ABSTRACT. The main result we obtain is that given \( \pi : N \to M \) a \( T^a \)-subbundle of the generalized Hopf fibration \( \hat{\pi} : H^{2n+s} \to CP^n \) over a Cauchy-Riemann product \( i : M \subseteq CP^n \), i.e. \( I : N \subseteq H^{2n+s} \) is a diffeomorphism on fibres and \( \hat{\pi} \circ i = i \circ \pi \), if \( s \) is even and \( N \) is a closed submanifold tangent to the structure vectors of the canonical \( \mathcal{R} \) structure on \( H^{2n+s} \) then \( N \) is a Cauchy-Riemann submanifold whose Chen class is non-vanishing.

KEY WORDS AND PHRASES. Principal toroidal bundle, \( \mathcal{R} \) manifold, generalized Hopf fibration, framed C.R. submanifold, characteristic form.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE: 53C40, 53C55

1.- INTRODUCTION AND STATEMENT OF RESULTS.

As a tentative of unifying the concepts of complex and anti-invariant submanifolds of an almost Hermitian manifold, A. BEJANCU, [1], has introduced the notion of Cauchy-Riemann (C.R.) submanifold. This has soon proved to possess a largely rich number of geometrical properties; e.g. by a result of D.E.BLAIR & B. Y.CHEN, [2], any C.R. submanifold of a Hermitian manifold is a Cauchy-Riemann manifold, in the sense of A.ANDREOTTI & C.D.HILL, [3].

Let \( M^{2n+s} \) be a real \((2n+s)\)-dimensional manifold carrying a metrical \( f \)-structure \((f, \xi, \eta, \mathcal{R})\), \( 1 \leq a \leq s \), with complemented frames, cf. [4]. A submanifold \( j : N \to M^{2n+s} \) is said to be a **framed C.R. submanifold** if it is tangent to each structure vector \( \xi_a \) of \( M^{2n+s} \) and it carries a pair of complementary (with respect to \( G = J^a \mathcal{R} \)) smooth distributions \( \mathcal{D}, \mathcal{D}^\perp \) such that \( f_x (\mathcal{D}^\perp) \subseteq \mathcal{D}^\perp \), \( f_x (\mathcal{D}^\perp) \subseteq T_x (N)^\perp \), for all \( x \in N \), where \( T(N)^\perp \to N \) stands for the normal bundle of \( j \). Cf. I.MIHAI, [5], L.ORNEA, [6]. Since \( f \)-structures are known to generalize both almost complex \((a=0)\) structures and almost contact \((a=1)\) structures, the notion of framed C.R. submanifold contains those of a C.R. submanifold (see e.g. [7], p.83) of an almost Hermitian manifold and of a
contact C.R. submanifold (see e.g. [7], p.48) of an almost contact metrical manifold.

Let \( \tilde{\pi} : H^{2n+s} \to \mathbb{C}P^n \) be the generalized Hopf fibration, as given by D.E. Blair, [8]. Leaving definitions momentarily aside we may formulate the following:

**THEOREM A**

i) Let \( N \) be a framed C.R. submanifold of an \( \mathbb{S} \)-manifold \( M^{2n+s} \). Then the \( \tau \)-anti-invariant distribution \( D^\perp \) of \( N \) is completely integrable.

ii) Any framed C.R. submanifold of \( H^{2n+s} \), (carrying the standard \( \mathbb{S} \)-structure) is either a C.R. submanifold (\( s \) even) or a contact C.R. submanifold (\( s \) odd). The converse holds.

iii) Let \( N \) be an \( \tau \)-invariant (i.e. \( D^\perp = 0 \)) submanifold of \( H^{2n+s} \). Then \( N \) is totally-geodesic if and only if it is an \( \mathbb{S} \)-manifold of constant \( \tau \)-sectional curvature \( 1 - \frac{3}{4} s \).

iv) Any \( \tau \)-invariant submanifold of \( H^{2n+s} \) having a parallel second fundamental form is totally-geodesic.

It is known that compact regular contact manifolds are \( S^1 \)-principal fibre bundles over symplectic manifolds, cf. W.M. Boothby & H.C. Wang, [9]. Eversince this (today classical) paper has been published, several "Boothby-Wang type" theorems have been established, cf. e.g. A. Morimoto, [10], for the case of normal almost contact manifolds, S. Tanno, [11], for contact manifolds in the non-compact case; more recently, we may cite a result of I. Vaisman, [12], asserting that compact generalized Hopf manifolds with a regular Lee field may be fibred over Sasakian manifolds, etc.

There exists today a large literature, cf. K. Yano & M. Kon, [7], concerned with the study of the geometry (of the second fundamental form) of a C.R. submanifold of a Kaehlerian ambient space. In particular, following the method of Riemannian fibre bundles (such as introduced by H.B. Lawson, [13], towards studying submanifolds of complex space-forms, and developed successively by Y. Maeda, [14], M. Okumura, [15]), K. Yano & M. Kon, [16], have taken under study contact C.R. submanifolds of a Sasakian manifold \( M^{2n+1} \) (where \( M^{2n+1} \) is previously fibred over a Kaehlerian manifold \( M^2 \) which are themselves \( S^1 \)-fibrations over C.R. submanifolds of \( M^2 \).

