Some Results on $\pi$-Solvable and Supersolvable Groups

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Abstract. For a finite group $G$, $\Phi_p(G)$, $S_p(G)$, $L(G)$ and $S_\pi(G)$ are generalizations of the Frattini subgroup of $G$. We obtain some results on $\pi$-solvable, $\pi'$-solvable and supersolvable groups with the help of the structures of these subgroups.

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1. Introduction.

Many authors have considered various generalizations of the Frattini subgroup of a finite group. Deskins [6] considered the subgroup $\Phi_p(G)$, Mukherjee and Bhattacharya [4] the subgroup $S_p(G)$ and Bhatia [3] the subgroup $L(G)$. In [7], we introduced the subgroup $S_\pi(G)$ and investigated its influence on solvable group. In this paper, our aim is to prove some results which imply a finite group $G$ to be $\pi'$-solvable, $\pi$-solvable and supersolvable. All groups are assumed to be finite. We use standard notations as found in Gorenstein [8] and denote a maximal subgroup $M$ of $G$ by $M \lhd G$.

2. Preliminaries.

Definition. Let $H$ and $K$ be two normal subgroups of a group $G$ with $K \triangleleft H$. Then the factor group $H/K$ is called a chief factor of $G$ if there is no normal subgroup $N$ of $G$ such that $K \subseteq N \subseteq H$, with proper inclusion. Let $M$ be a maximal subgroup of $G$. Then $H$ is said to be a normal supplement of $M$ in $G$ if $M H = G$. The normal index of $M$ in $G$ is defined as the order of a chief factor $H/K$, where $H$ is minimal in the set of all normal supplements of $M$ in $G$ and is denoted by $\eta(G : M)$.

(2.1) (Deskins [6, (2.1)], Beidleman and Spencer [2, Lemma-1])
If $M$ is a maximal subgroup of a group $G$ then $\eta(G : M)$ is uniquely determined.

(2.2) (Beidleman and Spencer [2, Lemma-2])
If $N$ is a normal subgroup of a group $G$ and $M$ is a maximal subgroup of $G$ such that $N \subseteq M$ then $\eta(G/N : M/N) = \eta(G : M)$

(2.3) (Mukherjee [9, Theorem-1])
If $M$ is a maximal subgroup of a group $G$ and $M \triangleleft G$ then $\eta(G : M) = [G : M]$, a prime.

(2.4) (Baer [1, Lemma-3])
If the group $G$ possesses a maximal subgroup with core 1 then the following properties of $G$ are equivalent.
(1) The indices in $G$ of all the maximal subgroups with core 1 are powers of one and the same prime $p$.

(2) There exists one and only one minimal normal subgroup of $G$ and there exists a common prime divisor of all the indices in $G$ of all the maximal subgroups with core 1.

(3) There exists a non-trivial solvable normal subgroup of $G$.

**Definition.** Let $G$ be a group and $p$ be any prime. The four characteristic subgroups of $G$, which are analogous to the Frattini subgroup $\Phi(G)$, are defined as follows:

- $\Phi_p(G) = \{ M : M \leq G, [G:M]_p = 1 \} \subseteq G$
- $\Phi_p(G) = \{ M : M \leq G, [G:M]_p = 1 \} \subseteq G$
- $\Lambda(G) = \{ M : M \leq G, [G:M] \text{ is composite} \} \subseteq G$
- $\Sigma_p(G) = \{ M : M \leq G, [G:M]_p = 1 \text{ and } \gamma_p(G:M) \text{ is composite} \} \subseteq G$

In case $\Sigma_p(G)$ is empty then we define $G = S_p(G)$ and the same thing is done for the other three subgroups.

(2.5) If $H$ is a subgroup with finite index $n$ in a group $G$ then $\text{core}_G H$ has finite index dividing $n!$

(2.6) (Dutta and Bhattacharyya [7, Theorem-3.5])

If $G$ is $p$-solvable then $S_p(G)$ is solvable.

**Definition.** Let $M$ be a maximal subgroup of a group $G$. Then $M$ is said to be c-maximal if $[G:M]_p \neq 1$ and $[G:M]_p$ is composite.

