A NOTE ON QUASI AND BI-IDEALS IN TERNARY SEMIGROUPS

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ABSTRACT. In this paper we have studied the properties of Quasi-ideals and Bi-ideals in ternary semi groups. We prove that every quasi-ideal is a bi-ideal in $T$ but the converse is not true in general by giving several example in different context.

KEY WORDS AND PHRASES. Quasi-ideal, Bi-ideal, Ternary Semi group.

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1. INTRODUCTION.

D.H. Lehmer [4] gave the definition of a ternary semi group as follows:

DEFINITION 1.1. A non-empty set $T$ is called a ternary semigroup if a ternary operation $[~]_T$ on $T$ is defined and satisfies the associative law

$$[[x_1 x_2 x_3]x_4 x_5] = [x_1[x_2 x_3 x_4]x_5] = [x_1 x_2[x_3 x_4 x_5]]$$

for all $x_i \in T$, $1 \leq i \leq 5$.

Banach showed by an example that a ternary semi group does not necessarily reduce to an ordinary semi group. This has been shown by the following example.

EXAMPLE 1.2. Let $T = \{-1, 0, 1\}$ be a ternary semi group under the multiplication over complex number while $T$ is not a binary semi group under the multiplication over complex number.

Los [5] showed that any ternary semi group however may be embedded in an ordinary semi group in such a way that the operation in the ternary semi group is an (ternary) extension of the (binary) operation of the containing semi group.

Dudek [1], Feizullaer [2], Kim and Roush [3], Lyapin [6] and Sioson [7] has also studied the properties of the ternary semi groups.

We give the following definitions of ideals [7] as follows:

DEFINITION 1.3. A left (right, lateral) ideal of a ternary semi group $T$ is a non-empty subset $L(R,M)$ of $T$ such that
DEFINITION 1.4. If a non-empty subset of T is a left, right and lateral ideal of T, then it is called an ideal of T.

DEFINITION 1.5. For each element t in T, the left, right and lateral ideal generated by 't' are respectively given by:

\[(t)_L = \{t\} \cup [TtT] \]
\[(t)_R = \{t\} \cup [TTt] \]
\[(t)_M = \{t\} \cup [TtT] \cup [TTtT] \]

Due to associative law in T, one may write Sioson [7]

\[[x_1, x_2, \ldots, x_{2n+1}] = [x_1 \ldots, x_m, x_{m+1}, \ldots, x_{m+4}, \ldots, x_{2n+1}], m \leq n \]

DEFINITION 1.6. Quasi-ideal in a ternary semi group [7] is also a subset Q of T (possibly empty) satisfying following two conditions:

1. \([QTT] \cap [TQT] \cap [TQ] \subseteq Q\)
2. \([QTT] \cap [TQT] \cap [TQ] \subseteq Q\)

REMARK 1.7. Every right, left and lateral ideal is a quasi-ideal. But every quasi-ideal is not a right, a left and a lateral ideal of T. This follows from the following example

EXAMPLE 1.8. Let T = \{(0,0),(1,0),(0,1),(0,0),(0,0),(0,0),(0,0),(0,0)\} be the ternary semi group under matrix multiplication. Then Q = \{(0,0),(0,1)\} be the quasi-ideal of T, which is neither a left, nor a right nor a lateral ideal of T.

DEFINITION 1.9. A ternary sub-semi group is a subset S of a ternary semi group T such that

\[[SSS] \subseteq S\]

DEFINITION 1.10. A ternary semi group T is said to be a ternary group if it satisfies the following property that for all x, y and z in T, there exists unique a, b, c in T such that

\[[x+ y] = c, [y + z] = c, [x + z] = c\]

DEFINITION 1.11. A ternary group T is said to be a ternary group with 0 if for all a, b, c in T

\[[o + a] = 0 = [a + o] = [a + o] = [o + o] = [o + c].\]

DEFINITION 1.12. A ternary semi group T is with identity if there exists an
idempotent e in T such that

\[ [a ae] = [e aa] = [ae a] = a, \forall a \in T. \]

2. SOME RESULTS ON QUASI-IDEAL IN T WHICH ARE TRIVIALLY TRUE

PROPOSITION 2.1. A ternary group T with 0 and \([TTT] \neq 0\) has no proper quasi-ideal.

PROPOSITION 2.2. The intersection of a quasi-ideal Q and a ternary sub semi-group A of a ternary semi group T is either empty or a quasi-ideal of A.

