A NOTE ON RINGS WITH CERTAIN VARIABLE IDENTITIES

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ABSTRACT. It is proved that certain rings satisfying generalized-commutator constraints of the form \([x^m, y^n, y^n, \ldots, y^n] = 0\) with \(m\) and \(n\) depending on \(x\) and \(y\), must have nil commutator ideal.

KEY WORDS AND PHRASES. Commutator ideal, periodic ring.

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1. INTRODUCTION.

Let \([x_1, x_2]\) denote \(x_1 x_2 - x_2 x_1\), and for \(k > 2\), let \([x_1, x_2, \ldots, x_k]\) = 
\([x_1, \ldots, x_{k-1}, x_k]\). For \(x_1 = x\) and \(x_2 = x_3 = \ldots = x_k = y\), denote 
\([x, y, \ldots, y]\) by \([x, y]_k\). A result of Herstein [1] and of Anan'in and Zyabko [2] 
asserts that if for any \(x\) and \(y\) in a ring \(R\), there exist positive integers \(m = m(x, y), n = n(x, y)\) such that \(x^m y^n = y^n x^m\), then the commutator ideal of \(R\) is nil. Recently, 
Herstein [3] proved that a ring \(R\) in which for any \(x, y, z \in R\) there exists positive 
integers \(m = m(x, y, z), n = n(x, y, z),\) and \(q = q(x, y, z)\) such that 
\([x^m, y^n, z^q] = 0\) must have nil commutator ideal. More recently Klein, Nada and 
Bell [4] raised the following conjecture which arises naturally from the above 
mentioned work.

CONJECTURE. Let \(k > 1\). If for each \(x, y \in R\), there exists positive integers \(m\) and 
\(n\) such that \([x^m, y^n]_k = 0\), then the commutator ideal of \(R\) is nil.

In [4], Klein, Nada and Bell proved the conjecture for rings with identity 1.
Given the complexity of [1] and [3], it would appear that no proof of this conjecture 
is in sight. Our objective is to prove the conjecture for certain classes of rings 
and to generalize a result of Herstein in [3] and some results in [4] and [5].

A ring \(R\) is called periodic if for each \(x\) in \(R\), there exists distinct positive 
integers \(m\) and \(n\) for which \(x^m = x^n\). In preparation for the proofs of our main 
theorems, we start with the following lemma which is known [5] and we omit its proof.
LEMMA 1. If $R$ is a periodic ring, then for each $x$ in $R$, there exists a positive integer $k = k(x)$ such that $x^k$ is idempotent.

2. MAIN RESULTS.

The following theorem shows that the conjecture is true for Artinian rings.

THEOREM 1. Let $k > 1$, and let $R$ be an Artinian ring such that for each $x, y$ in $R$, there exists positive integers $m$ and $n$ such that $[x, y]_k = 0$. Then the commutator ideal of $R$ is nil.

PROOF. To prove that the commutator ideal of $R$ is nil it is enough to show that if $R$ has no nonzero nil ideals then it is commutative. So we suppose that $R$ has no nonzero nil ideals. Since $R$ is Artinian, the Jacobson radical $J$ of $R$ is nilpotent. So $J = 0$, and hence $R$ is semisimple Artinian. This implies that $R$ has an identity element and now, $R$ is commutative by Theorem 3 of [4].

Next, we prove Theorem 2 which shows that the conjecture is true for periodic rings. This result generalizes a result of Bell in [5].

THEOREM 2. Let $k > 1$ and let $R$ be a periodic ring such that for each $x, y$ in $R$ there exists positive integers $m$ and $n$ such that $[x^m, y^n]_k = 0$. Then the commutator ideal of $R$ is nil.

PROOF. If $k = 2$, then the result follows by the theorem in [1]. So assume $k > 2$ and let $x$ be any element of $R$ and let $e$ be any idempotent of $R$. By hypothesis, there exists integers $m$ and $n$ such that $[x^m, e]_k = 0$. This implies that $[x^m, e]_k = 0$, and hence $[x^m, e]_k = e[x^m, e]_k$. Multiplying by $e$ from the right and using the fact that $[x^m, e]_k = 0$ we obtain $[x^m, e]_k = 0$. Hence $0 = ([x^m, e]_{k-2} - e[x^m, e]_{k-2})e = [x^m, e]_{k-2}e$. Continuing this way we get $[x^m, e] = 0$ which implies that $x^m e = x^m e$. Similarly, we get $x^m e = x^m e$. This implies that $x^m e = x^m e$, $x \in R$, $e$ any idempotent and $m = m(x, e)$. (2.1)

Let $y$ be any element of $R$. Since $R$ is periodic, Lemma 1 implies that $y^p$ is idempotent for some positive integer $p = p(y)$. So (2.1) implies that for each $x, y$ in $R$ there exists positive integers $m$ and $n$ such that $x^m y^n = y^p x^m$. Now, the result follows by the well-known theorem in [1] or [2].