3. SOME RESULTS ON $p$-SOLVABLE AND $p'$-SOLVABLE GROUPS.

**Theorem 3.1.** Let $p$ be the largest prime dividing $|G|$ and $\Sigma_p(G) \neq \emptyset$. Then $G$ is $p$-solvable if and only if $\eta(G:M)_p = [G:M]_p$ for each $M$ in $\Sigma_p(G)$.

**Proof.** Let $G$ satisfy the hypothesis of the theorem. Then $G$ is not simple. For, otherwise $|G|_p = \eta(G:M)_p = [G:M]_p = 1$, where $M$ belongs to $\Sigma_p(G)$, which contradicts the fact that $p$ divides $|G|$. Let $N$ be a minimal normal subgroup of $G$. If $p$ does not divide $|G/N|$ then $G/N$ is a $p'$-group and hence it is $p$-solvable. If $p$ divides $|G/N|$ then $p$ is the largest prime dividing $|G/N|$. If $\Sigma_p(G/N) = \emptyset$ then $G/N = S_p(G/N)$. By Theorem-8(1) [10], $S_p(G/N)$ is solvable and hence $G/N$ is $p$-solvable. We now assume that $\Sigma_p(G/N) \neq \emptyset$. By Lemma-2 [2], we obtain $\eta(G/N:M/N)_p = [G/N : M/N]_p$ for each $M/N$ in $\Sigma_p(G/N)$. So by induction, $G/N$ is solvable. We note that $S_p(G) \neq G$, since $\Sigma_p(G) \neq \emptyset$. If $N \leq S_p(G)$ then $N$ is solvable and so it is $p$-solvable and consequently $G$ is $p$-solvable. If $N \nsubseteq S_p(G)$ then there exists $M$ in $\Sigma_p(G)$ such that $N \nsubseteq M$ and so $G = MN$. By hypothesis $|N|_p = \eta(G:M)_p = [G:M]_p = 1$ and so $N$ is $p$-solvable and hence $G$ is $p$-solvable.

The converse follows directly from Theorem 1 [2].
THEOREM 3.2. Let p be the largest prime dividing |G|. Then G is p-solvable if the following hold.

(i) G has a p-solvable c-maximal subgroup M with $\gamma(G:M)_p = [G:M]_p$

(ii) If $M_1$ and $M_2$ are c-maximal subgroups of G with $\gamma(G:M_1)_p = \gamma(G:M_2)_p$ then $[G:M_1]_p = [G:M_2]_p$

REMARK 3.3. The converse of the above theorem is not necessarily true. Let G be a p-group, where p is any prime. Then G is p-solvable, but it has no c-maximal subgroup and so G does not satisfy the hypothesis (i) of the above theorem. If the group G has a c-maximal subgroup then the converse of Theorem 3.2 follows from Theorem 1 [2].

THEOREM 3.4. Let G be a p-solvable group and $\Sigma_p(G) \neq \emptyset$. Then G is $\Pi^t$-solvable if and only if $\gamma(G:M)_\Pi = [G:M]_\Pi$ for each M in $\Sigma_p(G)$.

