A FIXED POINT THEOREM FOR A NONLINEAR TYPE CONTRACTION

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ABSTRACT. A well-known result of Boyd and Wong [1] on nonlinear contractions is extended. Several other known results are obtained as special cases.

INTRODUCTION.

In this paper, we extend a well-known result of Boyd and Wong [1] and obtain as consequences several other known results (see [2], [3], [4], [5]).

Throughout this paper, let (X,d) be a complete metric space, \( \mathbb{R}^+ \) the nonnegative reals and \( \phi = \phi(t_1, t_2, t_3, t_4, t_5): (\mathbb{R}^+)^5 \to \mathbb{R}^+ \) a function which is (a) continuous from right in each coordinate variable (b) nondecreasing in \( t_2, t_3, t_4, t_5 \), and satisfies the inequality (c) \( \phi(t, s, s, as, bs) < \max\{t, s\} \) if \( \max\{t, s\} > 0 \) where \( \{a, b\} \subseteq \{0, 1, 2\} \) with \( a + b = 2 \). Note that (c) implies that \( \phi(t, t, t, t, t) < t \) for any \( t > 0 \).

2. MAIN RESULTS.

The following is the main result of this paper.

THEOREM 1. Let \( f, g: X \to X \) be two commutative mappings such that

(i) \( fX \subseteq gX \),

(ii) \( g \) is continuous,

(iii) \( d(fx, fy) \leq \phi(d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)) \),

for each \( x, y \in X \). Then, there exists a unique \( u \in X \) with \( fu = gu = u \).
We first prove the following lemma which simplifies the proof of the above theorem.

**LEMA.** Under the conditions of Theorem 1, if there exists a \( v \in X \) such that \( f(v) = g(v) \), then there exists a unique \( u \in X \) with \( f(u) = g(u) = u \).

**PROOF.** We show that for any \( w \in X \)

\[
    f(w) = g(w) \implies f(v) = f(w) \tag{2.1}
\]

Suppose \( t = d(fv, fw) > 0 \). Then it follows by (iii) that

\[
t < \phi(t, 0, 0, t, t) \leq \phi(t, t, t, t, t) < t,
\]

a contradiction. Thus \( f(v) = f(w) \). Now, since \( f(w) = g(w) \), therefore, \( f(fw) = g(fw) \) and consequently by (2.1)

\[
f(w) = f(fw) = g(fw).
\]

Thus, if we set \( u = f(w) \), then \( f(u) = g(u) = u \). The uniqueness of \( u \) now follows from (2.1).

**PROOF OF THEOREM 1.** Let \( x \) be an arbitrary point in \( X \). Construct a sequence \( \{ y_n \} \) in \( X \) as follows. Let \( y_0 = fx_0 \). By (i) there exists a \( x_1 \in X \) such that

\[
y_o = gx_1.
\]

Set \( y_1 = fx_1 \). Thus, if \( y_0, y_1, \ldots, y_n \) are obtained with \( y_n = fx_n \), there exists by (i) a \( x_{n+1} \in X \) such that \( y_n = gx_{n+1} \). Let \( y_{n+1} = fx_{n+1} \). Thus, for each \( n \in \mathbb{I} \) (nonnegative Integers),

\[
y_n = fx_n = gx_{n+1} \tag{2.2}
\]

We shall show that \( \{ y_n \} \) is a Cauchy sequence in \( X \). For this, let for each \( n \in \mathbb{I} \), \( d_n = d(y_n, y_{n+1}) \). Then by (i) and (b),

\[
d_{n+1} = d(fx_{n+1}, fx_{n+2}) \leq \phi(d_n, d_n, d_{n+1}, 0, d_n, d_{n+1}) \tag{2.3}
\]

Now, if for some \( n \in \mathbb{I} \), \( d_{n+1} > d_n \), then by (b) and (c)

\[
d_{n+1} \leq \phi(d_n, d_{n+1}, d_{n+1}, 0, d_{n+1}, 0, 2d_{n+1}) < d_{n+1},
\]
a contradiction. Thus for each \( n \in I \), \( d_{n+1} \leq d_n \), that is \( \{d_n\} \) is a nonincreasing sequence of nonnegative reals and consequently there exists a \( d \in \mathbb{R}^+ \) such that \( \{d_n\} \to d \). Clearly \( d = 0 \), for otherwise by (2.3) and (c),

\[
    d < \phi(d,d,d,0,2d) < d,
\]
a contradiction. Thus,

\[
d_n \to 0. \tag{2.4}
\]

