ON EXTENSION OF PAIRWISE $\theta$-CONTINUOUS MAPS

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ABSTRACT. The aim of the paper is to find suitable conditions so as to ultimately establish the existence and uniqueness of the extension of a pairwise $\theta$-continuous map onto an arbitrary extension-space of a bitopological space.

KEY WORDS AND PHRASES. Bitopological extension, pairwise $\theta$-continuity, pairwise $\theta$-proper, pairwise $\ast$-free.

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1. INTRODUCTION.

The problem of extending continuous maps on a topological space $X$ to a given extension $X^{*}$ of $X$ has been dealt with extensively by many mathematicians. For example it is known (see [2]) that a continuous map $f : X \rightarrow Y$ with $Y$ a compact Hausdorff space can be extended by a continuous map onto an extension $X^{*}$ of $X$ iff for each pair of closed disjoint subsets $A$ and $B$ of $Y$, the closures of $f^{-1}(A)$ and $f^{-1}(B)$ in $X^{*}$ are disjoint. One of the different interesting generalizations of this result arises when continuity and compactness are replaced by $\theta$-continuity and quasi-$\mathcal{H}$-closedness (QHC) respectively under a suitably changed condition (see Rudolf [5]).

Bitopological versions of QHC spaces and $\theta$-continuous functions have been introduced by Mukherjee [4], and Bose and Sinha [1] respectively. It is our purpose here to further generalize the above extension theorem by Rudolf [5]. For this we suitably modify and redefine the appliances used by Rudolf to ultimately establish the existence and uniqueness of the extension of a pairwise $\theta$-continuous map onto an arbitrary extension of a bitopological space under certain conditions.

By spaces $X$ and $Y$ we shall mean bitopological spaces $(X, Q_{1}, Q_{2})$ and $(Y, P_{1}, P_{2})$ respectively. For any $A \subseteq X$, $Q_{i}$-int$A$ and $Q_{i}$-cl$A$ will respectively stand for the interior and closure of $A$ in $(X, Q_{i})$, where $i=1,2$. A set $A$ is called an $ij$-regularly open set (Singal and Arya [6]) if $A = Q_{i}$-$\text{int} Q_{j}$-$\text{cl}A$, and complement of such a set is called $ij$-regularly closed where (and also in future discussion) $i,j=1,2$ and $i \neq j$. A space $X$ is called pairwise Hausdorff (Kelly [3]) if for $x, y \in X$ with $x \neq y$, there exist $U \in Q_{1}$ and $V \in Q_{2}$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. 
DEFINITION 1. (see Bose and Sinha [1]) A function (or map) \( f : (X, Q_1, Q_2) \to (Y, P_1, P_2) \) is called \( ij\)-\( \theta \)-continuous if for each \( x \in X \) and each \( P_1 \)-open neighbourhood (henceforth nbhd., for short) \( U \) of \( f(x) \), there is a \( Q_1 \)-open nbhd \( V \) of \( x \) with \( f(Q_1 \cl V) \subseteq P_\cl U \). \( f \) is called pairwise \( \theta \)-continuous if it is \( 12 \)- as well as \( 21 \)-\( \theta \)-continuous.

DEFINITION 2. (see Singal and Arya [6]) A subset \( A \) of a space \( (X, Q_1, Q_2) \) is said to be pairwise dense if every non-empty subset of \( X \) which is the intersection of a \( Q_1 \)-open set and a \( Q_2 \)-open set, has non-empty intersection with \( A \).

2. MAIN THEOREM AND ASSOCIATED RESULTS.

DEFINITION 3. A space \( (X*, Q_1*, Q_2*) \) is said to be an extension of a space \( (X, Q_1, Q_2) \) if \( Q_1*/X=Q_1 \), \( Q_2*/X=Q_2 \), and \( X \) is pairwise dense in \( X* \).

For an extension \( (X*, Q_1*, Q_2*) \) of \( (X, Q_1, Q_2) \), a map \( f : (X, Q_1, Q_2) \to (Y, P_1, P_2) \) and a point \( x \) of \( X* \) (of \( Y \)) let \( N_x^{i*} \) (resp. \( N_x^i \)) the family of all \( Q_i \)-open (\( P_i \)-open) nbds of \( x \) in \( X* \) (resp. in \( Y \)), for \( i=1,2 \). For \( x \in X* \), \( N_x^i \) (\( f(N_x^i) \)) shall denote the \( P_i \)-open filter on \( Y \) generated by the family \( \{ f(U \cap X) : U \in N_x^i \} \) (\( i=1,2 \)).