PROPOSITION 2.3. Let Q be any non-empty subset of a ternary semi-group T, then the following are true:

1. \( Q \cup [QT] \) is the smallest left ideal of T containing Q.
2. \( Q \cup [QT] \) is the smallest right ideal of T containing Q.
3. \( Q \cup [QT] \cup [TQIT] \) is the smallest lateral ideal of T containing Q.
4. If Q is a quasi-ideal of T. Then
   \[
   Q = (Q \cup [QT]) \cap (Q \cup [QT] \cup [TQIT]) \cap (Q \cup [QT]).
   \]

PROPOSITION 2.4. The intersection of arbitrary set of quasi-ideals in a ternary semi-group is either empty or a quasi-ideal of T.

DEFINITION 2.5. Let X be a non-empty subset of a ternary semi-group T. The quasi-ideal of T generated by X is intersection of all quasi-ideals \((X)_q\) of T containing X.

If the subset X consists of a single element x, then \((X)_q\) is the cyclic quasi-ideal of T.

PROPOSITION 2.6. Let X be a non-empty subset of a ternary semi-group T, then

\[
(X)_q = (X \cup [TX]) \cap (X \cup [TXT] \cup [TXITT]) \cap (X \cup [XIT])
\]

is the smallest quasi-ideal containing X.

PROOF. Sioson [7] shows that the intersection of a right, a left and a lateral ideal of a ternary semi-group T is a quasi-ideal. Therefore the proof easily follows by using 2.3.

From 2.6 it follows that

\[
(X)_q = (X \cup [TX]) \cap (X \cup [TXT] \cup [TXITT]) \cap (X \cup [XIT])
\]

is the smallest quasi-ideal of T containing X.

3. BI-IDEALS IN TERNARY SEMI GROUP

DEFINITION 3.1. A ternary sub semi-group B of a ternary semi-group T is a bi-ideal of T if \([BBB]B \subseteq B\).

PROPOSITION 3.2. Every quasi-ideal of a ternary semi-group T is a bi-ideal.
PROOF. Let \( Q \) be a quasi-ideal of \( T \). Then \( Q \) is a ternary semi group of \( T \). Now \([QTQTQ] \subseteq [QTT]T \subseteq [QT]T\).
Similarly \([QTQTQ] \subseteq [TT]Q \subseteq [TT]TQ\).
Therefore \([QTQTQ] \subseteq [TT]Q \subseteq [TT]TQ \subseteq [QTT]T \subseteq Q\).

PROPOSITION 3.3. Let \( A \) be an ideal and \( Q \) be a quasi-ideal of \( T \). Then \( A \cap Q \) is a bi-ideal and a quasi-ideal of \( T \).

PROOF. \( A \cap Q \subseteq A \cap T \subseteq A \cap 0 \subseteq A \cap Q \) implies that \( A \cap Q \) is a ternary sub semi group of \( T \). Also

\[ (A \cap Q) T \cap (A \cap Q) \subseteq [A[TAT]A] \subseteq Q \cap [AAA] \]

by (3.2) and the given hypothesis implies that L.H.S. \( \subseteq Q \cap A \). Thus \( A \cap Q \) is a bi-ideal of \( T \). Since \( A \) is an ideal of \( T \) and it is also a quasi-ideal of \( T \). Hence \( A \cap Q \) is a quasi-ideal of \( T \).

PROPOSITION 3.4. Let \( X, Y \) be non-empty subsets of ternary semi group \( T \), then \( N \subseteq [XTY] \) is a bi-ideal of \( T \).

PROOF. Clearly \( N \) is a ternary sub semi group of \( T \). Also

\[ [N]N \subseteq [XT][Y][T][T][Y] \subseteq [X][T][Y] \]

Then \( N \) is a bi-ideal of \( T \).

PROPOSITION 3.5. The intersection of arbitrary set of bi-ideals of \( T \) is either empty or a bi-ideal of \( T \).

We omit the trivial proof.

PROPOSITION 3.6. Every left, right or lateral ideal of \( T \) is a bi-ideal of \( T \).

PROOF. Trivial.

PROPOSITION 3.7. Let \( Q \) be a subset of a ternary semi group \( T \) and \( Y \) be a non-empty proper subset of \( T \) such that

1. \([TTQ] \cup [TQT] \cup [QTT] \cup [TTT] \subseteq Y\).
2. \( Y \subseteq Q\).

Then \( Y \) is an ideal of \( T \). Moreover \( Y \) is a bi-ideal of \( T \).

PROOF. It is obvious that \([TTY], [TTY], [TYT] \) and \([TTYT]\) are contained in \( Y \) under the condition (2) therefore \( Y \) is an ideal of \( T \). And hence a quasi-ideal of \( T \) which by 3.2 is a bi-ideal of \( T \).

