ON THE DUAL SPACE OF A WEIGHTED BERGMAN SPACE
ON THE UNIT BALL OF $\mathbb{C}^n$

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ABSTRACT. The weighted Bergman space $A^p_{\alpha}(B_n)(0 < p < 1)$, of the holomorphic functions on the unit ball $B_n$ of $\mathbb{C}^n$ forms an $F$-space. We find the dual space of $A^p_{\alpha}(B_n)$ by determining its Mackey topology.

KEY WORDS AND PHRASES. Hardy space, Bergman space, Mackey topology.

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1. INTRODUCTION.
Let $B_n$ be the unit ball of $\mathbb{C}^n$, $\nu$ be the normalized Lebesgue measure and $\sigma$ be the rotation invariant positive Borel measure on $S$, the boundary of $B_n$, with $\sigma(S) = 1$. The weighted Bergman space $A^p_{\alpha}(B_n)(0 < p < \infty, \alpha \geq -1)$ consists of all functions holomorphic in $B_n$ for which

$$\|f\|_{p,\alpha}^p = \left\{ \int_0^1 M^p_{p}(r;f)(1-r)^\alpha \, \sigma(\zeta) 2nr^{2n-1}dr < \infty, \quad \text{if} \quad \alpha > -1, \right.$$ \[ \sup_{0 \leq r < 1} M^p_{p}(r;f) < \infty, \quad \text{if} \quad \alpha = -1, \]

where $M^p_{p}(r;f) = \int_S |f(r\zeta)|^p \sigma(\zeta)$. Note that the weighted Bergman space $A^p_{\alpha}(B_n)$ is, in fact, the Hardy space $H^p(B_n)$ if $\alpha = -1$ (See [1]).

The purpose of this paper is to compute the dual space $(A^p_{\alpha}(B_n))^*$ for $0 < p \leq 1$ by determining the Mackey topology of $A^p_{\alpha}(B_n)$. The corresponding problems for the case $n = 1$ are settled by Duren, Romberg and Shields [2], Shapiro [3] and Ahern [4]. Our computations are very similar to those of them.

Throughout this work, $C_{\alpha, \beta,...}$ denotes a positive constant depending only on $\alpha, \beta,...$ which may vary in the various places, and the notation $a(z) - b(z)$ means that the ratio $a(z)/b(z)$ has a positive finite limit as $|z| 

2. SOME PRELIMINARY RESULTS.

Lemma 2.1. If $f \in A^p_{\alpha}(B_n)(0 < p < \infty, \alpha \geq -1)$, then

$$|f(z)| \leq C_{n,p,\alpha} \|f\|_{p,\alpha}(1 - |z|)^{-n+a+1}.$$
PROOF. The case $\alpha = -1$ is proved in [1, Thm 7.2.5]. For the proof of the case $\alpha > -1$, it is enough to prove the result for $\frac{1}{2} \leq |z| < 1$ since $|f(z)|$ is bounded for $|z| \leq 1$. For this range of $\rho$ we have:

$$
\|f\|_{p,\alpha}^p = \int_0^1 \int_S |f(rz)|^p (1 - r)^{\alpha 2n - 1} dr d\sigma(z)
$$

$$
\geq C_{n,p} \int_0^1 \int_S |f(rz)|^p (1 - r)^{\alpha} d\sigma(z) dr
$$

$$
\geq C_{n,p} M_p(p;f) \int_0^1 (1 - r)^{\alpha} dr
$$

$$
= C_{n,p,\alpha} M_p(p;f) (1 - \rho)^{n+1}.
$$

By the result of the case $\alpha = -1$ and the above result, we get

$$
|f(\rho z)| \leq C_{n,p} M_p(p;f) (1 - |z|) \frac{n}{p}
$$

$$
\leq C_{n,p,\alpha} \|f\|_{p,\alpha} (1 - \rho) \frac{\alpha + 1}{p} (1 - |z|) \frac{n}{p}.
$$