DEFINITION 4. A map \( f : X \to Y \) is \( ij\)-\( \theta \)-proper if for each \( x \in X* \), \( N_x^j(f(N_x^j)) \) is non-void \( P_i \)-adherence, where \( (X*, Q_1*, Q_2*) \) is an extension of \( (X, Q_1, Q_2) \). The map \( f \) is called pairwise \( \theta \)-proper if for each \( x \in X* \), \( N_x^j(U \cap X) : U \in N_x^j \) \( (f(N_x^j) \cap (1-U) \cap Y) \) then \( f(U) \cap Y \) is not empty, a contradiction because \( X \) is pairwise dense in \( X* \).

THEOREM 1. Let \( (X*, Q_1*, Q_2*) \) be an extension of a space \( (X, Q_1, Q_2) \) and \( f^*: (X*, Q_1*, Q_2*) \to (Y, P_1, P_2) \) be an \( ij\)-\( \theta \)-continuous extension of an \( ij\)-\( \theta \)-continuous map \( f : (X, Q_1, Q_2) \to (Y, P_1, P_2) \). Then for each \( x \in X* \), \( f^*(x) \cap (1-U) \cap Y \) is not empty, a contradiction because \( X \) is pairwise dense in \( X* \).

LEMMA 1. Let \( (X*, Q_1*, Q_2*) \) be an extension of a space \( (X, Q_1, Q_2) \) and let \( f : (X, Q_1, Q_2) \to (Y, P_1, P_2) \) be an arbitrary map. Then for each \( x \in X* \) and each \( y \in Y \), \( x \in f^{-1}(U \cap X) \) if and only if \( x \in f^{-1}(U \cap X) \).

THEOREM 2. Let \( (X*, Q_1*, Q_2*) \) be an extension of \( (X, Q_1, Q_2) \). Then for each pairwise \( \theta \)-proper map \( f : (X, Q_1, Q_2) \to (Y, P_1, P_2) \), \( f \) is called pairwise \( \theta \)-proper if it is \( 12 \)- as well as \( 21 \)-\( \theta \)-free.
extension of pairwise 0-continuous maps

**PROOF.** Let \( x \in X^*-X. \) Since \( f \) is pairwise \( \theta \)-proper, suppose that \( y \in \{ P_1-\text{cl} U : U \in N( f, x \} \} \). By Lemma 1, \( x \in \{ P_2-\text{cl} V : V \in N( f, x \} \} \). We consider a point \( y \in Y \) such that \( y \neq P_2-\text{cl} V \). Now, \( f \) being \( 12*-\text{free} \) there exists a \( P_2 \)-open nbd \( U \) of \( y \) such that \( x \in Q_2-\text{cl} f^{-1}(P_2-\text{cl} V) \). Since \( x \in \{ P_2-\text{cl} V : V \in N( f, x \} \} \), \( x \in Q_2-\text{cl} f^{-1}(P_2-\text{cl} V) \), and thus \( y \in f^{-1}(P_2-\text{cl} V) \) (by Lemma 1).

**LEMMA 2.** For an \( l-j \)-continuous map \( f : X \rightarrow Y \) and \( U \in \mathcal{N} \), \( f(\mathcal{Q}_2^* \cap f^{-1}(U)) \subset P_1-\text{cl} U \).

We are now in a position to prove the main theorem of this paper as follows:

**THEOREM 3.** Let \((X^*, Q_1^*, Q_2^*)\) be an extension of \((X, Q_1, Q_2)\). Then each pairwise \( \theta \)-continuous, pairwise \( \theta \)-proper function \( f : X \rightarrow Y \) possesses a pairwise \( \theta \)-continuous extension \( f^* : X^* \rightarrow Y \). The extension is unique if \( Y \) is pairwise Hausdorff.

**PROOF.** For \( x \in X \) we take \( f^*(x) = f(x) \), and for each \( x \in X^*-X \) we choose and fix a point of \( \{ P_1-\text{cl} U : U \in N( f, x \} \} \) and define it to be \( f^*(x) \); the latter choice is possible since \( f \) is pairwise \( \theta \)-proper.