In the following example we show that if both or either of the conditions (1) and (2) of above proposition are not satisfied then \( Y \) is neither a left, a right, a lateral, a quasi nor a bi-ideal of \( T \).

EXAMPLE 3.8. Let \( T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \).

Then \( T \) is a ternary semi group under matrix multiplication.
(1) Take \( Y = \{(0 0), (1 0)\} \) and the quasi-ideal
\[
Q = \{(0 0), (1 0)\}
\)
of \( T \).
We see that \( Y \subseteq Q \), and
\[
[TTQ] \cup [TTQ] \cup [TTQ] \cup [TTQ] = \{(0 0), (1 0), (0 0), (0 1), (0 0)\}
\]
\( \subseteq Y \)
Also each of \( [TTQ] \), \( [TTQ] \), \( [TTQ] \) and \( [TTQ] \) is not in \( Y \).
Therefore \( Y \) is neither a left, nor a right nor a lateral ideal of \( T \).
Moreover \( [TTQ] \cap [TTQ] \cap [TTQ] \cap [TTQ] \cap Y \).
So, \( Y \) is not a quasi-ideal of \( T \).

(2) Take \( Y = \{(0 0), (1 0)\} \) and \( Q = \{(0 0), (1 0)\} \). Then \( Y \subseteq Q \).
Again \( [TTQ] \cup [TTQ] \cup [TTQ] \cup [TTQ] \subseteq Y \).
Since each of \( [TTQ] \), \( [TTQ] \), \( [TTQ] \) contains \( (0 0) \), they are not contained in \( Y \).
Hence \( Y \) is neither a left, a lateral nor a right ideal of \( T \).
Also \( [TTQ] \cap [TTQ] \cap [TTQ] \cap Y \).
So \( Y \) is not a quasi-ideal of \( T \).
Further \( [TTQ] \subseteq Y \) implies \( Y \) is not a bi-ideal of \( T \).

(3) Now we take \( Y = \{(0 0), (1 0), (1 0), (0 1), (0 0)\} \) and \( Q = \{(0 0)\} \) of \( T \). Then \( Y \subseteq Q \).
\[
[TTQ] \cup [TTQ] \cup [TTQ] \cup [TTQ] = \{(0 0)\} \subseteq Y.
\]
We find that \( (0 0) \in [TTQ], [TTQ] \) and \( [TTQ] \).
But \( (0 0) \in Y \). So \( Y \) is either a left, a lateral nor a right ideal of \( T \). Similarly
\( Y \) is neither a quasi nor a bi-ideal of \( T \).

**Theorem 3.9.** Let \( X, Y \) and \( Z \) be three non-empty subsets of a ternary semi-
group \( T \) and \( N = [XYZ] \). Then \( N \) is a bi-ideal of \( T \) if one of the following conditions
holds:

(1) \( X, Y \subseteq Z \) and \( Z \) is a bi-ideal of \( T \).
(2) \( Y, Z \subseteq X \) and \( X \) is a bi-ideal of \( T \).
(3) \( X, Z \subseteq Y \) and \( Y \) is a bi-ideal of \( T \).
(4) At least one of \( X, Y, Z \) is a right, or a left or a lateral ideal of \( T \).

**Proof.** (1) \( [NNN] \subseteq [XYZ][ZZZ] \)
\[
\subseteq [XY][ZZZ] \subseteq N
\]
and \( [NNN] \subseteq [XY][ZZZ] \subseteq N \).
Similar proofs establish (2) and (3).
(4) Assume \( X \) is a right ideal of \( T \). Then
Similar proofs can be given when either X or Y or Z is a left, or a lateral or a right ideal of T.

**Definition 3.10** [7]. An element 't' in a ternary semi group T is said to be regular if there exists x, y in T such that

\[ [txyt] = t. \]

If all the elements of T are regular then it is said to be regular ternary semi group.

**Example 3.11.** This example shows that there exists a ternary semi group while T is not a regular ternary semi group such that T has a minimal right, a minimal lateral and a minimal left ideal of T.

Let \( T = \{0, e, a, b\} \) be the ternary semi group under the operation \( (\, ) \), (given below in the table)

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\( \forall a, b, c \in T, \ [abc] = a(bc) = (ab)c. \)

Hence \( \{0\} \) is a minimal right, a minimal left and a minimal lateral ideal of T.

Since a and b are not the regular elements of T. Therefore T is not a regular ternary semi group.

Now we use theorem 3.9 to give an example of a ternary semi group in which a bi-ideal is not a quasi-ideal.

**Example 3.12.** Let T be a ternary semi group such that T is not regular, X, Y, Z be respectively a minimal right, a minimal lateral and a minimal left ideal of T satisfying the condition of 3.9. Thus \( N = [XYZ] \) is a bi-ideal of T. We will show that N is not a quasi-ideal of T.

