambda_d \), for any \( x,y \in X \) : \( d(x,y) = \lambda_d(\text{stack } y)(x) \).

REMARK. By Theorems 5.1 and 5.2, the pre-approach space \((X,\delta_d)\) is identical to \((X,\delta_d)\) where \( \delta_d \) is the pre-distance derived from \( \lambda_d \), i.e. for all \( x \in X \) and \( A \subset X \):

\[ \delta_d(x,A) = \inf_{U \in \mathcal{U}(\text{stack } A)} \inf_{y \in U} \sup_{x \in U} d(x,y). \] (9.1)
This rather complicated expression for $\delta_d$ can however be much simplified using the following lemma.

**Lemma 9.1.** Given $(X,d) \in |pq_{s-MET}|$, $U \in U(X)$ and $x \in X$ we have

$$\sup_{U \in U} \inf_{y \in U} d(x,y) = \inf_{U \in U} \sup_{y \in U} d(x,y).$$

**Proof.** The inequality $\leq$ follows from the fact that for any $U_1, U_2 \in U$:

$$\inf_{y \in U_1} \sup_{y \in U_2} d(x,y) \leq \sup_{y \in U_1} \inf_{y \in U_2} d(x,y).$$

To show the other one, suppose

$$\inf_{U \in U} \sup_{y \in U} d(x,y) > 0.$$

Then for all $U \in U$ there exists $y_U \in U : d(x,y_U) > 0$. Clearly

$$W := \{y_U | U \in U\} \in U$$

and $\inf_{y \in W} d(x,y) \geq 0$ which proves our claim. \hfill \blacksquare

**Theorem 9.2.** Given $(X,d) \in |pq_{s-MET}|$ the pre-distance $\delta_d$ associated with $\lambda_d$ is given by

$$\delta_d(X,A) = \inf_{a \in A} d(x,a), \quad x \in X, A \subseteq X.$$

**Proof.** Immediate from (9.1) and Lemma 9.1. \hfill \blacksquare

**Theorem 9.3.** $pq_{s-MET}$ is a bicoreflective subcategory of PRAP.

**Proof.** Let $(X,d) \in |PRAP|$ and define the map

$$d_\delta : X \times X \to \mathbb{R}_+^*$$

$$(x,y) \mapsto \delta(x,\{y\}).$$

It is clear that $(X,d_\delta) \in |pq_{s-MET}|$. The remainder of the proof now is exactly the same as in Theorem 6.7 [17], where it was shown that $pq_{s-MET}$ is a bicoreflective subcategory of AP, and so we omit this. \hfill \blacksquare

Analogous to the characterization of $pq_{s-MET}$ in AP [17], we have the next result, the verification of which we leave to the reader.

**Proposition 9.1.** A space $(X,d) \in |PRAP|$ is an $s$-$pq_{s}$-metric space, if and only if for any $x \in X$ and $A \subseteq X$:

$$\delta(x,A) = \inf_{a \in A} \delta(x,a).$$

**Theorem 9.4.** $pq_{s-MET}$ is a hereditary topological construct.

**Proof.** This is an immediate consequence of Corollary 5.1, Theorem 9.3 and of Theorem 6 [11]. \hfill \blacksquare

**Remark.** Initial structures in $pq_{s-MET}$ are obtained as follows. Let

$$\begin{array}{c}
\xymatrix{ X \ar[r]^{f_j} & (X_j,d_j) } \end{array}$$

be a source, then the initial $s$-$pq_{s}$-metric on $X$ is given by

$$\delta_{\text{initial}} = \inf_{j \in J} \delta_{(X_j,d_j)}.$$
We leave the verification to the reader.

THEOREM 9.5. \( pqs-MET \) is cartesian closed.

PROOF. For \( (X,d_X), (Y,d_Y) \in pqs-MET \) let \( HOM(X,Y) \) stand for all non-expansive maps from \( (X,d_X) \) to \( (Y,d_Y) \). For any \( f,g \in HOM(X,Y) \) put

\[
C(f,g) := \{ \alpha \in \mathbb{R}_+ | d_Y(f \circ g) \leq d_X \circ \alpha \}
\]

and define

\[
d : HOM(X,Y) \times HOM(X,Y) \longrightarrow \mathbb{R}_+
(f,g) \longmapsto \inf C(f,g)
\]