The last piece of the mosaic we are going to mend is the concept of canonical cohomology class (here after referred to as the Chen class) of a C.R. submanifold. Cf. B.Y. Chen, [17], with any C.R. submanifold \( M \) of a Kaehlerian manifold there may be associated a cohomology class \( c(M) \in H^p(M; \mathbb{R}) \), where \( p \) stands for the complex dimension of the holomorphic distribution of \( M \). Although the canonical Hermitian structure (cf. [18]) of \( H^{2n+s} \) is never Kaehlerian (cf. [8], p.174) we show that the Chen class of a C.R. submanifold may be constructed as well and obtain the following:

**THEOREM B**

Let \( j : N \to H^{2n+s} \) be a closed (i.e. compact, orientable) submanifold tangent to the vector fields \( \xi_a \), \( 1 \leq a \leq s \), of the canonical \( \mathbb{S} \)-structure on \( H^{2n+s} \) and assume there exists a \( T^s \)-principal bundle \( \pi : N \to M \) over a Cauchy-
Riemann product $(M, \mathcal{D}, \mathcal{D}^\perp)$, $i: M \to \mathbb{CP}^n$, $(\mathcal{D}$ is the holomorphic distribution), such that $\tilde{\pi} \circ j = i \circ \pi$ and $j$ is a diffeomorphism on fibres. If $s$ is even then $N$ is a C.R. submanifold whose totally-real foliation is normal to the characteristic field of $\mathbb{H}^{2n+s}$ and whose Chen class $c(N) \in H^{2n+s}(N; \mathbb{R})$, $p = \dim_c \mathcal{D}$, is non-vanishing.

2.- NOTATIONS, CONVENTIONS AND BASIC FORMULAE.

Let $M^{2n+s}$ be a real $(2n+s)$-dimensional $C^\infty$-differentiable connected manifold. Let $\mathcal{F}$ be an $\mathcal{F}$-structure on $M^{2n+s}$, i.e. a $(1,1)$-tensor field such that $\mathcal{F}^2 + \mathcal{F} = 0$ and rank($\mathcal{F}$) = $2n$ everywhere on $M^{2n+s}$, cf. K.YANO, [19]. Assume that $\mathcal{F}$ has complemented frames, i.e. there exist the differential 1-forms $\eta^a$ and the dual vector fields $\xi^a$ on $M^{2n+s}$, i.e. $\eta^a(\xi^b) = \delta^a_b$, $1 \leq a, b \leq s$, such that the following formulae hold:

$$\eta^a \circ \mathcal{F} = 0, \quad \mathcal{F}(\xi^a) = 0, \quad \mathcal{F}^2 = -I + \eta^a \otimes \xi^a.$$  \hspace{1cm} (2.1)

Throughout, one adopts the convention $\eta^a = \eta^{*a}$, $\xi^a = \xi^{*a}$. The $\mathcal{F}$-structure is normal if $[\mathcal{F}, [\mathcal{F}, \mathcal{F}] + (d\eta^a) \otimes \xi^a = 0$, where $[\mathcal{F}, \mathcal{F}]$ denotes the Nijenhuis torsion of $\mathcal{F}$, see e.g. H.NAKAGAWA, [20]. Let $\mathcal{B}$ be a compatible Riemannian metric on $M^{2n+s}$, i.e. one satisfying:

$$\mathcal{B}(\mathcal{F}X, \mathcal{F}Y) = \mathcal{B}(X, Y) - \eta^a(X) \eta^{*a}(Y).$$ \hspace{1cm} (2.2)

Compatible metrics always exist, cf. D.E.BLAIR, [4]. Such $(\mathcal{F}, \xi^a, \eta^a, \mathcal{B})$ has often been called a \textit{metrical $\mathcal{F}$-structure with complemented frames}. Let $\mathcal{F}(X, Y) = \mathcal{B}(X, Y)$ be its fundamental 2-form. Throughout we assume $M^{2n+s}$ to be an $\mathcal{P}$-manifold, cf. the terminology in [4], i.e. the given $\mathcal{F}$-structure is normal, its fundamental 2-form is closed and there exist $s$ smooth real-valued functions $\alpha^a \in C^\infty(M^{2n+s})$, $1 \leq a \leq s$, such that:

$$d \eta^a = \alpha^a \mathcal{F}.$$ \hspace{1cm} (2.3)

We shall need, cf. [4], [21], the following result. Let $M^{2n+s}$, $n > 1$, be a connected manifold carrying the $\mathcal{P}$-structure $(\mathcal{F}, \xi^a, \eta^a, \mathcal{B})$, $1 \leq a \leq s$. Then $\alpha^a$ are real constants, $\xi^a$ are Killing vector fields (with respect to $\mathcal{B}$) and the following relations hold:

$$\mathcal{D}_X \xi^a = -\frac{1}{2} \alpha^a \mathcal{F} X$$ \hspace{1cm} (2.4)

$$(\mathcal{D}_X \mathcal{F} Y = \frac{1}{2} \alpha^a \{[\mathcal{B}(X, Y) - \eta^a(X) \eta^{*b}(Y)] \xi^a - [X - \eta^a(X) \xi^b] \eta^{*b}(Y)\}$$ \hspace{1cm} (2.5)

for any tangent vector fields $X$, $Y$ on $M^{2n+s}$. Here $\mathcal{D}$ denotes the Riemannian connection of $(M^{2n+s}, \mathcal{B})$, and $\alpha^a = \alpha^a$, $1 \leq a \leq s$.