PROOF. Let the condition of the theorem hold. Let G be simple. Then it immediately follows that either G is a p'-group or is of prime order p. If G is of prime order p then it is solvable and hence $\Pi^t$-solvable. If G is a p'-group then $|G|_p = 1$. Also |G| is composite. For, otherwise, G is cyclic and hence it is $\Pi^t$-solvable. Let $|G|_\Pi \neq 1$ and $p_1, p_2, \ldots, p_n$ be the set of prime divisors of |G|, which belong to $\Pi$. Let $S(p_i)$ (i = 1, 2, ..., n) denote the Sylow $p_i$-subgroup of G. Then $S(p_i)$ $\not\subset$ G for $i = 1, 2, \ldots, n$. For, otherwise, G is solvable and hence G is $\Pi^t$-solvable. Let $M_i$ be the maximal subgroups of G such that $S(p_i) \subset M_i \subset G$ and so $[G:M_i]_{p_i} = 1$ (i = 1, 2, ..., n). By hypothesis $|G|_{\Pi} = \gamma(G:M_1)_{\Pi} = [G:M_1]_{\Pi}$ (i = 1, 2, ..., n). As each $p_i \not\in \Pi$, it follows that $|G|_\Pi = 1$, a contradiction. So $|G|_{\Pi} = 1$ and hence G is $\Pi^t$-solvable. We now suppose that G is not simple. Let N be a minimal normal subgroup of G. Then G/N is a p-solvable group. If $\Sigma_p(G/N) = \emptyset$ then G/N = $S\Sigma_p(G/N)$ and so by (2.6), it follows that G/N is solvable and hence it is $\Pi^t$-solvable. Now we assume that $\Sigma_p(G/N) \neq \emptyset$. Using Lemma 2 [2], we obtain $\gamma(G/N : M/N)_{\Pi} = [G/N : M/N]_{\Pi}$ for each M/N in $\Sigma_p(G/N)$. By induction, G/N is $\Pi^t$-solvable. Let $N_0$ be another minimal normal subgroup of G. Then G/N_0 is $\Pi^t$-solvable. Since G = G/N $\cap$ N_0 is isomorphic to a subgroup of the $\Pi^t$-solvable group G/N x G/N_0, it follows that G is $\Pi^t$-solvable. We may now assume that N is the unique minimal normal subgroup of G. We shall now show that N is $\Pi^t$-solvable. We note that $S\Sigma_p(G) \neq \emptyset$, since $\Sigma_p(G) \neq \emptyset$. If $N \not\in S\Sigma_p(G)$ then by (2.6) it follows that N is solvable and hence it is $\Pi^t$-solvable. If $N \not\in S\Sigma_p(G)$ then there exists $M_0$ in $\Sigma_p(G)$ such that $N \not\in M_0$ and so G = $M_0$N and core$_G(M_0) = \langle 1 \rangle$. Let M be any maximal subgroup of G with core 1. Then N $\not\subset$ M and so G = MN. Clearly M belongs to $\Sigma_p(G)$. By hypothesis $|N|_{\Pi} = \gamma(G:M)_{\Pi} = [G:M]_{\Pi}$. If $|N|_{\Pi} = 1$ then N is $\Pi^t$-solvable. If $|N|_{\Pi} \neq 1$ then there exists a common prime divisor of all the indices in G of all the maximal subgroups with core 1. So by (2.4), N is solvable and hence it is $\Pi^t$-solvable. Thus G/N and N are both $\Pi^t$-solvable. So G is $\Pi^t$-solvable. The converse follows directly from Theorem 2 [9].

THEOREM 3.5. Let G be a group with $\lambda(G) \neq \emptyset$. Then G is $\Pi^t$-solvable if and only if $\gamma(G:M)_{\Pi} = [G:M]_{\Pi}$ for each M in $\lambda(G)$, where $\lambda(G) = \{M : M \not\subset G, \gamma(G:M) = \text{composite} \}$.

THEOREM 3.6. Let G be a group with $|\lambda(G)| \geq 2$. Then G is $\Pi^t$-solvable if and only if $\gamma(G:M_1)_{\Pi} = \gamma(G:M_2)_{\Pi}$ implies $[G:M_1]_{\Pi} = [G:M_2]_{\Pi}$ for any $M_1, M_2$ in $\lambda(G)$.