Consequently we have

$$
|f(z)| = |f(\sqrt{r}rz)| \quad (z = rz)
$$

$$
\leq C_{n,p,\alpha} \|f\|_{p,\alpha} (1 - \sqrt{r}) \frac{\alpha + 1}{p} (1 - \sqrt{r}) \frac{n}{p}
$$

$$
\leq C_{n,p,\alpha} \|f\|_{p,\alpha} (1 - r) \frac{n + \alpha + 1}{p}.
$$

COROLLARY 2.2. (a) The convergence of $A_p^\alpha(B_n)$ with its invariant metric

$$d(f,g) = \begin{cases} 
\|f - g\|_{p,\alpha}, & (0 < p < 1), \\
\|f - g\|_{p,\alpha}^p, & (p \geq 1)
\end{cases}
$$

implies the uniform convergence on any compact subset of $B_n$.

(b) $A_p^\alpha(B_n)$ is an F-space if $0 < p < 1$ and a Banach space if $p \geq 1$.

PROOF. (a) follows immediately from Lemma 2.1. The proof of (b) is routine and is omitted.

COROLLARY 2.3. $A_p^\alpha(B_n) \subset A_q^\alpha(B_n)$ if $0 < p < q$ and $\frac{n + \alpha + 1}{p} = \frac{n + \beta + 1}{q}$.

In particular,

$$A_p^\alpha(B_n) \subset A_0^\alpha(B_n), \text{ where } \sigma = \frac{n + \alpha + 1}{p} - (n+1).
$$

PROOF. First, we prove the case $\alpha > -1$. We use Lemma 2.1 in the first inequality of the following.
This completes the proof of the case \( a > -1 \). The remaining case is essentially a result of Hardy and Littlewood, but we give a proof using Ahern's technique in [5]. Let \( f \in H^p(B_n) \). By Lemma 2.1, we have

\[
|f(z)| \leq K_{n,p} (1 - |z|)^{-n} \|f\|_p.
\]

Set

\[
Mf(\zeta) = \sup_{0<r<1} |f(r\zeta)|.
\]

Then we have

\[
\int_0^1 |f(r\zeta)|(1 - r)^{\frac{-n-1}{p}} 2nr^{2n-1} dr \leq K_{n,p} \|f\|_p \int_0^1 (1 - r)^{-n-1} dr + c_{n,p} Mf(\zeta) \int_0^1 (1 - r)^{\frac{-n-1}{p}} dr
\]

\[
\leq K_{n,p} \|f\|_p \frac{(1 - \lambda)^{-n}}{n} + c_{n,p} Mf(\zeta) \frac{(1 - \lambda)^{\frac{-n}{p}}}{\frac{1}{p} - 1}.
\]

If \( Mf(\zeta) \leq K_{n,p} \|f\|_p \), by setting \( \lambda = 0 \) in (2.2), (2.1) is dominated by \( C_{n,p} \|f\|_p \).

If \( Mf(\zeta) \geq K_{n,p} \|f\|_p \), by setting

\[
\lambda = 1 - \left( \frac{K_{n,p} \|f\|_p}{Mf(\zeta)} \right)^{\frac{p}{n}}
\]

in (2.2), (2.1) is dominated by

\[
C_{n,p} \|f\|_p^{1-p} Mf(\zeta)^p.
\]

Hence, for any \( \zeta \in S \),

\[
(2.1) \leq C_{n,p} \|f\|_p + C_{n,p} \|f\|_p^{1-p} Mf(\zeta)^p.
\]

Integrating (2.3) with respect to \( d\sigma(\zeta) \) over \( S \) and using the complex maximal theorem [1, Thm. 5.6.5], we obtain
3. THE HACKEY TOPOLOGY OF $A^p(B_n)$.

In this section, we will show that the Mackey topology of $A^p(B_n)$ is the restriction of the topology of $A^1(B_n)$, where $\sigma = \frac{n+\alpha+1}{p} - (n+1)$.