Let $M^{2n+s}$ be an $\mathcal{P}$-manifold with the structure tensors $(\mathcal{F}, \xi^a, \eta^a, \mathcal{B})$. Let $\mathcal{A}$ be the smooth s-distribution on $M^{2n+s}$ spanned by $\xi^a$, $1 \leq a \leq s$. By normality one has $[\xi^a, \xi^b] = 0$, i.e. $\mathcal{A}$ is involutive. If both $\mathcal{A}$ and the structure vector fields $\xi^a$ are regular (in the sense of R.PALAIS, [22]) then the $\mathcal{P}$-structure itself is termed \textit{regular}. We shall need the main result of D.E. BLAIR & G.D.LUDDEN & K.YANO, ([21], p.175). That is, let $M^{2n+s}$ be a compact connected $(2n+s)$-dimensional, $n > 1$, $\mathcal{P}$-manifold; then there is a $T^a$-principal fibre bundle $\tilde{\pi}: M^{2n+s} \to M^{2n} = M^{2n+s}/\mathcal{A}$ and $M^{2n}$ is a Kaehlerian
manifold. Here $M^{2n}$ denotes the leaf space of the $s$-dimensional foliation $\mathcal{F}$ and $T^a$ is the $s$-torus. Also, cf. ([21], p.178), $\gamma = (\eta_1^*, \ldots, \eta_s^*)$ is a connection 1-form in $M^{2n+s}(\tilde{\pi}, T^a)$. If $X$ is a tangent vector field on $M^{2n}$, let $X^H$ denote its horizontal lift with respect to $\gamma$. The Kaehlerian structure $(J, g)$ of $M^{2n}$ is expressed by:

$$ J X = \tilde{\pi}_* f X^H $$

(2.6)

$$ \tilde{g}(X, Y) = \Theta(X^H, Y^H). $$

(2.7)

Let $\mathcal{D}$ be the smooth 2n-distribution on $M^{2n+s}$ defined by the Pfaffian equations $\eta_a^* = 0$, $1 \leq a \leq s$. Then $\mathcal{D}$ is precisely the horizontal distribution of $\gamma$. Since $\eta_a^* \circ f = 0$, the $f$-structure preserves the horizontal distribution. Therefore (2.6) may be also written $(J X)^H = f X^H$. Let $\nabla$ be the Riemannian connection of $(M^{2n}, \tilde{g})$. By ([21], p.179) one has:

$$ D_X Y^H = (\nabla_X Y)^H + \frac{1}{2} \alpha^a \Theta(f X^H, Y^H) \xi_a^* . $$

(2.8)

REMARK

Let $\pi : N \to M$ be a Riemannian submersion, cf. B.O'NEILL, [23]. Then $\text{Ker}(\pi_*)$ is the vertical distribution, while its complement (with respect to the Riemannian metric of $N$) is the horizontal distribution of the Riemannian submersion. As to our $\tilde{\pi} : M^{2n+s} \to M^{2n}$ a number of important coincidences occur. Firstly, if $M^{2n}$ is assigned the Riemannian metric (2.7), then $M^{2n+s} \to M^{2n}$ is a Riemannian submersion. Moreover $\mathcal{F} = \text{Ker}(\tilde{\pi}_*)$ and therefore the horizontal distribution of the Riemannian submersion is precisely $\mathcal{D}$.

Let $N$ be an $(m+s)$-dimensional submanifold of $M^{2n+s}$, and $M$ an $m$-dimensional submanifold of $M^{2n}$, such that there exists a fibering $\pi : N \to M$ such that $\tilde{\pi} \circ j = i \circ \pi$ and $j$ is a diffeomorphism on fibres. Both $i : M \to M^{2n}$, $j : N \to M^{2n+s}$ stand for canonical inclusions. Let $g = i^* \tilde{g}$, $G = j^* \Theta$ be the induced metrics on $M$ and $N$, respectively. Also we denote by $\nabla, D, V, W$ the corresponding Riemannian connections of $(M, g)$ and $(N, G)$, respectively. Let $B$ (resp. $h$) be the second fundamental form of $i$ (resp. $j$) and denote by $A$ (resp. $W$) the Weingarten forms. Let $T(M)^\perp \to M$ (resp. $T(N)^\perp \to N$) be the normal bundle of $i$ (resp. $j$). We put $\xi_a = \tan(\xi_a^*)$, $\xi_a^\perp = \nor(\xi_a^*)$, where $\tan_x$, $\nor_x$ stand for the projections associated with the direct sum decomposition $T_x(M^{2n+s}) = T_x(N) \oplus T_x(N)^\perp$, $x \in N$. Then the Gauss and Weingarten formulae, (cf. e.g. [24],p.39-40), of $i, j$ and our (2.8) lead to:

$$ D_X Y^H = (\nabla_X Y)^H + \frac{1}{2} \alpha^a \Theta(f X^H, Y^H) \xi_a^* $$

(2.9)

$$ h(X^H, Y^H) = B(X, Y)^H + \frac{1}{2} \alpha^a \Theta(f X^H, Y^H) \xi_a^* $$

(2.10)

$$ W_V Y^H = (A_V X)^H - \frac{1}{2} \alpha^a \Theta(f X^H, V^H) \xi_a^* $$

(2.11)

$$ D_X Y^H = (\nabla_X Y)^H + \frac{1}{2} \alpha^a \Theta(f X^H, V^H) \xi_a^\perp $$

(2.12)

for any tangent vector fields $X, Y$ on $M$, respectively any cross-section $V$ in $T(M)^\perp \to M$. Here $\nabla, D, V$ stand for the normal connections of $i, j$. Of course, towards obtaining our (2.9) - (2.12) one exploits the fact that $(i_* X)^H$ is tangent to $N$, while $V^H$ is a cross-section in $T(N)^\perp \to N$. 