We first prove that for each \( P_j \)-open set \( U \) of \( Y \),

\[
f^*((X^*-X) \cap Q_1^* \cap f^{-1}(P_1-\text{cl} U) \subset P_1-\text{cl} U \tag{2.1}
\]

If not, then for some \( x \in X^*-X \), there exists a \( P_j \)-open set \( U \) in \( Y \) with \( x \in (X^*-X) \cap Q_1^* \cap f^{-1}(P_1-\text{cl} U) \) but \( f^*(x) = y \), say \( f^*(x) \subset P_1-\text{cl} U \). Since \( f \) is \( ij*-\text{free} \), there exists a \( P_j \)-open nbd \( V \) of \( y \) such that \( x \in Q_j^* \cap f^{-1}(P_j-\text{cl} V) \cap Q_1^* \cap f^{-1}(P_1-\text{cl} U) \). Now since \( y = f^*(x) \subset \{ f(x) \in N( f, x \} \} \), Lemma 1 gives \( x \in Q_j^* \cap f^{-1}(P_j-\text{cl} V) \) which implies \( x \in Q_1^* \cap f^{-1}(P_1-\text{cl} U) \), contradicting the choice of \( x \). This proves (2.1).

Now to prove the pairwise \( \theta \)-continuity of \( f^* \), we first consider \( x \in X \). Suppose \( f^*(x) = y \), and let \( U \) be an arbitrary \( P_j \)-open nbd of \( y \). By \( ij*-\text{continuity} \) of \( f \) there exists a \( Q_j \)-open nbd \( U \) of \( x \) such that \( f(Q_j-\text{cl} U) \subset P_1-\text{cl} U \), i.e.,

\[
Q_1^* \cap \text{cl} f^{-1}(P_1-\text{cl} U) \subset P_1-\text{cl} U \tag{2.2}
\]

Define \( U_x = \{ x \cap U \} \), which is a \( Q_j \)-open nbd of \( x \). Then using the pairwise denseness of \( X \) we have

\[
Q_1^* - \text{cl} f^{-1}(P_1-\text{cl} U) \subset f(Q_1^* - \text{cl} f^{-1}(P_1-\text{cl} U)) \tag{2.3}
\]

Again, \( f(Q_1^* - \text{cl} f^{-1}(P_1-\text{cl} U)) = \{ f((X^*-X) \cap Q_1^* - \text{cl} f^{-1}(P_1-\text{cl} U) \} \cap f((X^*-X) \cap Q_1^* - \text{cl} f^{-1}(P_1-\text{cl} U)) \) for \( y \) by virtue of (2.1), (2.2) and (2.3).

Next we consider \( x \in X^*-X \), and let \( U \) be a \( ji*-\text{regularly open} \) set containing \( f^*(x) \) (= \( y \)). Hence \( y \in Y - U \), where \( Y - U \) is a \( ji*-\text{regularly closed} \) set. Since \( y \in \{ f(x) \in N( f, x \} \} \), Lemma 1 gives \( x \in Q_j^* - \text{cl} f^{-1}(P_1-\text{cl} U) \). Then by \( ji*-\text{regularness} \) of \( f \), \( x \in Q_j^* - \text{cl} f^{-1}(Y - U) \). Then \( U \subset Q_j^* - \text{cl} f^{-1}(Y - U) \), and thus \( x \in f(Q_j-\text{cl} U) \subset P_1-\text{cl} U \). This \( x \in Q_1^* - \text{cl} f^{-1}(P_1-\text{cl} U) \). If not, let \( x \in X^*-X \) but \( x \notin R.H.S. \). Let \( V \) be any \( Q_j \)-open nbd of \( x \). Since \( x \notin (X^*-X) \cap f^{-1}(Y - U) \) (note that \( x \notin R.H.S. \)) and \( X \) is pairwise dense in \( X \), \( V \cap (X^*-X) \cap f^{-1}(Y - U) \) is a \( Q_j \)-open nbd of \( x \), which gives \( V \cap f^{-1}(P_1-\text{cl} U) = \emptyset \). Hence \( x \notin Q_1^* - \text{cl} f^{-1}(P_1-\text{cl} U) \), a contradiction. Now since \( U \subset Q_j^* - \text{cl} f^{-1}(Y - U) = \emptyset \), \( U \subset Q_1^* - \text{cl} f^{-1}(P_1-\text{cl} U) \), i.e.,