First we give necessary definitions.

**DEFINITION 3.1.** The Hackey topology of a non-locally convex topological vector space $(X, \tau)$ is the unique locally convex topology $m$ on $X$ satisfying the following conditions:

1. $m$ is weaker than $\tau$,
2. the $\tau$-closure of the absolutely convex hull of each $\tau$-neighborhood of the origin contains an $m$-neighborhood of the origin (See [6, Thm 1]).

**DEFINITION 3.2.** For $\beta > -n$ and $z, w \in B_n$, we define

$$K^p_{\beta}(z, w) = \left[\frac{n + \beta}{n}\right] \frac{(1 - |w|^2)^{\beta}}{(1 - <z, w>)^{\beta + n + 1}}$$

and

$$J^p_{\beta, c}(w)(z) = (1 - |w|^2)^{-c} K^p_{\beta}(z, w).$$

The following proposition is useful in the sequel:

**PROPOSITION 3.3.** [1, p. 120] If $\beta > -n$, then $K^p_{\beta}(z, w)$ is a reproducing kernel for the holomorphic functions in $L^1((1 - |w|^2)^{\beta}dv(w))$. In other words, if $f$ is holomorphic on $B_n$ and integrable with respect to the measure $(1 - |w|^2)^{\beta}dv(w)$, then

$$f(z) = \int_{B_n} K^p_{\beta}(z, w) f(w) dv(w).$$

**LEMMA 3.4.** [1, Prop. 1.4.10] For $z \in B_n$ and $c$ real, we define

$$I^c(z) = \int S |<z, \zeta|^{n+c} \frac{d\sigma(\zeta)}{\delta(1 - <z, \zeta>)^{n+c}}.$$ 

If $c > 0$, then $I^c(z) - (1 - |z|^2)^{-c}$. 

**LEMMA 3.5.** [7, Lemma 6] If $0 < r, \rho < 1$ and $\alpha - \beta + 1 < 0$, then

$$\int_0^1 (1 - r)^{\alpha}(1 - \rho r)^{\beta - 1} dr \leq C_{\alpha, \beta}(1 - \rho)^{\alpha - \beta + 1}$$

for some positive constant $C_{\alpha, \beta}$.

The next lemma is an easy application of the above two lemmas.

**LEMMA 3.6.** Let $0 < p < 1$ and fix $\beta > \frac{n+\alpha+1}{p} - (n+1) \equiv \sigma$. Then

$$\sup \{ \|J^p_{\beta, c}(w)\|_{p, \sigma} : w \in B_n \} < \infty.$$ 

**PROOF.** We only prove the case $\alpha > -1$. Let $w \in B_n$, and $0 < r < 1$. 

Then we have, by Lemma 3.4 and 3.5,

\[
\|J_{\beta,\sigma}(\omega)\|_{p,\alpha}^P = \int_0^1 \int_S |J_{\beta,\sigma}(\omega)(rz)|^p (1 - r)^{\alpha} 2^{n-1} drd\sigma(t)
\]

\[
\leq C_n,p,\beta(1 - |w|^2)^{(\beta-\sigma)p} \int_0^1 (1 - r)^{\alpha} \int_0^S \frac{d\sigma(t)}{|1 - rz, w|^{(\beta+n+1)p}} dr
\]

\[
\leq C_n,p,\alpha,\beta(1 - |w|^2)^{(\beta-\sigma)p}(1 - |w|^2)^{\alpha+n+1-(\beta+n+1)p}
\]

Thus we have

\[
\sup\{\|J_{\beta,\sigma}(\omega)\|_{p,\alpha}^P : \omega \in B_n\} < \infty.
\]

The proofs of the following theorems are essentially the same as those of [3] (Prop. 4.4 and Prof. 4.5) and are omitted.