---

**L.M. ABATANGELO AND S. DRAGOMIR**
REMARKS

1) Let $H(i) = \frac{1}{m} \text{Trace}(B)$ (resp. $H(j) = \frac{1}{m+s} \text{Trace}(h)$) be the mean curvature vector of $i$ (resp. $j$). As an application of our (2.9) - (2.12) one may derive:

$$(m+s) H(j) = m H(i) + \sum_{a=1}^{s} \left[ \frac{1}{2} \alpha^a \text{nor}(\xi_{a}) - D_{\xi_{a}} \xi_{a} \right]$$  \hspace{1cm} (2.13)

provided that $\{\xi_{a} : 1 \leq a \leq s\}$ consists of mutually orthogonal unit vector fields. In particular, if $N$ is tangent to each structure vector $\xi_{a}$, then $N$ is minimal if and only if $M$ is minimal. Indeed, if $X$ is tangent to $N$, then (2.4) and the Gauss - Weingarten formulae lead to:

$$D_{X} \xi_{a} = W_{X} \perp X - \frac{1}{2} \alpha^a \tan(f X)$$ \hspace{1cm} (2.14)

$$h(X, \xi_{a}) + D_{X} \xi_{a} = - \frac{1}{2} \alpha^a \text{nor}(f X).$$ \hspace{1cm} (2.15)

Now, if $\{\xi_{a} : 1 \leq a \leq s\}$ are orthonormal, one uses a frame $\{X_{i} \xi_{a}^{H}\}$ (where $\{X_{i} : 1 \leq i \leq m\}$ is an orthonormal tangential frame of $M$) such as to compute $H(j)$.

2) Generally, if $N$ is a submanifold of the manifold $M^{2n+s}$ and $N$ is normal to some $\xi_{a}$ with $\alpha_{a} = 0$ then tangent spaces at points of $N$ are $\mathbb{F}$-anti-invariant, i.e. $\xi_{a}(T_{x}(N)) \subseteq T_{x}(N) \perp x \in N$. Indeed, by (2.4) and the Weingarten formula of $N$ in $M^{2n+s}$, one has $\Theta(\alpha_{a}, f X, Y) = -2 \Theta(D_{X} \xi_{a}, Y) = 2 \Theta(W_{X} \perp X, Y)$ where from $W_{X} \perp = 0$ and $fX$ is normal to $N$.

3. MANIFOLDS AS HERMITIAN OR NORMAL ALMOST CONTACT METRICAL MANIFOLDS.

We denote by $cP^{n}$ the complex projective space with constant holomorphic sectional curvature 1 (with Fubini - Study metric) and complex dimension $n$, and by $S^{2n+1}$ the $(2n+1)$-dimensional unit sphere carrying the standard Sasakian structure. Let $\pi^{1} : S^{2n+1} \to cP^{n}$ be the Hopf fibration and set

$$H^{2n+s} = \{(p_{1}, \ldots, p_{s}) \in S^{2n+1} \times \cdots \times S^{2n+1} | \pi^{1}(p_{a}) = \cdots = \pi^{1}(p_{a})\}.$$

We define a principal toroidal bundle by the commutative diagram:

$$\begin{array}{ccc}
H^{2n+s} & \xrightarrow{\hat{\Lambda}} & S^{2n+1} \times \cdots \times S^{2n+1} \\
\pi \downarrow & & \pi^{1} \times \cdots \times \pi^{1} \\
cP^{n} & \xrightarrow{\Lambda} & cP^{n} \times \cdots \times cP^{n}
\end{array}$$

where $\Lambda$ denotes the diagonal map, while $\hat{\Lambda}$ stands for the canonical inclusion. Let $\eta^{'}$ be the standard contact 1-form on $S^{2n+1}$. We put $\eta^{\#} = \hat{\Lambda}^{*} \Lambda^{*} \eta^{'}$, 1 \leq a \leq s$ where $\Lambda_{a} : S^{2a+1} \to S^{2a+1}$ are natural projections. Let $\Omega$ be the Kaehler 2-form of $cP^{n}$. Then on one hand $\gamma = (\eta, \ldots, \eta)$ is a connection 1-form in $H^{2n+s}(cP^{n}, \pi, T^{*})$, and on the other $d\eta^{'} = \pi^{*} \Omega$, such that one may apply theorem 3.1 of [8], (p.163) such as to yield a natural $\mathcal{F}$-structure on $H^{2n+s}$. (Cf also [4], p.173). Let $(f, \xi_{a}, \eta_{a}, \Theta)$ be the canonical $\mathcal{F}$-structure
of $H^{2n+s}$. If $s$ is even one sets:
\[ \mathcal{J} = \xi + \sum_{i=1}^{s/2} \{ \eta_i \circ \xi^* \circ \eta_i \circ \xi^* \} \]  
(3.1)
where $i^* = i + \frac{s}{2}$, $1 \leq i \leq \frac{s}{2}$. If $s$ is odd, one labels the 1-forms $\eta_i$ as follows:
\[ \eta_0, \eta_1, \eta_{s-1}, i^* = i+r, 1 \leq i \leq r, \ s = 2r+1, \]  
and similarly for the tangent vector fields $\xi_i$. We consider:
\[ \varphi = \xi + \sum_{i=1}^{r} \{ \eta_i \circ \xi^* \circ \eta_i \circ \xi^* \}. \]  
(3.2)
The characteristic 1-form of $H^{2n+s}$, $s$ even, is defined by:
\[ \omega = 2 \sum_{i=1}^{s/2} \{ \eta_i \circ \eta_i^* \}. \]  
(3.3)
Let $B = \omega^+$ be the characteristic field, where $\pm$ means raising of indices by $\mathcal{B}$.