Suppose, now that \( \{y_n\} \) is not a Cauchy sequence. Then there exists an \( E > 0 \) such that for each \( k \in I \), there exist integers \( n(k), m(k) \) with \( k \leq n(k) < m(k) \) satisfying

\[
    E_k = d(y_{n(k)}, y_{m(k)}) > E.
\]

Let \( m(k) \) be the least integer greater than \( n(k) \) such (2.4) holds. This implies that for each \( k \in I \), \( d(y_{n(k)}, y_{m(k)-1}) \leq E \). Consequently, for each \( k \in I \),

\[
    E < E_k \leq d(y_{n(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) \leq E + d_k. \tag{2.5}
\]

Hence, it follows by (2.4) that as \( k \to \infty \), \( E_k \to E \).

However, for each \( k \in I \),

\[
    E_k \leq d_n(k) + d(fx_{n(k)}+1, fx_{m(k)+1}) + d_m(k),
\]

\[
    \leq 2d_k + \phi(E_k, d_k, d_k, E_k + d_k, E_k + d_k),
\]

Therefore, as \( k \to \infty \),

\[
    E \leq \phi(E, 0, 0, E, E) < E,
\]

contradicting the existence of \( E > 0 \). Thus, \( \{y_n\} \) is a Cauchy sequence in \( X \).

Consequently, there is a \( v \in X \) such that \( \{y_n\} \to v \), that is

\[
    fx_n = gx_{n+1} \to v. \tag{2.6}
\]

We show that for this \( v \),

\[
    \alpha = d(fv, gv) = 0.
\]
Suppose \( \alpha > 0 \). Now by (ii) and (2.6) we have,
\[
f_{gx_n} = g_{fx_n} \Rightarrow \gamma v \quad \text{and} \quad g_{2x_n} \Rightarrow \gamma v.
\]

Also, it follows by (b) and (iii) that,
\[
d(f(gx_n),fv) \leq \phi(d(g^2x_n,gv), d(fgx_n,g^2x_n), \alpha, d(fgx_n,gv), \alpha + d(gv,g^2x_n)).
\]

Therefore, as \( n \to \infty \), the above inequality yields that
\[
\alpha = d(gv,fv) \leq \phi(0,0,\alpha,\alpha) < \alpha,
\]
a contradiction. Thus \( fv = gv \) and hence by the above lemma, there is a unique \( u \in X \) satisfying \( fu = gu = u \).

In the special case when \( g \) is taken to be the identity map of \( x \) in Theorem 1, we have

**COROLLARY 1.** Let \( f:X \to X \) satisfy either of the following conditions: for all \( x,y \in X \),
\[\]
(A). \( d(fx, fy) \leq \phi(d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)). \]
(B). \( d(fx, fy) \leq \alpha(d(x,fx) + d(y,fy)) + \beta(d(x,fy) + d(y,fx)) + \gamma(d(x,y)) \)
\[\]
where \( \alpha > 0, \beta > 0 \) and \( \gamma: \mathbb{R}^+ \to \mathbb{R}^+ \) is a right continuous function satisfying
\[\]
\( \gamma(t) < (1-2\alpha-2\beta)t \) if \( t > 0 \). Then \( f \) has a unique fixed point in \( X \).

**PROOF.** The conclusion is an obvious consequence of Theorem 1 if (A) holds.
In case of condition (B), let \( \phi:(\mathbb{R}^+)^5 \to \mathbb{R}^+ \) be defined by
\[
\phi(t_1,t_2,t_3,t_4,t_5) = \gamma(t_1) + \alpha(t_2 + t_3) + \beta(t_4 + t_5).
\]
then \( \phi \) satisfies conditions (a), (b) and (c). Thus the conclusion again follows by Theorem 1.

It may be remarked that if \( \alpha = \beta = 0 \) in (B) then Corollary 1 yields a well-known result of Boyd and Wong [1]. If \( \gamma(t) = at \), then Corollary 1 yields certain results of Hardy and Rogers [2], Kannan [3], Reich [4], Sehgal [5]. All these results are special cases of Theorem 1.
REFERENCES


KEY WORDS AND PHRASES. Nonlinear type contraction, Fixed point theorems.

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