DUAL INTEGRAL EQUATIONS—REVISITED

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Dual integral equations with trigonometric kernel are reinvestigated here for a solution. The behaviour of one of the integrals at the end points of the interval complementary to the one in which it is defined plays the key role in determining the solution of the dual integral equations. The solution of the dual integral equations is then applied to find an exact solution of the water wave scattering problems.

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1. Introduction. Boundary value problems with mixed boundary conditions arising in different branches of mathematical physics can be reduced to dual integral equations. A mixed boundary condition is the one in which one condition is prescribed at one part of the boundary while some other condition is prescribed at the remaining part of the boundary. The solution of the dual integral equations essentially depends on the behaviour of one of the integrals at the end points of the interval complementary to the one in which it is defined [1, 4]. This behaviour is dictated by the physics of the problem.

In the present paper, we consider the following dual integral equations:

\[
\begin{align*}
\int_{0}^{\infty} A_j(k)L(k,y)\,dk &= -R_j \exp(-Ky), \quad y \in G_j, \\
\int_{0}^{\infty} kA_j(k)L(k,y)\,dk &= iK(1-R_j) \exp(-Ky), \quad y \in B_j,
\end{align*}
\]  

(1.1)

where

\[
L(k,y) = k\cos ky - K\sin ky,
\]

(1.2)

\[G_j = (0,\infty) - B_j,\]

A(k) is an unknown function, and R is an unknown constant. This integral equation arises in the well-known problem of scattering water waves by a vertical barrier under the assumption of linearised theory [5, 6, 7, 8]. The vertical barrier may be (i) partially immersed in deep water, (ii) completely submerged and extending infinitely downwards in deep water, (iii) a vertical wall with a gap, or (iv) a submerged plate. The solution of (1.1) has been obtained here by noting the behaviour of the second equation of (1.1) at the end points of the interval \(G_j\), which can be determined from physical consideration. Equation
(1.1) was then reduced to a singular integral equation whose kernel involves Cauchy and logarithmic type singularity. The solution of this singular integral equation is known (cf. [3, 4, 6, 8]). The solution of (1.1) was then obtained by utilizing the solution of aforesaid singular integral equation. Knowing the solution of (1.1), the solution of the corresponding scattering problems was obtained in a closed form. In Section 2, we consider the genesis of dual integral equation (1.1), and in Section 3, we find the solution of (1.1) and hence the solution of the corresponding scattering problems.

2. Genesis of the dual integral equations. The two-dimensional problem of the scattering of surface waves by a vertical barrier present in deep water under the assumption of linearised theory consists in solving mixed two-dimensional boundary value problem given as follows: \( \phi_j \) satisfies

\[
\nabla^2 \phi_j = 0 \quad \text{in } -\infty < x < \infty, \ y \geq 0, \quad (2.1)
\]

the free surface condition

\[
K \phi_j + \phi_{jy} = 0 \quad \text{on } y = 0, \ K = \frac{\sigma^2}{g}, \ \text{a constant}, \quad (2.2)
\]

the condition on the barrier,

\[
\frac{\partial \phi_j}{\partial x} = 0 \quad \text{on } x = 0 \ y \in B_j, \ j = 1, 2, 3, 4. \quad (2.3)
\]

Here, \( B_j \) represents the vertical barrier. (i) For \( j = 1 \), the barrier is partially immersed to a depth \( a_1 \) below the mean free surface \( y = 0 \) so that \( B_1 = (0, a_1) \). (ii) For \( j = 2 \), the vertical barrier is completely submerged and extends infinitely downwards, so \( B_2 = (a_2, \infty) \). (iii) For \( j = 3 \), the vertical barrier is in the form of a wall with a gap, so \( B_3 = (0, a_3) + (a_4, \infty) \). (iv) For \( j = 4 \), the barrier is in the form of a plate submerged in deep water, so \( B_4 = (a_5, a_6) \). The bottom condition is given by

\[
\nabla \phi_j \to 0 \quad \text{as } y \to \infty. \quad (2.4)
\]