REMARKS
1) If $s$ is even then $(H^{2n+s}, \mathcal{J}, \mathcal{B})$ is a Hermitian non-Kählerian manifold and its characteristic form is parallel. Indeed, if $s$ is even, then $\mathcal{J}$ given by (3.1) is a complex structure and $(H^{2n+s}, \mathcal{J}, \mathcal{B})$ turns to be a Hermitian manifold, (cf. prop.4.1 in [8], p.174). Let $F(X, Y) = \mathcal{B}(X, \mathcal{J} Y)$ be its Kaehler 2-form. By (3.1) it follows that $F = F - 2 \sum_{i=1}^{s/2} \eta_i \wedge \eta_i^*$; consequently (3.3) leads to
\[ dF = \omega \wedge F \]  
(3.4)
i.e. $\mathcal{B}$ is not a Kaehler metric. Now our (2.4) yields $\mathcal{D} \omega = \frac{1}{2} \sum_{i=1}^{s/2} (\alpha_i - \tilde{\alpha}_i) F$ on an arbitrary $\mathcal{F}$-manifold, provided $s$ is even. Yet for $H^{2n+s}$ one has $\alpha_i = \ldots = \alpha_s$ (cf.[8],p.173), i.e. $\omega$ is parallel.
2) Since $d \eta_i^* = \tilde{\pi}^* \Omega, 1 \leq a \leq s$, it follows that $\omega$ is closed. Therefore $H^{2n+s}$, $s$ even, admits the canonical foliation $\mathcal{F}$ defined by the Pfaffian equation $\omega = 0$. Each leaf of $\mathcal{F}$ is a totally-geodesic real hypersurface normal to the characteristic field of $H^{2n+s}$.
3) Consider the submanifolds $i : M \to \mathbb{C}P^n$ and $j : N \to H^{2n+s}$ and assume that a $T^s$-subbundle $\pi : N \to M$ of the generalized Hopf fibration, i.e. $\tilde{\pi} \circ j = i \circ \pi$ and $j$ is a diffeomorphism on fibres. Suppose $N$ is tangent to the structure vectors $\xi_i$ of the $\mathcal{B}$-manifold $H^{2n+s}$. Then $M$ is a C.R. submanifold of $\mathbb{C}P^n$ if and only if $N$ is either a C.R. submanifold of $(H^{2n+s}, \mathcal{J}, \mathcal{B})$ or a contact C.R. submanifold of $(H^{2n+s}, \varphi, \xi_i, \eta_i, \mathcal{B})$. Note firstly that, if $s$ is odd, then $(\varphi, \xi_i, \eta_i, \mathcal{B})$ is a normal almost contact metrical (a. ct. m.) structure on $H^{2n+s}$, (cf. [8], p.175). If $\xi_i = 0, 1 \leq a \leq s$, and $s$ is even then:
\[ \mathcal{J} \xi_i = \xi_i, \quad \mathcal{J} \xi_i^* = - \xi_i, \quad \mathcal{J} X^H = (J X)^H \]  
(3.5)
for any tangent vector field $X$ on $M$, cf.(2.6). Let us define $\mathcal{P} Y = \tan (\mathcal{J} Y), \mathcal{P}^\perp Y = \nor (\mathcal{J} Y)$, for any tangent vector field $Y$ on $N$. Then:
\[ \mathcal{P}^\perp \mathcal{J} \xi_i = 0, \quad \mathcal{P}^\perp \mathcal{J} \xi_i^* = 0, \quad \mathcal{P}^\perp \mathcal{J} X^H = (F P X)^H \]  
(3.6)
where $F, P$ are defined by (1.1) in [7] (p.76). Suppose for instance that $(M, \mathcal{B}, \mathcal{P}^\perp)$ is a C.R. submanifold of $\mathbb{C}P^n$. Then $P$ is $\mathcal{B}$-valued, while $F$ vanishes on
4. Framed Cauchy-Riemann Submanifolds

S. Goldberg, [25], has inaugurated a program of unifying the treatment of the cases $s$ even, and $s$ odd, and studied $f$-invariant submanifolds of codimension 2 of an $\mathcal{F}$-manifold. To make the terminology precise, let $(N, \mathcal{D}, \mathcal{D}^\perp)$ be a framed C.R. submanifold of $M^{2n+1}$; we call $N$ an $f$-invariant (resp. $f$-anti-invariant) submanifold if $\mathcal{D}_x(0)$ (resp. if $\mathcal{D}_x(0)$), for any $x \in N$.

Let $M^{2n+1}$ be an $\mathcal{F}$-manifold; let $x \in M^{2n+1}$ and $p \in T_x(M^{2n+1})$ a 2-plane. (Cf. [8], p.159), $p$ is an $f$-section if it is spanned by $\{X, fX\}$ for some unit tangent vector $X \in T_x$. The Riemannian sectional curvature of $(M^{2n+1}, \mathcal{F})$ restricted to $f$-sections is referred to as the $f$-sectional curvature of the $\mathcal{F}$-manifold. (Cf. also [21], p. 183).