Clearly \( d \) is a well-defined \( \ast \)-pqs-metric on \( HOM(X,Y) \). In order to show that

\[
ev : (X,d_X) \times (HOM(X,Y),d) \longrightarrow (Y,d_Y)
(x,f) \longmapsto f(x)
\]

is non-expansive, let \( x,y \in X \) and \( f,g \in HOM(X,Y) \) then

\[
d_Y(f(x),g(y)) \leq \inf d_X(x,y) \vee \alpha | \alpha \in C(f,g))
= d_X(x,y) \vee d(f,g)
= d_X \circ d((x,y),(f,g)).
\]

Next, if \( (Z,d_Z) \in pqs-MET \), \( f \in HOM(X \times Z,Y) \) then consider the map

\[
f^* : (Z,d_Z) \longrightarrow (HOM(X,Y),d)
z \longmapsto f^*(z)
\]

where \( f^*(z) \) is defined by \( f^*(z)(x) := f(x,z) \).

In order to show \( f^* \) is non-expansive, let \( z,z' \in Z \) then since \( f \in HOM(X \times Z,Y) \) we obtain for all \( x,x' \in X \):

\[
d_Y \circ (f^*(z) \times f^*(z'))(x,x')
= d_Y(f(x,z),f(x',z'))
\leq d_X(x,x') \vee d_Z(z,z').
\]

Consequently \( d_Z(z,z') \in C(f^*(z),f^*(z')) \) and \( d(f^*(z),f^*(z')) \leq d_Z(z,z') \).

The combined results of Theorem 9.4 and 9.5 now give us the following theorem.

THEOREM 9.6. \( pqs-MET \) is a quasitopos.

As a final result in this section, we shall now show that \( pq-MET \) (which is a bi-coreflective subcategory of \( AP \) [17]) is a bireflective subcategory of \( pqs-MET \).

THEOREM 9.7. \( pq-MET \) is a bireflective subcategory of \( pqs-MET \).

PROOF. Given \( (X,d) \in pqs-MET \) define

\[
\hat{d} : X \times X \rightarrow \mathbb{R}_+
\]

as follows. For any \( x,y \in X \), put
\[ S_n(x,y) := \{ (x_j)_{j=0}^n \in X^n \mid x_j = x, x_n = y \} \]

and then define
\[
\hat{d}(x,y) = \inf_{n \in \mathbb{N}_0} \inf \{ \frac{1}{n} \sum_{j=1}^n d(x_{j-1},x_j) \mid (x_j)_{j=0}^n \in S_n(x,y) \}.
\]

Verification of the facts that \( \hat{d} \) satisfies the triangle inequality and that
\[
(X,d) \xrightarrow{\text{id}_X} (X,\hat{d})
\]
is the pq-MET\(^m\)-reflection of \((X,d)\), we leave to the reader. 

10. COUNTEREXAMPLES

COUNTEREXAMPLE 1. We construct a finitely generated topological space, a CONV-quotient of which is not in PRETOP. Let \( X := \bigcup_{n \geq 3} ]n,n+1[ \). Consider the partition of \( X \) by splitting each interval \( ]n,n+1[ \) in three subintervals \( ]n,n \[, ]n,n+1[, \]n+1,n+1[ \) and let \( \mathcal{U} \) be the associated partition topology on \( X \). Since \( X \) is then a co-product of indiscrete spaces it is finitely generated [9]. Let \( Y := ]0,1[ \) and consider the map
\[
f : X \rightarrow Y \\
x \mapsto x-\lfloor x \rfloor.
\]

The CONV-quotient \( q \) on \( Y \) is characterized by stating that \((\mathcal{F},y) \in q\), if and only if there exist \( x_1,\ldots,x_n \in f^{-1}(y) \) such that
\[
f(\mathcal{U}(x_i)) \subset \mathcal{F}
\]
where for any \( x \in X \) : \( \mathcal{U}(x) \) is the \( \mathcal{U} \)-neighborhoodfilter of \( x \). Consequently for all \( n \leq 3 \):
\[
(f(\mathcal{U}(n\frac{1}{2}))),\frac{1}{2}) \in q
\]
but
\[
(\cap_{n \geq 3} f(\mathcal{U}(n\frac{1}{2}))),\frac{1}{2}) \notin q
\]
and thus \( q \) is not pre-topological. 

COUNTEREXAMPLE 2. We construct a finite \( =\)-pq-metric space, the \( =\)-pq-metric of which attains only the values 0 and \( =\) and a pqs-MET\(^m\)-quotient of which is not in pq-MET\(^m\). Let \( X := \{u,v,x,y\} \) and let \( d \) be the \( =\)-pq-metric (i.e. \( d \) is symmetric) defined by
\[
d(u,y) = d(v,x) = 0 \\
d(u,v) = d(v,y) = d(y,x) = d(x,u) = 1.
\]

Let \( Y := \{u,v,z\} \) and define the map
\[
f : X \rightarrow Y
\]
by \( f(u) = u, f(v) = v, f(x) = f(y) = z \).