**THEOREM 3.7.** Let \(0 < p < 1\) and \(\beta > (n+1)\). Then there exists \(C_n,p,\alpha,\beta < \infty\) such that for each \(f \in A_1^p(B_n)\) there exist a sequence \((\omega_j)\) of points in \(B_n\) and a sequence \((\lambda_j)\) of the complex numbers such that

\[
\sum_j |\lambda_j| \leq C_n,p,\alpha,\beta \|f\|_{1,\sigma}
\]

and

\[
f = \sum_j \lambda_j J_{\beta,\sigma}(\omega_j),
\]

where the last series converges in \(A_1^p(B_n)\).

**THEOREM 3.8.** The Mackey topology of \(A_1^p(B_n)\) is the restriction of the topology of \(A_1^p(B_n)\) where \(\sigma = \frac{n+\alpha+1}{p} - (n+1)\).

4. THE DUAL SPACE OF \(A_1^p(B_n)\).

Finally, we will find the dual space of \(A_1^p(B_n)\). For the proof of this main result, the following definition is needed:

**DEFINITION 4.1.** (Radial fractional derivatives of holomorphic functions in \(B_n\)) Let \(g(z) = \sum_{k=0}^\infty G_k(z)\) be the homogeneous expansion of \(g\). For any real number \(q\), the radial fractional derivative of \(g\) of order \(q\) is defined by

\[
R^q g(z) = \sum_{k=0}^\infty (k+1)^q G_k(z).
\]

Let

\[
f(z) = \sum_{k=0}^\infty F_k(z) = \sum_{k=0}^\infty \sum_{|\gamma|=k} c(\gamma) z^\gamma,
\]

where \(c(\gamma)\) are complex numbers.
and
\[ g(z) = \sum_{k=0}^{\infty} \sum_{\gamma=0}^{\infty} c(\gamma) d(\gamma) \frac{(n-1)!k!}{(n-1+k)!} \frac{2n(k+1)^q}{(k+n)^q} \rho^k \]

be the homogeneous expansions of \( f \) and \( g \), respectively. We note that for \( q > 0 \), \( 0 \leq \rho < 1 \), we have
\[
\sum_{k=0}^{\infty} \sum_{\gamma=0}^{\infty} c(\gamma) d(\gamma) \frac{(n-1)!k!}{(n-1+k)!} \frac{2n(k+1)^q}{(k+n)^q} \rho^k = 2 q \int_0^1 (\log \frac{1}{r})^{q-1} \int S R^{-1} f(r(\gamma)) R^{q+1} g(r(\gamma)) 2nr^{n-1} d\sigma(d). \tag{4.1} \]

We can now prove the duality relation. We use the idea of Ahern [4] in the proof of the following.

**Theorem 4.2.** Let \( 0 < p < 1 \) and \( \sigma = \frac{n+q+1}{p} - (n+1) \). Then
\[
(A_p^\alpha(B_n))^* = \{ f \in H(B_n) : \sup(l - |z|)|R^{q+2}f(z)| = \| f \|_{A_\sigma} < \infty \}. \]

**Proof.** By Theorem 3.8, \((A_\alpha^p)^* = (A_0^1)^* \). It suffices to compute \((A_0^1)^* \). For simplicity we assume \( \sigma = 0 \). Take \( g \) such that
\[
\sup_{z \in B_n} (1 - |z|)|R^2g(z)| < \infty
\]
and let \( f \) be a polynomial. Then by (4.1)
\[
\sum_{k=0}^{\infty} \sum_{\gamma=0}^{\infty} c(\gamma) d(\gamma) \frac{(n-1)!k!}{(n-1+k)!} \frac{2n(k+1)^q}{(k+n)^q} \rho^k
\]
\[
= 2 \int_0^1 \int S R^{-1} f(r(\gamma)) R^{q+2} g(r(\gamma)) 2nr^{n-1} d\sigma(d)
\]
\[
= 4 \int_0^1 \int S R^{-1} (\log \frac{1}{r}) R^2 g(r(\gamma)) 2nr^{n-1} d\sigma(d). \tag{4.2} \]