At this point we may establish i) of theorem A. Let $X, V$ be respectively a tangent vector field on $N$ and a cross-section in $T(N)^\perp \to N$. We set $P X = \tan(f X)$, $F X = \nor(f V)$ and $f V = \nor(f V)$, $j X = \nor(f V)$. The following identities hold as direct consequences of definitions:

$$P^2 + f F = -I + \eta^\perp \cdot \xi^\perp, \quad F P + f F = 0, \quad P t + t f = 0,$$

$$F t + f \xi^\perp = -I, \quad f \xi^\perp = P \xi^\perp, \quad F \xi^\perp = 0,$$

$$P \xi^\perp = 0.$$  \hspace{1cm} (4.1)

Using (2.5) and the Gauss-Weingarten formulae of $N$ in $M^{2n+1}$ one obtains:

$$(D_X) Y = W_{FY} X + t h(X, Y) +$$

$$+ \frac{1}{2} \alpha^s \{[G(X, Y) - \eta^s(X) \eta^s(Y)] \xi^\perp - [X - \eta^s(X) \xi^\perp] \eta^s(Y)\}$$  \hspace{1cm} (4.2)

for any tangent vector fields $X, Y$ on $N$. Let $X, Y \in \mathcal{D}^\perp$. As $D$ is torsion-free.
and by (4.2) one obtains:

\[ P[X, Y] = W_{FY}X - W_{FY} Y + \alpha^a \{ \frac{1}{2} (X \wedge Y) \xi_a + (\eta_a \wedge \eta_b) (X, Y) \xi^b \} \] (4.3)

At this point we may establish the following:

**LEMMA**

Let \( (N, \mathcal{D}, \mathcal{D}^\perp) \) be a framed C.R. submanifold of the \( \mathcal{P} \)-manifold \( M^{2n+s} \). Then:

\[ W_{FY}X = W_{FY} X + \alpha^a \{ \eta_a(X) Y - \eta_a(Y) X - [\eta_a(X) \eta_b(Y) - \eta_a(Y) \eta_a(X)] \xi^b \} \] (4.4)

for any \( X, Y \in \mathcal{D}^\perp \).

**Proof.** By (4.1), \( P \) vanishes on \( \mathcal{D}^\perp \). Using (4.2) for any \( X, Y \in \mathcal{D}^\perp, Z \in T(N) \), one has:

\[ 0 = G((D_Z P)X, Y) = G(W_{FY}Z, Y) + G(\theta(Z, X), Y) + \frac{1}{2} \alpha^a \{ G(Z, X) \eta_a(Y) - G(Z, Y) \eta_a(X) + [\eta_a(X) \eta_b(Y) - \eta_a(Y) \eta_a(X)] \eta_b(Z) \} \]

and finally \( G(\theta(Z, X), Y) = - G(W_{FY}X, Z) \) leads to (4.4).

By (4.3) and the above lemma we conclude \( P[X, Y] = 0 \), i.e. \( D^\perp \) is involutive.

Let us prove now ii) in theor. A. We analyse for instance the case \( s \) even. Let \( N \) a framed C.R. submanifold of \( H^{2n+s} \). Let

\[ \mathcal{F} = P + \sum_{i=1}^{s/2} \eta_i \circ \xi_i, \quad \mathcal{F}^\perp = F \] (4.5)

Next \( \mathcal{F}^\perp = F \), \( P = 0 \), and one applies theor.3.1 of [7], p.87. The case \( s \) odd being similar is left as an exercise to the reader. To prove the converse of ii) in theor.A we need to characterize framed C.R. submanifolds as follows.

Let \( N \) be a framed C.R. submanifold of an \( \mathcal{F} \)-manifold \( M^{2n+s} \). Then (4.1) leads to \( P \neq P, F P = 0, f F = 0 \), etc. One obtains the following statement. Let \( N \) be a submanifold of the \( \mathcal{F} \)-manifold \( M^{2n+s} \) such that \( N \) is tangent to the structure vectors \( \xi_i \). Then \( N \) is a framed C.R. submanifold of \( M^{2n+s} \) if and only if \( P \neq 0 \). We have proved the necessity already. Viceversa, let us put by definition \( \ell = - P^2 + \eta_a \circ \xi_a \), \( \ell^\perp = 1 - \ell \). Since \( F P = 0 \), the projections \( \ell, \ell^\perp \) make \( N \) into a framed C.R. submanifold, Q.E.D. Now the converse of ii) in theor. A is easily seen to hold, i.e. both C.R. submanifolds of \( (H^{2n+s}, \mathcal{F}, \mathcal{D}) \), \( s \) even, and contact C.R. submanifolds of \( (H^{2n+s}, \varphi, \xi_0, \eta_0, \mathcal{D}) \), \( s \) odd, are framed C.R. submanifolds.

**REMARKS**

1) Let \( (N, \mathcal{D}, \mathcal{D}^\perp) \) be a framed C.R. submanifold of \( H^{2n+s} \). By (4.5) one obtains:

\[ \mathcal{F}^2 = P^2 - \eta_a \circ \xi_a \] (4.6)

Now the notion of framed C.R. submanifold appears to be essentially on old concept. For not only \( N \) becomes a C.R. submanifold of the Hermitian manifold \( H^{2n+s} \), if for instance \( s \) is even, but its holomorphic and totally-real distributions are precisely \( \mathcal{D}, \mathcal{D}^\perp \). Indeed, by (4.6) one has \( \ell \simeq \ell, \quad Q.E.D. \)

2) Due to (3.4) there is a certain similarity between \( \mathcal{F} \)-manifolds and locally conformal Kaehler manifolds, cf. P.LIBERMANN, [26]. See also [12]. For instance, we may use the ideas in [2] (cf. also theor. 3.4 of [7], p.89) to
give an other proof of the integrability of the f-anti-invariant distribution of a framed C.R. submanifold. Indeed, let \( N \) be a framed C.R. submanifold of \( H^{2n^+} \), \( s \) even. Let \( X \in \mathcal{D}, \ Z, W \in \mathcal{D}^\perp \). By (3.4) one has \( \widehat{0} = 3(d F)(X, Y, W) = - G([Z, W], J X) \). Hence \( [Z, W] \in \mathcal{D}^\perp \). Note that, although \( N \) is C.R. in the usual sense one could not apply theor.3.4 or theor.4.1 of [7] (p.89-90) since \( H^{2n^+} \) is neither locally conformal Kaehler nor Kaehler.

