GENERALIZED COMMON FIXED POINT THEOREMS FOR
A SEQUENCE OF FUZZY MAPPINGS

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ABSTRACT. We obtain generalized common fixed point theorems for a sequence of fuzzy mappings, which is a generalization of the result of Lee and Cho [6].

KEY WORDS AND PHRASES. Fuzzy set, fuzzy mapping, upper semi-continuous, common fixed point.

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1. INTRODUCTION. Heilpern [3] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings, which is a fuzzy analogue of the fixed point theorems for multi-valued mappings ([2], [4], [9]) and the well-known Banach fixed point theorem. Bose and Sahani [1], in their first theorem, extended Heilpern’s result for a pair of generalized fuzzy contraction mappings. They also, in their second theorem, proved a fixed point theorem for non-expansive fuzzy mappings on a compact star-shaped subset of a Banach space. Lee and Cho [5] proved a fixed point theorem for a contractive-type fuzzy mapping which is an extension of the result of Heilpern [3]. Also, they [6] obtained common fixed point theorems for a sequence of fuzzy mappings which are generalizations of their result in [5]. Lee et al. [7] obtained a common fixed point theorem for a sequence of fuzzy mappings satisfying certain conditions, which is a generalization of the second theorem of Bose and Sahani. They also showed common fixed point theorems for a pair of fuzzy mappings in [8], which is an extension of the first theorem of Bose and Sahani [1].

In this paper, we prove generalized common fixed point theorems for a sequence of fuzzy mappings satisfying certain conditions which are generalizations of the result of Lee and Cho [6].

2. PRELIMINARIES.

Let \((X,d)\) be a linear metric linear space. A fuzzy set \(A\) in \(X\) is a function from \(X\) into \([0,1]\). If \(x \in X\), the function value \(A(x)\) is called the grade of membership of \(x\) in \(A\). The \(\alpha\)-level set of \(A\), denote by \(A_\alpha\), is defined by

\[
A_\alpha = \{x : A(x) \geq \alpha\} \quad \text{if} \quad \alpha \in (0,1], \quad A_0 = \{x : A(x) > 0\},
\]

where \(\alpha\) is a real number.
where $\bar{B}$ denotes the closure of the nonfuzzy set of $B$.

Let $W(X)$ be the collection of all the fuzzy sets $A$ in $X$ such that $A_\alpha$ is compact and convex for each $\alpha \in [0,1]$, and $\sup_{x \in X} A(x) = 1$. For $A, B \in W(X), A \subseteq B$ means $A(x) \leq B(x)$ for each $x \in X$.

**DEFINITION 2.1.** Let $A, B \in W(X)$ and $\alpha \in [0,1]$. Then we define

$$P_\alpha(A, B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x, y), \quad P(A, B) = \sup_\alpha P_\alpha(A, B)$$

and

$$D(A, B) = \sup_\alpha d_H(A_\alpha, B_\alpha),$$

where $d_H$ is the Hausdorff metric induced by the metric $d$. We note that $P_\alpha$ is a nondecreasing function of $\alpha$ and $D$ is a metric on $W(X)$.

**DEFINITION 2.2.** Let $X$ be an arbitrary set and $Y$ be any linear metric space. $F$ is called a fuzzy mapping if and only if $F$ is a mapping from the set $X$ into $W(Y)$.

In the following section, we will use the following lemmas.

**LEMMA 2.1** [5]. Let $(X,d)$ be a complete linear metric space, $F$ a fuzzy mapping from $X$ into $W(X)$ and $x_0 \in X$, then there exists $x_1 \in X$ such that $\{x_1\} \subseteq F(x_0)$.

**LEMMA 2.2** [8]. Let $A, B \in W(X)$. Then for each $\{z\} \subset A$ there exists $\{y\} \subset B$ such that $D(\{z\}, \{y\}) \leq D(A, B)$.

We can easily prove the following lemma.

**LEMMA 2.3.** Let $x \in X$ and $B \in W(X)$. If $\{y\} \subset B$, then $P(\{x\}, B) \leq d(x, y)$.

**3. COMMON FIXED POINTS THEOREMS FOR A SEQUENCE OF FUZZY MAPPINGS.**

**THEOREM 3.1.** Let $g$ be a non-expansive mapping from a complete linear metric space $(X,d)$ into itself. If $(F^i)_{i=1}^\infty$ is a sequence of fuzzy mappings from $X$ into $W(X)$ satisfying the following condition: For each pair of fuzzy mappings, $F_i, F_j$ and for any $x \in X, \{u_i\} \subset F_i(x)$, there exists $\{v_j\} \subset F_j(y)$ for all $y \in X$ such that

$$D(\{u_i\}, \{v_j\}) \leq a_1d(g(x), g(u_i)) + a_2d(g(y), g(v_j)) + a_3d(g(x), g(y)) + a_4d(g(y), g(v_j)) + a_5d(g(x), g(y)),$$

where $a_1, a_2, a_3, a_4, a_5$ are nonnegative real numbers, $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and $a_3 \geq a_4$. Then there exists $p \in X$ such that $\{p\} \subseteq \bigcap_{i=1}^\infty F_i(p)$.