At the sharp edges of the barrier, we must have

\[
r^{1/2} \nabla \phi_j \quad \text{bounded as } r \to 0, \quad (2.5)
\]

where \( r \) denotes the distance from sharp edges \( a_j \) of the barrier, \( j = 1, \ldots, 6 \)

\[
\phi_j \sim \begin{cases} R_j \exp(-K y - iKx) + \exp(-K y + iKx) & \text{as } x \to -\infty, \\ T_j \exp(-K y + iKx) & \text{as } x \to \infty, \end{cases} \quad (2.6)
\]
where \( T_j, R_j \) are unknown complex constants. The function \( \phi_j, j = 1, 2, 3, 4 \), represents the velocity potential for two-dimensional irrotational motion corresponding to various scattering problems. The function \( \exp(-Ky + iKx) \) (dropping the time dependent factor \( \exp(-i\sigma t) \) where \( \sigma \) is the circular frequency \( K = \sigma^2/g, g \) being acceleration due to gravity) represents the wave propagating from the negative \( x \)-direction incident upon the barrier \( B_j \). The complex constants \( R_j \) and \( T_j \) are the reflection and transmission coefficients, respectively.

By Havelock expansion of water wave potential, a suitable representation of \( \phi_j \) satisfying (2.1), (2.2), (2.4), and (2.6) is

\[
\phi_j = \begin{cases} 
R_j \exp(-Kx + iKy) + \exp(-Kx - iKy) & x < 0, \\
T_j \exp(-Kx + iKy) + \int_0^\infty B_j(k)L(k,y)\exp(kx)dk, & x > 0,
\end{cases}
\] (2.7)

where (cf. [8])

\[
T_j + R_j = 1, \quad A_j(k) = -B_j(k).
\] (2.8)

By condition (2.3), using (2.7) we have

\[
\int_0^\infty kA_j(k)L(k,y)dk = iK(1 - R_j)\exp(-ky), \quad y \in B_j.
\] (2.9)

Also, \( \phi_j \) is continuous across the gap \( G_j \) below/above/between the barrier so that

\[
\phi_j(+0, y) = \phi_j(-0, y), \quad y \in G_j.
\] (2.10)

Using (2.7), we have

\[
\int_0^\infty A_j(k)L(k,y)dk = R_j\exp(-ky), \quad y \in G_j.
\] (2.11)

Here, \( G_1 = (a_1, \infty) \), \( G_2 = (0, a_2) \), \( G_3 = (a_3, a_4) \), and \( G_4 = (0, a_5) + (a_6, \infty) \). Equations (2.9) and (2.11) give the required integral equations. In the following section, we determine the solution of (1.1).

3. The solution of (1.1). Let

\[
iK(1 - R_j)\exp(-Ky) - \int_0^\infty kA_j(k)L(k,y)dk = \begin{cases} 
0, & y \in B_j, \\
h_j(y), & y \in G_j,
\end{cases}
\] (3.1)
where \( h_j(y) \) is the unknown function. In view of (2.9), (2.3), and (2.4),

\[
\begin{align*}
    h_1(y) & \sim \begin{cases} 
        O\left( |y - a_1|^{-1/2} \right) & \text{as } y \to a_1, \\
        \to 0 & \text{as } y \to \infty,
    \end{cases} \\
    h_2(y) & \sim \begin{cases} 
        O\left( |y - a_2|^{-1/2} \right) & \text{as } y \to a_2, \\
        \text{bounded} & \text{as } y \to 0,
    \end{cases} \\
    h_3(y) & \sim \begin{cases} 
        O\left( |y - a_i|^{-1/2} \right) & \text{as } y \to a_i, \ i = 3, 4, \\
        \text{bounded} & \text{as } y \to 0, \ i = 3, 4,
    \end{cases} \\
    h_4(y) & \sim \begin{cases} 
        O\left( |y - a_i|^{-1/2} \right) & \text{as } y \to a_i, \ i = 5, 6, \\
        \to 0 & \text{as } y \to \infty, \\
        \text{bounded} & \text{as } y \to 0.
    \end{cases}
\end{align*}
\]

By Havelocks’ expansion theorem [8], we have from (3.1)

\[
\begin{align*}
    i(1 - R_j) &= 2 \int_{G_j} h_j(t) \exp(-Kt) dt, \\
    kA_j(k) &= \frac{2}{\pi} \frac{1}{K^2 + k^2} \int_{G_j} h_j(t) L(k, t) dt.
\end{align*}
\]