To establish iii) let \( N \) be an f-invariant submanifold of \( H^{2n^+} \). As a consequence of (2.5), for any tangent vector fields \( X, Y \) on \( N \) one has:

\[
(D_X f) Y = \frac{1}{2} \alpha_s \{ [G(X, Y) - \eta_b(X) \eta_b(Y)] \xi_s - [X - \eta_b(X) \xi_s] \eta_b(Y) \} \quad (4.7)
\]

\[
h(X, f Y) = f h(X, Y). \quad (4.8)
\]

Let \( k(X, Y) \) be the Riemannian sectional curvature of the 2-plane spanned by the orthonormal pair \( \{X, Y\} \) on \( N \); using the Gauss equation, i.e. equation (2.6) in [24], (p.45), and the notations in [4], (p.161), i.e. \( H(X) = k(X, fX), X \in \mathcal{D}, \ X \in \mathcal{D}^\perp \), one obtains:

\[
1 - \frac{1}{4} s = H(X) + 2 \| h(X, X) \|^2 \quad (4.9)
\]

as \( H^{2n^+} \) has constant f-sectional curvature, (cf.[8], p.173). By (2.15) and f-invariance one has \( h(X, \xi_s) = - \frac{1}{2} \alpha_s \text{nor}(X) = 0 \); a standard argument based on (4.8) leads to the proof.

To prove iv) one uses \( D h = 0 \), (2.15) and f-invariance, i.e. one has \( h((D_X \xi_s, Y) = 0. \) Thus \( \alpha_s h(f X, Y) = 0, \) by (2.14). For some \( \alpha_s = 0 \) one uses (4.7). Finally, apply once more \( f \) and notice that \( \eta_s \) vanish on normal vectors. Thus \( h = 0. \)

REMARK

Let \( \mathcal{F} \) be the canonical foliation of \( H^{2n^+} \). Let \( N \) be a framed C.R. submanifold of \( H^{2n^+} \), as above. Then \( \mathcal{D}^\perp \subseteq \mathcal{F}, \) i.e. the totally-real foliation of \( N \) (regarded as a C.R. submanifold, \( s \) even) is normal to the characteristic field \( \sum_{s=1}^{2} (\xi_s) \) of \( H^{2n^+} \). Indeed, since \( \xi_s \in \mathcal{D}^\perp \), the \( \eta_s \) vanish on \( \mathcal{D}^\perp \). Thus \( \omega \circ \mathcal{L} = 0. \)

5.- THE CHEN CLASS OF A CAUCHY-RIEMANN SUBMANIFOLD.

Let \( M \) be a C.R. submanifold of \( cP^n \). Let \( \pi : N \rightarrow M \) be a \( T^2 \)- fibration, as in theor. B. Assume \( s \) is even. Then \( N \) is a C.R.submanifold of \( H^{2n^+} \) and its totally-real distribution is integrable. We shall need the following:

LEMA

The holomorphic distribution of \( N \) is minimal.

Proof.

Note that we may not use lemma 4. in [17] (p.169) since its proof makes essential use of the Kaehler property. Neither could one use corollary 2.3 of [27] (p.291), (although \( \mathcal{D}^\perp \subseteq \mathcal{F} \) since \( \mathcal{F} \) fails to be locally conformal Kaehler. Now (2.4) - (2.5), (3.1) lead to:

\[
(D_X \mathcal{F}) Y = \frac{1}{2} \{ [\Phi(X, Y) - \eta_b(X) \eta_b(Y)] \xi - \ [X - \eta_b(X) \xi_s] \eta_b(Y) \} - \frac{1}{4} \{ E(X, Y) B + \omega(Y) f X \} \quad (5.1)
\]
where \( \eta = \sum \eta_s \), \( \xi = \eta^+ \). Let \( X \in \mathcal{D}_N \), \( Z \in \mathcal{D}^\perp_N \). Using (5.1) we have:

\[
(\mathcal{Z}, D_X X) = \mathcal{G}(\mathcal{Z}, Z, D_X X) = \mathcal{G}(\mathcal{Z}, D_X X, X) = \mathcal{G}(W, \mathcal{X}, \mathcal{Z}).
\]

Thus: \( \mathcal{G}(Z, D_X X + D_X X) = 0 \) and \( \mathcal{D}^\perp_N \) follows to be minimal. Let \( p = \dim_x \mathcal{D} \).