**Proof.** \( [XYZ] \subseteq [XTT] \subseteq X, [XYZ] \subseteq Y, [XYZ] \subseteq Z. \) So, \( [XYZ] \subseteq X \cap Y \cap Z \) which is a minimal quasi-ideal of T [7].

If we assume that \( [XYZ] \) is a quasi-ideal then \( [XYZ] = X \cap Y \cap Z \) which (by Sioson [7]) thus implies that T is a regular ternary semi group. Hence it contradicts the hypothesis. So \( [XYZ] \) is not a quasi-ideal but bi-ideal by Theorem 3.9.

**Proposition 3.13.** In a regular ternary semi group every bi-ideal is a quasi-ideal.

**Proof.** Sioson [7] shows that a subset Q of a regular ternary semi group T is a quasi-ideal if and only if
Since a bi-ideal of \( T \), clearly satisfies the above condition, so we get the proof.

**PROPOSITION 3.14.** Let \( C \) be a non-empty subset of a ternary semi group \( T \) without identity. Then \( C \cup \langle CCC \rangle \cup \langle CTICIC \rangle \) is the smallest bi-ideal of \( T \) containing \( C \).

**PROOF.** Let \( x \) be any element of \( C \cup \langle CCC \rangle \cup \langle CTICIC \rangle \). Then either \( x = x_1 \) for \( x_1 \) in \( C \) or \( x = [c_1c_2c_3] \in \langle CCC \rangle \) for all \( c_i \) in \( C \), \( i = 1,2,3 \) or \( x = [c_1t_1c_2t_2c_3] \) \( \langle CTICIC \rangle \) for all \( c_i \) in \( C \), \( i = 1,2,3 \), \( t_i \) in \( T \), \( i = 1,2 \).

We will consider the elements of \( \langle CTICIC \rangle \). The other two cases will be done in a similar manner. Let \( x,y,z \in \langle CTICIC \rangle \).

i.e., \( x = [c_1t_1c_2t_2c_3], y = [c_4t_3c_5t_4c_6], z = [c_7t_5c_8t_6c_9], c_i \in C, \)

\( i = 1,2,\ldots,9, t_i \in T, i = 1,2,\ldots,6. \)

Then

\[
\{xyz\} = \left\{ [c_1t_1c_2t_2c_3][c_4t_3c_5t_4c_6][c_7t_5c_8t_6c_9] \right\} = \left\{ [c_1][\{t_1c_2t_2\}[c_3c_4t_3][c_5c_6][c_7t_5c_8t_6c_9] \right\} = \left\{ [c_1t_7c_7c_8t_6c_9] \right\} \text{ where} \]

\( t_7 = \left\{ [t_1c_2t_2][c_3c_4t_3][c_5t_6c_9] \right\} \)

\( t_8 = \left\{ [c_5c_8t_6c_9] \right\} \)

so \( \{xyz\} \in C \cup \langle CCC \rangle \cup \langle CTICIC \rangle \).

Further, \( \{xt_9yt_10z\} = \left\{ [c_1t_1c_2t_2c_3][t_9c_4c_5t_4c_6][t_10c_7t_5c_8t_6c_9] \right\} \)

\( = \left\{ [c_1][\{t_1c_2t_2\}[c_3c_4t_3][t_9c_5t_4][t_10c_7c_8t_6c_9] \right\} = \left\{ [c_1t_1c_6t_2c_9] \right\} \)

where

\( t_{11} = \left\{ [t_1c_2t_2][c_3c_4t_3][t_9c_5t_4] \right\} \)

\( t_{12} = \left\{ [t_1c_2t_2][c_3c_4t_3][t_5c_8t_6c_9] \right\} \)

Thus

\( \{xt_9yt_10z\} \in C \cup \langle CCC \rangle \cup \langle CTICIC \rangle \).

Hence \( C \cup \langle CCC \rangle \cup \langle CTICIC \rangle \) is a bi-ideal of \( T \) containing \( C \).

Suppose there exists a bi-ideal \( R \) of \( T \) containing \( C \) such that

\( R \subseteq C \cup \langle CCC \rangle \cup \langle CTICIC \rangle \).

Then \( R \) being a bi-ideal implies that

\( R \subseteq C \cup \langle CCC \rangle \cup \langle CTICIC \rangle \subseteq R \cup \langle RRR \rangle \cup \langle RTRTR \rangle \subseteq R. \)

Thus \( R = C \cup \langle CCC \rangle \cup \langle CTICIC \rangle \) is the smallest bi-ideal of \( T \) containing \( C \).
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