THEOREM 3. Let $k > 1$. If $R$ is a prime ring having a nonzero idempotent element such that for each $x, y$ in $R$ there exists positive integers $m$ and $n$ such that $[x^m, y^n]_k = 0$. Then $R$ is commutative.

PROOF. The argument used in Theorem 2 to reach statement (2.1) in the proof shows that a ring satisfying the generalized commutator constraint $[x^m, y^n]_k = 0$ must have its idempotent elements in the center. For let $e_1$ and $e_2$ be idempotent elements in $R$. (2.1) implies that $e_1 e_2 = e_2 e_1$ and hence the idempotents of $R$ commute. This implies that the idempotents of $R$ are central in $R$ [6, Remark 2]. Let $e$ be a nonzero
idempotent of \( R \). Then \( e \) is a nonzero central idempotent in the prime ring \( R \). Hence \( e \) is an identity element of \( R \) since it can not be a zero divisor. Now \( R \) is commutative by Theorem 3 of [4].

The proof of Theorem 4 below was done by Kezlan in the proof of his main theorem in [7]. So we omit its proof here.

**THEOREM 4.** Let \( k > 1 \). If \( R \) is a prime ring with a nontrivial center such that for each \( x, y \in R \) there exists positive integers \( m \) and \( n \) such that \( [x^m, y^n]_k = 0 \), then \( R \) is commutative.

The following result generalizes Theorem 1 of [4].

**THEOREM 5.** Let \( R \) be a ring and let \( M \) be a fixed positive integer. Suppose that for each \( x, y \in R \) there exist positive integers \( m = m(x, y) < M \) and \( n = n(x, y) \) such that \( [x^m, y^n] \) belongs to the center of \( R \). Then the commutator ideal of \( R \) is nil.

**PROOF.** Again, we suppose that \( R \) has no nil ideals and hence \( R \) is a subdirect product of prime rings satisfying the above hypothesis of \( R \). So we may assume that \( R \) is prime. Let \( Z \) be the center of \( R \). If \( Z = 0 \), then for each \( x, y \in R \), \( [x^m, y^n] = 0 \) where \( m = m(x, y) < M \), and \( n = n(x, y) \). This implies that \( R \) is commutative by Theorem 4 of [4]. So we may assume that \( R \) has a nontrivial center, and hence \( R \) is commutative by Theorem 4 above.

The following result generalizes Theorem 8 in [3].

**THEOREM 6.** Let \( R \) be a ring in which, for each \( x, y, z \in R \), there exists positive integers \( m = m(x, y, z) \) \( n = n(x, y, z) \) and \( q = q(x, y, z) \) such that \( [x^m, y^n, z^q] \) belongs to the center of \( R \). Then the commutator ideal of \( R \) is nil.

**PROOF.** Again, we may assume that \( R \) is a prime ring satisfying the above hypothesis. Let \( Z \) be the center of \( R \). If \( Z = 0 \), then for each \( x, y, z \in R \), \( [x^m, y^n, z^q] = 0 \), where \( m = m(x, y, z) \), \( n = n(x, y, z) \) and \( q = q(x, y, z) \). This implies that \( R \) is commutative by Theorem 8 of [3]. So we may assume that \( R \) has a nontrivial center. For any \( x, y \in R \), \( [[x^m, y^n], y^q] \in Z \) where \( m, n, q \) are each functions of the variables \( x \) and \( y \). So \( [[[x^m, y^n], y^q], y] = 0 \), which implies that \( [[[x^m, y^n], y^q], y^{nq}] = 0 \). Hence \( R \) is commutative by Theorem 4 above.

**REMARK.** The result in Theorem 6 can be generalized as follows. Let \( R \) be a ring such that for each \( x, y, z \in R \), there exists positive integers \( m = m(x, y, z) \) \( n = n(x, y, z) \) and \( q = q(x, y, z) \) such that \( [x^m, y^n, z^q, r_1, r_2, \ldots, r_k] = 0 \) for all elements \( r_1, \ldots, r_k \) in \( R \). Then the commutator ideal of \( R \) is nil. This can be done by induction on \( k \) and using the argument in Theorem 6. We omit the details of the proof.

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