Let \( \{X_A^0, 1 \leq A \leq 2p\} \) be a real orthonormal frame of \( \mathcal{D} \), where \( X_{i+p} = X_i \), \( 1 \leq i \leq p \). Then \( \{X^0_A, \xi^s\} \) is an orthonormal frame of \( \mathcal{D}^\perp_N \). Let \( \lambda^A, 0 \leq A \leq 2p \), be differential 1-forms on \( N \) defined by \( \lambda^A(X_b) = \delta^A_b \), \( \lambda^A(Y) = 0 \), for any \( Y \in \mathcal{D}^\perp_N \). Let \( \lambda = \lambda^1 \wedge \ldots \wedge \lambda^p \wedge \eta^1 \wedge \ldots \wedge \eta^s \). Then \( \lambda \) is a globally defined \((2p+s)\)-form on \( N \), as \( \mathcal{D}^\perp_N \) is orientable. We leave it as an exercise for the reader to follow the ideas in [17] (p.170) and show that since \( \mathcal{D}^\perp_N \) is minimal and \( \mathcal{D}^\perp_N \) integrable the \((2p+s)\)-form \( \lambda \) is closed. Thus \( \lambda \) determines a cohomology class \( c(N) = [\lambda] \in H^{2p+s}(N; \mathbb{R}) \) refered to as the *Chen class* of \( N \).

To prove theorem B suppose \( M \) is a C.R. product, i.e. \( M \) is locally a product of a complex submanifold and a totally-real submanifold of \( \mathbb{C}P^s \), see e.g. [28], (p.63). Now C.R. products have an integrable holomorphic distribution and a minimal totally-real distribution. By (2.8), for any tangent vector fields \( X, Y \) on \( \mathbb{C}P^s \) one has:

\[
[X^H, Y^H] = [X, Y]^H - \alpha^s \mathcal{F}(X^H, Y^H) \xi^s. \tag{5.2}
\]

Then (5.2) used for \( X = X_A^0 \), \( Y = X_B^0 \) leads to \( [X^H_A, X^H_B] \in \mathcal{D}_N \). Next, as \( \mathcal{D}^\perp_N \)

\[
[X^H_A, \xi^s] = (D_{\xi^s} \mathcal{D}^\perp) X^H_A - \mathcal{D}^\perp_A \mathcal{D}X^H \xi^s. \tag{5.3}
\]

We need the following:

**Lemma**

The covariant derivative \( (D_X \mathcal{D}^\perp) \) \( Y = D^\perp_X Y - \mathcal{D}^\perp X \mathcal{Y} \) of \( \mathcal{D}^\perp \) is expressed by:

\[
(D_X \mathcal{D}^\perp) Y = - h(X, \mathcal{Y}) + f (X, Y) - \frac{1}{4} \omega(Y) F X \tag{5.4}
\]

for any tangent vector fields \( X, Y \) on \( N \). Here \( f \) \( V = \mathrm{nor}(\mathcal{G}) V \) for any cross-section \( V \) in \( T(N)^{\perp^s} + N \).

**Proof.**

Let also \( V = \tan(\mathcal{G}) V \). Using the Gauss and Weingarten formulae of \( N \) in \( \mathbb{H}^{2n+s} \) one has:

\[
(D_X \mathcal{G}) Y = (D_X \mathcal{G} Y - W \mathcal{D}^\perp Y X - th(X, Y) + (D_X \mathcal{D}^\perp) Y + h(X, \mathcal{G} Y) - f h(X, Y) \tag{5.5}
\]

Let us use (5.1) to substitute in (5.5); a comparisson between the normal components in (5.5) leads to (5.4), Q.E.D.

Now we may use the above lemma to end the proof of the involutivity of \( \mathcal{D}_N \). Indeed, by (5.4) and (2.4) our (5.3) turns into:

\[
\mathcal{D}^\perp[X^H_A, \xi^s] = - h(\xi^s, \mathcal{P} X^H_A) + f h(\xi^s, X^H_A) - \frac{1}{4} \omega(X^H_A) F \xi^s + \frac{1}{2} \alpha^s \mathcal{D}^\perp f X^H_A \tag{5.6}
\]

and by (2.15) one obtains \( \mathcal{D}^\perp [X^H_A, \xi^s] = 0 \).

The last step is to establish minimality of \( \mathcal{D}^\perp_N \). Let \( q = \dim_x \mathcal{D}^\perp, x \in M \).
If \( \{ E_i : 1 \leq i \leq q \} \) is an orthonormal frame of \( \mathcal{D}^1 \) then (2.8) yields:

\[
\left\langle \sum_{i=1}^{q} \mathcal{D} E_i^H \mathbf{E}_i \right\rangle = \left\langle \sum_{i=1}^{q} \nabla_i E_i^H \right\rangle.
\]

(5.7)

But \( \mathcal{D}^1 \) is minimal, so the right hand member of (5.7) is zero. Finally, one may follow the ideas in [17], (p.170) to show that since \( \mathcal{D}_N \) is integrable and \( \mathcal{D}_N \) minimal the \((2p+s)\)-form \( \lambda \) is coclosed. As \( N \) is compact, \( \lambda \) is harmonic. Thus \( c(N) \neq 0 \), and our theor. B is completely proved.

REFERENCES


19 K. YANO, On a structure defined by a tensor field f of type (1,1) satisfying $f^2 + f = 0$, *Tensor, N.S.*, 14 (1963), 99-109.


Special Issue on
Singular Boundary Value Problems for Ordinary Differential Equations

Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal’s Author Guidelines, which are located at http://www.hindawi.com/journals/bvp/guidelines.html. Authors should follow the Boundary Value Problems manuscript format described at the journal site http://www.hindawi.com/journals/bvp/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>August 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O’Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie

Hindawi Publishing Corporation
http://www.hindawi.com