If \( e \) denotes the pqs-MET\(^m\)-quotient on \( Y \) then the \( e \)-distance between two points in \( Y \) is equal to the \( d \)-distance between their fibres. Consequently
\[
e(v,z) = d(\{v\},\{x,y\}) = d(v,x) \land d(v,y) = 0 \\
e(u,z) = d(\{u\},\{x,y\}) = d(u,x) \land d(u,y) = 0
\]
but since
\[ e(u,v) = d((u),(v)) = d(u,v) = \infty \]
\( e \) does not fulfil the triangle inequality and thus is not an \(-pq\)-metric.

From [17] and the foregoing sections we obtain the following diagram where r (resp. c) means the smaller category is bireflective (resp. bicoreflective) in the larger one.

![Diagram](image)

We shall now use the results of both counterexamples to show that this diagram is complete in the sense that no other reflectivity or coreflectivity arises except those indicated and those obtained by transitivity.

In [17] it was shown that \( TOP \cap pq\text{-MET}^\infty \) consists precisely of all finitely generated spaces and that a distance attains at most the value 0 and \( \infty \), if and only if it is associated with a topology. Consequently in both counterexamples the original space \( X \) is at the same time in \( TOP \), in \( pq\text{-MET}^\infty \) and a fortiori also in \( AP \).

The first counterexample then shows that \( Y \), the \( CONV \)-quotient (= \( CAP \)-quotient) of \( X \), is not in \( PRETOP \) and since \( CONV \cap PRAP = PRETOP \) (see Proposition 6.1 and 7.1) also not in \( PRAP \). This gives us first the known results that neither \( TOP \) nor \( PRETOP \) are coreflective in \( CONV \), second that neither \( AP \) nor \( PRAP \) are coreflective in \( CAP \) and third that neither \( pq\text{-MET}^\infty \) nor \( pqs\text{-MET}^\infty \) are coreflective in \( CAP \).

The second counterexample gives us \( Y \), the \( pqs\text{-MET}^\infty \)-quotient (= \( PRAP \)-quotient) of \( X \), which is not in \( pq\text{-MET}^\infty \). However since the pre-distance on \( Y \) again attains only the values 0 and \( \infty \), \( Y \) is in \( PRETOP \). On the other hand, since \( pqs\text{-MET}^\infty \cap AP = pq\text{-MET}^\infty \) (see Theorem 6.20 [17] and Proposition 9.1) \( Y \) is not in \( AP \) and thus also not in \( TOP \). This then gives us first again the known result that \( TOP \) is not coreflective in \( PRETOP \), second that \( AP \) is not coreflective in \( PRAP \) and third that \( pq\text{-MET}^\infty \) is not coreflective in \( pqs\text{-MET}^\infty \). Further in [17] it was seen that \( pq\text{-MET}^\infty \) is not reflective in \( AP \) from which it follows that \( pqs\text{-MET}^\infty \) is neither reflective in \( PRAP \) nor in \( CAP \), and that \( pq\text{-MET}^\infty \) is not reflective in \( CAP \). This completes our argumentation.

The authors would like to thank J. ADAMEK and L.D. NEL for bringing the problem of cartesian closedness in the setting of \( AP \) to their attention.

REFERENCES
3. BINZ E., Continuous convergence on \( C(X) \), Springer Lecture Notes in Math. 469 (1975).


Special Issue on
Boundary Value Problems on Time Scales

Call for Papers

The study of dynamic equations on a time scale goes back to its founder Stefan Hilger (1988), and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics; moreover, it often reveals the reasons for the discrepancies between two theories.

In recent years, the study of dynamic equations has led to several important applications, for example, in the study of insect population models, neural network, heat transfer, and epidemic models. This special issue will contain new researches and survey articles on Boundary Value Problems on Time Scales. In particular, it will focus on the following topics:

- Existence, uniqueness, and multiplicity of solutions
- Comparison principles
- Variational methods
- Mathematical models
- Biological and medical applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/ade/guidelines.html. Authors should follow the Advances in Difference Equations manuscript format described at the journal site http://www.hindawi.com/journals/ade/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>April 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>July 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>October 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Alberto Cabada, Departamento de Análise Matemática, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain; alberto.cabada@usc.es

Guest Editor

Victoria Otero-Espinar, Departamento de Análise Matemática, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain; mvictoria.oter@usc.es