\[
Q_1^* - \text{cl} f^{-1}(P_1-\text{cl} U) \subset Q_1^* - \text{cl} f^{-1}(P_1-\text{cl} U) \tag{2.4}
\]
Again, \( U \cap X = X - Q_j \ast \text{cl} f^{-1}(Y - U_y) \subseteq X - f^{-1}(Y - U_y) = f^{-1}(U_y) \). Thus
\[
Q_j - \text{cl}(U \cap X) \subseteq Q_j - \text{cl} f^{-1}(U_y) \tag{2.5}
\]

Now, \( f^*(Q_j - \text{cl} U_x) = f^*((X - X) \cap Q_j - \text{cl} U_x) \cup f^*((X - X) \cap Q_j - \text{cl} U_x) \cup f(Q_j - \text{cl}(U \cap X)) \) (as \( X \) is pairwise dense in \( X \subseteq P_j - \text{cl} U_y \) (by (2.1), (2.4), (2.5) and Lemma 2, noting that \( f \) is \( i j \)-\( @ \)-continuous). If \( U_y \) be any \( P_j \)-open nbd of \( f^*(x) \), then \( U_y = P_j - \text{int} P_j - \text{cl} U_y \) is a \( ji \)-regularly open set containing \( y \). Thus by what we have obtained so far, there is a \( Q_j \ast \text{open nbd} U_x \) of \( x \) with \( f^*(Q_j - \text{cl} U_x) \subseteq P_j - \text{cl} U_y = P_j - \text{cl} U_y' \). Hence \( f^* \) is \( ji \)-\( @ \)-continuous at each point of \( X^* - X \). Thus we infer that \( f^* : X^* \rightarrow Y \) is \( ji \)-\( @ \)-continuous. The \( ij \)-\( @ \)-continuity of \( f^* \) can similarly be dealt with. The uniqueness of the extension \( f^* \) of \( f \) follows from Theorems 1 and 2.

REMARK 1. Putting \( Q_1 = Q_2 \) and \( P_1 = P_2 \) in the above theorem, we get Theorem 3.1 of Rudolf [5]. If \( X \) and \( Y \) are topological spaces, then the \( \theta \)-properness of a map \( f : X \rightarrow Y \) is ensured by the \( QHC \) property of \( Y \) (see [5] for details). In bitopological setting, the definition of pairwise \( QHC \) property of \( Y \) (cf. [4]) implies that \( \bigcap \{ P_j - \text{cl} U : U \in T(f, N_x) \} \neq \emptyset \), for \( i, j = 1, 2 \) (\( i \neq j \)). But it is not necessary that \( \bigcap \{ P_j - \text{cl} U : U \in T(f, N_x) \} \neq \emptyset \). Hence in our case, the role of pairwise \( \theta \)-properness of \( f \) in Theorem 3 cannot be replaced, in general, by pairwise \( \ast \)-Hausdorffness of \( (Y, P_1, P_2) \). Nevertheless, taking \( Q_1 = Q_2 \) and \( P_1 = P_2 \) we see that every \( \ast \)-free \( \theta \)-continuous map from a topological space \( X \) to any \( H \)-closed topological space \( Y \) can be extended uniquely over any extension space \( X^* \) of \( X \).

EXAMPLE 1. Let \( X^* = Y = R (= \) the set of reals), \( Q_1 = P_1 = \) the usual topology on \( R \) and \( Q_2 = P_2 = \) the lower limit topology on \( R \). If \( X = \) the set of rationals and \( Q_i = Q_1 \ast /X \), for \( i = 1 \) and 2, then clearly \( (X^*, Q_1, Q_2) \) is an extension of \( (X, Q_1, Q_2) \) and also, the map \( f : (X, Q_1, Q_2) \rightarrow (Y, P_1, P_2) \), defined by \( f(x) = x \) (\( x \in X \)), is pairwise \( \theta \)-continuous and pairwise \( \ast \)-proper. Since \( (Y, P_1, P_2) \) is pairwise Hausdorff, \( f \) has a unique pairwise \( \theta \)-continuous extension over \( X^* \), by Theorem 3.

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