Substituting \( A_j(k) \) from (3.7) into (2.11), we have

\[
\begin{align*}
    \frac{2}{\pi} \int_{G_j} h_j(t) \int_{0}^{\infty} \frac{L(k, t) L(k, y)}{k(K^2 + k^2)} dk dt &= R_j \exp(-Ky), \quad y \in G_j. 
\end{align*}
\]

Simplifying (3.8) and applying \((d/dy + K)\), we have

\[
\int_{G_j} h_j(t) \left[ K \ln \left| \frac{y - t}{y + t} \right| + \frac{1}{y + t} + \frac{1}{y - t} \right] dt = 0, \quad y \in G_j.
\]

This is a singular integral equation in \( h_j(t) \), whose kernel involves a combination of Cauchy type and logarithmic singularity. An appropriate solution of (3.9) can be obtained by considering the behaviour of \( h_j(t) \) at the end points of \( G_j \), which is given in (3.2), (3.3), (3.4), and (3.5) for various configurations of the barrier. Hence (3.6) and (3.7) show that the behaviour of \( h_j(t) \) at the end points of \( G_j \) plays the key role in determining the solution of (1.1).

Now, considering (3.2), (3.3), (3.4), and (3.5), we find \( h_j(t) \) for \( j = 1, 2, 3, 4 \) and hence \( A_j(k) \) and \( R_j \) for \( j = 1, 2, 3, 4 \).
(1) Knowing (3.2), \( h_1(t) \) is given by \((\text{cf. [8]})\)

\[
h_1(t) = C_1 \frac{d}{dy} \left\{ \exp(-ky) \int_{a}^{y} \frac{t \exp(Kt)}{(t^2 - a^2)^{1/2}} dt \right\}, \quad y \in G_1,
\]

where \( C_1 \) is a constant. Substituting \( h_1(t) \) in (3.6) and (3.7), we have

\[
A_1(k) = -a_1 C_1 \frac{K_2}{K^2 + k^2} J_1(ka), \quad R_1 = 1 + ia_1 C_1 K_1(Ka).
\]

(3.11)

To find \( C_1 \), \( A_1(k) \) and \( R_1 \) are substituted in the first equation of (1.1) to get

\[
C_1 = \frac{1}{a_1 \Delta_1}, \quad \Delta_1 = \pi I_1(Ka_1) - iK_1(Ka_1).
\]

(3.12)

So that

\[
A_1(k) = -\frac{J_1(ka_1)}{\Delta_1(K^2 + k^2)}, \quad R = \frac{\pi I_1(ka_1)}{\Delta_1}.
\]

(3.13)

(2) For \( j = 2 \),

\[
h_2(y) = C_2 \frac{d}{dy} \left\{ \exp(-ky) \int_{b}^{y} \frac{\exp(kv)}{(b^2 - v^2)^{1/2}} dv \right\} \quad (\text{cf. [6]}),
\]

where \( C_2 \) is a constant. Substituting in (3.6) and (3.7)

\[
A_2(k) = -\frac{C_2}{K^2 + k^2} J_0(ka_2), \quad R_2 = 1 + i\pi C_2 I_0(Ka_2).
\]

(3.15)

The constant \( C_2 \) is determined by substituting \( A_2(k) \), \( R_2 \) in first equation of (1.1). On simplification, this gives

\[
C_2 = -\frac{1}{K_0(Ka_2) + i\pi I_0(Ka_2)}.
\]

(3.16)

(3) For \( j = 3 \) (cf. [3]),

\[
h_3(y) = \frac{d}{dy} \exp(-Ky) \int_{a_4}^{y} C_3 \exp(Ku) \lambda(u) du,
\]

where

\[
\lambda(u) = \frac{u}{R(u)} \left\{ \delta - \frac{2}{\pi} F_1(a_3, a_4, u) \right\},
\]