PROOF. Let the condition of the theorem hold. If $|\lambda(G)| = 1$ then G is a $\Pi^t$-group and hence it is $\Pi^t$-solvable. So we assume that $|\lambda(G)| \neq 1$. Let G be simple and $p_1, p_2, \ldots, p_n$ be the set of prime divisors of |G|, which belong to $\Pi$. Then as in the proof of Theorem 3.4, we can show that there exist maximal subgroups $M_i$ of G such that $[G:M_i]_{p_i} = 1$ (i = 1, 2, ..., n).
By hypothesis, $|G|_\pi = [G:M_1]_\pi = [G:M_2]_\pi = \cdots = [G:M_n]_\pi$. As each $p_i \in \mathcal{P}$, it follows that $|G|_\pi = 1$, a contradiction. So $G$ can not be simple. Let $N$ be a minimal normal subgroup of $G$. If $\lambda(G/N)$ is empty then $\Lambda(G/N)$ is also empty and so by definition, $L(G/N) = G/N$ and consequently by the supersolvability of the group $L(G/N)$, it follows that $G/N$ is $\pi$-solvable. If $\lambda(G/N)$ consists of only one element $M/N$, say, then either $\Lambda(G/N) = \{M/N\}$ then $M/N = L(G/N)$ and consequently $M/N$ is normal in $G/N$. So by Theorem 1 [9], $\eta(G/N:M/N) = [G:N:M/N] = $ a prime, a contradiction, since $M/N \notin \Lambda(G/N)$. We now assume that $|\Lambda(G/N)| \geq 2$. It can be shown that $G/N$ satisfies the hypothesis of the theorem. So by induction, $G/N$ is $\pi$-solvable. As before, we can assume that $N$ is the unique minimal normal subgroup of $G$. Also we see that $L(G) \neq G$. If $\eta(G) \neq L(G)$ then $N$ is solvable and hence it is $\pi$-solvable. If $N \notin L(G)$ then there exists $M_0$ in $\Lambda(G)$ such that $N \notin M_0$ and $G = M_0N$ and $\text{core}_{G}(M_0) = \{1\}$. Let $M$ be any maximal subgroup of $G$ with core 1. Then $N \notin M$ and so $G = MN$. Consequently $\eta(G:M) = |N| = \eta(G:M_0)$, whence it follows that $M$ belongs to $\Lambda(G)$. By hypothesis $[G:M]_\pi = |N|_\pi$. If $|N|_\pi = 1$ then $N$ is $\pi$-solvable. If $|N|_\pi \neq 1$ then using (2.4), we have $N$ is solvable and hence it is $\pi$-solvable. Thus $G/N$ and $N$ are both $\pi$-solvable and consequently $G$ is $\pi$-solvable.

The contrary follows directly from Theorem 5 [9].

**Theorem 3.7.** Let $G$ be a $\pi$-solvable group and $|\Sigma_\mathcal{P}(G)| \geq 2$. Then $G$ is $\pi$-solvable if and only if $\eta(G:M_1)_\pi = \eta(G:M_2)_\pi$ implies $[G:M_1]_\pi = [G:M_2]_\pi$ for any $M_1, M_2$ in $\Sigma_\mathcal{P}(G)$.

**Theorem 3.8.** Let $G$ be a $\pi$-solvable group and $|\Sigma_\mathcal{P}(G)| \geq 2$. Then $G$ is $\pi$-solvable if and only if the following hold.

1. $G$ has a $\pi$-solvable maximal subgroup $M$ with $\eta(G:M)_\pi = [G:M]_\pi$.
2. $\eta(G:M_1)_\pi = \eta(G:M_2)_\pi$ implies $[G:M_1]_\pi = [G:M_2]_\pi$ for any $M_1, M_2$ in $\Sigma_\mathcal{P}(G)$.

**Theorem 3.9.** Let $G$ be a group with $|\Sigma_\mathcal{P}(G)| \geq 2$. Then $G$ is $\pi$-solvable if and only if the following hold.

1. $G$ has a $\pi$-solvable maximal subgroup $M$ with $\eta(G:M)_\pi = [G:M]_\pi$.
2. $\eta(G:M_1)_\pi = \eta(G:M_2)_\pi$ implies $[G:M_1]_\pi = [G:M_2]_\pi$ for any $M_1, M_2$ in $\Sigma_\mathcal{P}(G)$.

**Proposition 3.10.** Let $G$ be a $\pi$-solvable group and $|\Sigma_\mathcal{P}(G)| \geq 2$. Then $G$ is $\pi$-solvable if $\eta(G:M_1)_\pi = \eta(G:M_2)_\pi = 1$ for all $M_1, M_2$ in $\Sigma_\mathcal{P}(G)$ with equal normal index.

**Proposition 3.11.** Let $G$ be a group with $\Lambda(G) \neq \emptyset$. Then $G$ is $\pi$-solvable if $\eta(G:M)_\pi = 1$ for each $M$ in $\Lambda(G)$.

**Proposition 3.12.** Let $G$ be a $\pi$-solvable group or $p$ be the largest prime dividing $|G|$ and $\Sigma_\mathcal{P}(G) \neq \emptyset$. Then $G$ is $\pi$-solvable if $\eta(G:M)_\pi = 1$ for each $M$ in $\Sigma_\mathcal{P}(G)$.

**Proposition 3.13.** Let $G$ be a $\pi$-solvable group and $|\Sigma_\mathcal{P}(G)| \geq 2$. Then $G$ is $\pi$-solvable if $\eta(G:M_1)_\pi = \eta(G:M_2)_\pi = 1$ for all $M_1, M_2$ belonging to $\Lambda(G)$ with equal normal index.