**PROOF.** Let $x_0 \in X$. Then we can choose $x_1 \in X$ such that $\{x_1\} \subseteq F_1(x_0)$ by Lemma 2.1. By our assumptions, there exists $x_2 \in X$ such that $\{x_2\} \subseteq F_2(x_1)$ and

$$D(\{x_1\}, \{x_2\}) \leq a_1d(g(x_0), g(x_1)) + a_2d(g(x_1), g(x_2)) + a_3d(g(x_1), g(x_1)) + a_4d(g(x_0), g(x_2)) + a_5d(g(x_0), g(x_1)) \leq a_1d(x_0, x_1) + a_2d(x_1, x_2) + a_3d(x_1, x_1) + a_4d(x_0, x_2) + a_5d(x_0, x_1).$$

Again we can find $x_3 \in X$ such that $\{x_3\} \subseteq F_3(x_2)$ and

$$D(\{x_2\}, \{x_3\}) \leq a_1d(x_2, x_3) + a_2d(x_2, x_3) + a_3d(x_2, x_3) + a_4d(x_1, x_3) + a_5d(x_1, x_2).$$

Inductively, we obtain a sequence $\{x_n\}$ in $X$ such that $\{x_{n+1}\} \subset F_{n+1}(x_n)$ and

$$D(\{x_n\}, \{x_{n+1}\}) \leq a_1d(x_{n-1}, x_n) + a_2d(x_n, x_{n+1}) + a_3d(x_n, x_n) + a_4d(x_{n-1}, x_n) + a_5d(x_{n-1}, x_n).$$

Since $D(\{x_n\}, \{x_{n+1}\}) = d(x_n, x_{n+1})$, by (3.1)

$$d(x_n, x_{n+1}) \leq a_1d(x_{n-1}, x_n) + a_2d(x_n, x_{n+1}) + a_3d(x_n, x_n) + a_4d(x_{n-1}, x_n) + a_5d(x_{n-1}, x_n).$$
Hence
\[ d(x_n, x_{n+1}) \leq \left[ (a_1 + a_4 + a_5)/(1 - a_2 - a_4) \right] d(x_{n-1}, x_n). \]

Let \( r = (a_1 + a_4 + a_5)/(1 - a_2 - a_4). \) Since \( a_3 \geq a_4, \) \( 0 < r < 1. \) Moreover, we have \( d(x_n, x_{n+1}) \leq r^n d(x_0, x_1). \) We can easily show that \( (x_n)_{n=1}^{\infty} \) is a Cauchy sequence in \( X. \) Since \( X \) is complete, there exists \( p \in X \) such that \( \lim_{n \to \infty} x_n = p. \) Let \( F_m \) be an arbitrary member of \( (F_i)_{i=1}^{\infty}. \) Since \( \{x_n\} \subset F_m(x_{n-1}) \) for all \( n, \) there exists \( v_n \in X \) such that \( \{v_n\} \subset F_m(p) \) for all \( n \) and
\[
D(\{x_n\}, \{v_n\}) \leq a_1 d(x_{n-1}, x_n) + a_2 d(p, v_n) + a_3 d(p, x_n) + a_4 d(x_{n-1}, v_n) + a_5 d(x_{n-1}, p). \tag{3.2}
\]

From (3.2), we have
\[
d(x_n, v_n) \leq a_1 d(x_{n-1}, x_n) + a_2 d(p, x_n) + a_3 d(x_n, v_n) + a_4 d(p, p) + a_5 d(x_n, x_n).
\]
Thus we have
\[
(1 - a_2 - a_4) d(x_n, v_n) \leq a_1 d(x_{n-1}, x_n) + a_2 d(p, x_n) + a_3 d(x_n, v_n) + a_4 d(p, p) + a_5 d(x_n, x_n).
\]
Since \( x_n \to p \) as \( n \to \infty, \) \( (1 - a_2 - a_4) d(x_n, v_n) \to 0 \) as \( n \to \infty. \) Hence \( d(x_n, v_n) \to 0 \) as \( n \to \infty. \) Since \( d(p, v_n) \leq d(p, x_n) + d(x_n, v_n), \) \( v_n \to p \) as \( n \to \infty. \) Since \( F_m(p) \in W(X), \) \( F_m(p) \) is upper semi-continuous and thus
\[
\lim_{n \to \infty} \sup\{F_m(p)(v_n)\} \leq \lim_{n \to \infty} \{F_m(p)(v_n)\} = 1. \]
Since \( \{v_n\} \subset F_m(p) \) for all \( n, \) \( \{F_m(p)(v_n)\} = 1. \) Hence \( \{v_n\} \subset F_m(p). \) Since \( F_m \) is arbitrary, \( \{v_n\} \subset \bigcap_{m=1}^{\infty} F_m(p). \)

Putting \( g(x) = x, \) we get the following corollary from Theorem 3.1.