(3.18)
\[C_3 \text{ is a constant,}
\]
\[F_1(a_3,a_4,u) = \int_0^{a_3} \frac{R(v)}{v^2-u^2} dv,
\]
\[R(u) = \left| a_3^2-u^2 \right|^{1/2}\left| a_4^2-u^2 \right|^{1/2},
\]
\[\delta = \frac{K^{-1} \exp(ka) + (2/\pi)\alpha_2(-K,F_1)}{\alpha_2(-K)},
\]
\[\alpha_i(K) = \alpha_i(K,1), \quad \alpha_i(K,F_1) = \int_{t_i} \frac{uF_1(a_3,a_4,u)}{R(u)} du,
\]
\[t_i = \begin{cases} (-a_3,a_3), & i = 1, \\ (a_3,a_4), & i = 2, \\ (a_4,\infty), & i = 3, \end{cases}
\]
and hence (3.6) and (3.7) give
\[A_3(k) = \frac{2}{\pi} \frac{C_3}{k(k^2+k^2)} \left\{ -\sin ka + k \int_{a_3}^{a_4} \lambda(u) \cos ku du \right\}, \quad R_3 = C_3I, \]
\[I = \delta \{ \alpha_1(K) - \alpha_3(K) \} - \frac{2}{\pi} \{ \alpha_1(K,F_1) - \alpha_3(K,F_1) \}.
\]
To find \(C_3\), substitute \(A_3(k)\) and \(R_3\) in the first equation of (1.1) to get
\[C_3 = \frac{i}{J+iI}, \quad (3.21)
\]
where
\[J = K^{-1} \exp(ka) + \delta \alpha_2(K) - \alpha_2(K,F_1). \quad (3.22)
\]
(4) For \(j = 4\) (cf. [2]),
\[h_4(y) = \begin{cases} \frac{d}{dy} \{ \exp(-Ky) \int_{a_5}^{y} \exp(Ku)P(u) du \}, & y < a_5, \\ \frac{d}{dy} \{ -\exp(-Ky) \int_{a_6}^{y} \exp(Ku)P(u) du \}, & y < a_6, \end{cases} \quad (3.23)
\]
where
\[P(u) = \frac{C_4}{R_0(u)}(d_0^2-u^2), \quad (3.24)
\]
\(C_4\) and \(d_0^2\) are constants,
\[R_0(u) = \left| u^2-a_5^2 \right|^{1/2}\left| u^2-a_6^2 \right|^{1/2}, \quad (3.25)
\]
and (3.6) and (3.7) give
\[ A_4(k) = \frac{J(k)}{K^2 + k^2} C_4, \quad J(k) = \int_a^b \left( \frac{d_0^2 - u^2}{R_0(u)} \right) \sin k u \, du, \]  
(3.26)
\[ R_4 = 1 - iC_4(\alpha_0 - \beta_0). \]  
(3.27)

To determine \( C_4 \) and \( d_0^2 \), we substitute \( A_4(k) \) in the first equation of (1.1) to get the relations
\[ -R_4 = C_4 y_0, \]  
(3.28)
\[ -R_4 = C_4 \left\{ y_0 - \int_{a_5}^{a_6} \left( \frac{d_0^2 - x^2}{R_0(x)} \right) \exp(Kx) \, dx \right\}, \]  
(3.29)
which yield
\[ \int_{a_5}^{a_6} \left( \frac{d_0^2 - x^2}{R_0(x)} \right) \exp(Kx) \, dx = 0. \]  
(3.30)

This determines \( d_0^2 \). Equating (3.26) and (3.28), we have
\[ C_4 = \frac{i}{\Delta_4}, \quad \Delta_4 = \alpha_0 - \beta_0 - iy_0, \]  
(3.31)
where
\[ \alpha_0 = \int_{a_5}^{a_6} \left( \frac{d_0^2 - x^2}{R_0(x)} \right) \exp(Kx) \, dx, \]  
\[ \beta_0 = \int_{a_6}^{\infty} \left( \frac{d_0^2 - x^2}{R_0(x)} \right) \exp(Kx) \, dx, \]  
(3.32)
\[ y_0 = \int_{a_5}^{a_6} \left( \frac{d_0^2 - x^2}{R_0(x)} \right) \exp(Kx) \, dx. \]

Thus, knowing \( A_j(k) \) and \( R_j \), the corresponding \( \phi_j(x, y) \) for \( j = 1, 2, 3, 4 \) are known from (2.7).

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**References**


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<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>August 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
</tr>
</tbody>
</table>

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