Hence
\[
\lim_{\rho \to 1} \left( \sum_{k=0}^{\infty} \sum_{\gamma=0}^{\infty} c(\gamma) d(\gamma) \frac{(n-1)!k!}{(n-1+k)!} \frac{2n(k+1)^q}{(k+n)^q} \rho^k \right)
\]
\[
\leq 4 \int_0^1 \int S \frac{1}{1-r} |f(r(\gamma))| \sup_{z \in B_n} (1 - r)|R^2 g(r(\gamma))| 2nr^{n-1} d\sigma(d). \tag{4.2} \]

Since \( \log \frac{1}{r} - 1 - r \) as \( r \to 1 \), (4.2) is dominated by
\[
C_{k,n} \| f \|_{A_0^1} \| g \|_{A_1}.
\]

Since polynomials are dense in \( A_0^1 \), the mapping
\[
\psi(f) = \lim_{\rho \to 1} \left( \sum_{k=0}^{\infty} \sum_{\gamma=0}^{\infty} c(\gamma) d(\gamma) \frac{(n-1)!k!}{(n-1+k)!} \frac{2n(k+1)^q}{(k+n)^q} \rho^k \right)
\]
extends to be a bounded linear functional on $A^1_0$. Conversely, let $\psi \in (A^1_0)^*$. Since $A^1_0 \subset L^1(2nr^{2n-1}drd\sigma(\zeta))$, by the Hahn-Banach theorem $\psi$ extends to be a bounded linear functional $\psi$ on the space $L^1(2nr^{2n-1}drd\sigma(\zeta))$. But since $(L^1)^* = L^\infty$, there exists $G$ in $L^\infty(2nr^{2n-1}drd\sigma(\zeta))$ such that

$$\psi(f) = \int_0^1 \int_S f(r\zeta)\frac{G(r\zeta)}{2nr^{2n-1}drd\sigma(\zeta)}$$

for each $f$ in $A^1_0$. Let

$$H(z) = \int_0^1 \int_S \frac{G(w)}{(1-<z,w>)^{n+1}} 2np^{2n-1}d\sigma(n)$$

be the holomorphic projection of $G$. If $f$ is a holomorphic polynomial, then

$$\psi(f) = \int_0^1 \int_S f(r\zeta)\frac{G(r\zeta)}{2nr^{2n-1}drd\sigma(\zeta)}$$

$$= \int_0^1 \int_S f(r\zeta)\frac{H(r\zeta)}{2nr^{2n-1}drd\sigma(\zeta)}$$

$$= \int_0^1 \int_S R^{-1} f(r\zeta)\frac{\overline{H(r\zeta)}}{R^{-2}(r\zeta)2nr^{2n-1}drd\sigma(\zeta)},$$

where $g$ is defined to be $R^{-1}H$. The proof will be complete if we can show that

$$\sup_{\zeta \in B_n} (1-|\zeta|)|R^1H(z)| < \infty.$$

Since

$$\frac{\partial H(r\zeta)}{\partial r} = \frac{1}{r} \int_0^1 \int_S \frac{(n+1)}{r^2} \frac{<r\zeta, \rho n>G(\rho n)}{(1-<r\zeta, \rho n>)^{n+2}} 2np^{2n-1}d\sigma(n),$$

we have

$$|\frac{\partial H(r\zeta)}{\partial r}| \leq C_n \|G\|_\infty \int_0^1 \int_S \frac{d\sigma(n)}{|1-<r\zeta, \rho n>|^{n+2}} dp$$

$$\leq C_n \|G\|_\infty \int_0^1 \frac{1}{(1-\rho r)^2} dp$$

$$= C_n \|G\|_\infty \frac{1}{1-\rho}.$$  \hspace{1cm} (4.3)

By (4.3) and $R^1H(r\zeta) = \frac{\partial H(r\zeta)}{\partial r} + H(r\zeta)$, we have

$$\sup_{\zeta \in B_n} (1-|\zeta|)|R^1H(z)| < \infty.$$
REFERENCES


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