**COROLLARY 3.1.** Let \((X, d)\) be a complete linear metric space. If \((F_i)_{i=1}^{\infty}\) is a sequence of fuzzy mappings from \( X \) into \( W(X) \) satisfying the following condition (*) for each pair of fuzzy mapping \( F_i, F_j \) and for any \( x \in X, \) \( \{u_n\} \subset F_i(x), \) there exists \( \{v_n\} \subset F_j(y) \) for all \( y \in X \) such that
\[
D(\{u_n\}, \{v_n\}) \leq a_1 d(x, u_n) + a_2 d(y, v_n) + a_3 d(y, u_n) + a_4 d(x, v_n) + a_5 d(x, y),
\]
where \( a_1, a_2, a_3, a_4, a_5 \) are nonnegative real numbers, \( a_1 + a_2 + a_3 + a_4 + a_5 < 1 \) and \( a_3 \geq a_4. \) Then there exists \( p \in X \) such that \( \{p\} \subset \bigcap_{i=1}^{\infty} F_i(p). \)

By Lemmas 2.2 and 2.3, we can obtain the following corollary from Corollary 3.1.

**COROLLARY 3.2.** Let \((X, d)\) be a complete linear metric space and let \((F_i)_{i=1}^{\infty}\) be a sequence of fuzzy mappings from \( X \) into \( W(X) \) satisfying the following condition (**) for each pair of fuzzy mappings \( F_i, F_j \),
\[
D(F_i(x), F_j(y)) \leq a_1 P(x, F_i(x)) + a_2 P(y, F_j(y)) + a_3 P(y, F_i(x)) + a_4 P(x, F_j(y)) + a_5 d(x, y),
\]
for all \( x, y \) in \( X, \) where \( a_1, a_2, a_3, a_4, a_5 \) are nonnegative real numbers, \( a_1 + a_2 + a_3 + a_4 + a_5 < 1 \) and \( a_3 \geq a_4. \) Then there exists \( p \in X \) such that \( \{p\} \subset \bigcap_{i=1}^{\infty} F_i(p). \)

The following example shows that the condition (*) in Corollary 3.1 does not imply the condition (**) in Corollary 3.2.

**EXAMPLE 3.1.** Let \((F_i)_{i=1}^{\infty}\) be a sequence of fuzzy mappings from \([0, \infty)\) into \( W([0, \infty)), \)
where \( F_i(x): [0, \infty) \to [0, 1] \) is defined as follows
\[
\begin{align*}
\text{if } x = 0, & \quad [F_i(x)](z) = \begin{cases} 1, & z = 0 \\
0, & z \neq 0, \end{cases} \\
\text{otherwise, } & \quad [F_i(x)](z) = \begin{cases} 1/2, & 0 \leq z \leq x/2 \\
1/2, & x/2 < z \leq ix \\
0, & z > ix. \end{cases}
\end{align*}
\]
Then the sequence $(F_n)_{n=1}^\infty$ satisfies the condition (*) when $a_1 = a_2 = a_3 = a_4 = 0$, but does not satisfy the condition (**).

Putting $a_1 = a_2 = a_3 = a_4 = 0$, we get the following corollary from Theorem 3.1.

**COROLLARY 3.3** [6]. Let $g$ be a non-expansive mapping from a complete linear metric space $(X,d)$ into itself and $(F_n)_{n=1}^\infty$ a sequence of fuzzy mappings from $X$ into $W(X)$ satisfying the following condition: There exists a constant $k$ with $0 < k < 1$ such that for each pair of fuzzy mappings $F_i, F_j$ and for any $x \in X, \{u_x\} \subset F_i(x)$, there exists $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that

$$D(\{u_x\}, \{v_y\}) \leq kd(g(x), g(y)).$$

Then there exists $p \in X$ such that $\{p\} \subset \bigcap_{n=1}^\infty F_n(p)$.

By Lemma 2.2, we get the following corollary from Corollary 3.3.

**COROLLARY 3.4** [6]. Let $g$ be a non-expansive mapping from a complete linear metric space $(X,d)$ into itself and $(F_n)_{n=1}^\infty$ a sequence of fuzzy mappings from $X$ into $W(X)$ satisfying the following condition: There exists a constant $k$ with $0 < k < 1$ such that for each pair of fuzzy mappings $F_i, F_j$,

$$D(F_i(x), F_j(y)) \leq kd(g(x), g(y)) \quad \text{for all } x, y \in X,$$

Then there exists $p \in X$ such that $\{p\} \subset \bigcap_{n=1}^\infty F_n(p)$.

Putting $g(x) = x$, we get the following corollary from Corollary 3.4.

**COROLLARY 3.5** [6]. Let $(X,d)$ be a complete linear metric space and $(F_n)_{n=1}^\infty$ be a sequence of fuzzy mappings from $X$ into $W(X)$ satisfying the following condition. There exists a constant $k$ with $0 < k < 1$ such that for each pair of fuzzy mappings, $F_i, F_j$,

$$D(F_i(x), F_j(y)) \leq kd(x, y) \quad \text{for all } x, y \in X,$$

Then there exists $p \in X$ such that $\{p\} \subset \bigcap_{n=1}^\infty F_n(p)$.

**REFERENCES**

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