**Proposition 3.14.** If a group $G$ has a $\pi$-solvable maximal subgroup $M$ with $\eta(G:M)_\pi = 1$ then $G$ is $\pi$-solvable.

**Proof.** Let $G$ satisfy the hypothesis of the proposition. Then $G$ is not simple. For, otherwise, $|G|_\pi = \eta(G:M)_\pi = 1$ and so $G$ is $\pi$-solvable. Let $N$ be a minimal normal subgroup of $G$. If $N \notin M$ then $N$ is $\pi$-solvable and also, by induction, $G/N$ is $\pi$-solvable and hence $G$ is $\pi$-solvable. If $N \notin M$ then $G=MN$ and since $G/N \not\cong M/MN$, $G/N$ is $\pi$-solvable.

**Some Results on Supersolvable Groups.**

**Theorem 4.1.** Let $G$ be a $\pi$-solvable group and suppose that for each $c$-maximal
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Let G be a group. Then G is supersolvable if and only if \( \eta(G:M) \) is square-free for each M in \( \Sigma_p(G) \).

**PROOF.** Let G satisfy the hypothesis of the theorem. We claim that \( \Sigma_p(G) \) is empty. If possible, let there exist M in \( \Sigma_p(G) \). Then G is not simple. For otherwise, \( |G| = \eta(G:M) \) is square-free and so G is supersolvable. Let \( \eta(G:M) = |H/K| \) where H/K is a chief factor of G and H is minimal in the set of normal supplements of M in G. By hypothesis \( |H/K| \) is square-free and hence H/K is supersolvable. Thus H/K is a solvable minimal normal subgroup of G/K. So H/K is an elementary abelian \( q \)-group for some prime \( q \). Consequently \( \eta(G:M) = |H/K| = q \), a prime, which is a contradiction. So \( \Sigma_p(G) \) is empty. By definition G = \( \Pi(G) \) and hence G is solvable. We shall now show that \( \Lambda(G) \) is empty. If possible, let there exist M in \( \Lambda(G) \). Then since \( \eta(G:M) = |G:M| \), \( [2, \text{Corollary of Theorem 1}] \), it follows that \( (G:M) \) is composite and hence \( p \) divides \( |G:M| \). Now the solvability of G implies that \( |G:M| \) is the power of the prime \( p \). By hypothesis, \( |G:M| = \eta(G:M)_p = p \), a prime, which is a contradiction. Hence \( \Lambda(G) \) is empty and consequently G = L(G). Hence G is supersolvable.

Conversely if G is supersolvable then \( \eta(G:M) = |G:M| = p \) for each maximal subgroup M of G and hence the assertion immediately follows.

**PROPOSITION 4.2.** Let \( p, q \) be two distinct primes. Suppose that G is either \( p \)-solvable or \( q \)-solvable. Then G is supersolvable if and only if \( \eta(G:M) \) is square-free for every M in \( \Sigma_p(G) \) or \( \Sigma_q(G) \).

**PROPOSITION 4.3.** If G contains a supersolvable maximal subgroup M such that \( \eta(G:M) = 1 \) and \( \eta(G:M) \) is square-free then G is supersolvable.

**PROOF.** Let G be simple. By hypothesis, \( |G| = \eta(G:M) \) is square-free. So G is supersolvable. We now assume that G is not simple. Let N be a minimal normal subgroup of G. Since \( \eta(G:M) = 1 \), it follows that \( N \not\in M \) and so \( G = MN \). By hypothesis \( |N| = \eta(G:M) \) is square-free and so N is supersolvable. Since \( G/N \sim M/M \), it follows that \( G/N \) is supersolvable. Thus G/N and N are both solvable. Hence G is solvable. Now since N is a minimal normal subgroup of the solvable group, it follows that N is an elementary abelian \( p \)-group for some prime \( p \). Hence \( |N| = p \) and consequently N is cyclic. Therefore G is supersolvable.

**PROPOSITION 4.4.** If G contains a supersolvable maximal subgroup M such that \( \eta(G:M) \) is square-free and the Fitting subgroup \( F(G) \), is not contained in M then G is supersolvable.

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