ON \( k \)-NEARLY UNIFORM CONVEX PROPERTY IN GENERALIZED CESÁRO SEQUENCE SPACES

WINATE SANHAN and SUTHEP SUANTAI

Received 19 January 2003

We define a generalized Cesàro sequence space \( \text{ces}(p) \), where \( p = (p_k) \) is a bounded sequence of positive real numbers, and consider it equipped with the Luxemburg norm. The main purpose of this paper is to show that \( \text{ces}(p) \) is \( k \)-nearly uniform convex (\( k \)-NUC) for \( k \geq 2 \) when \( \lim_{n \to \infty} \inf p_n > 1 \). Moreover, we also obtain that the Cesàro sequence space \( \text{ces}_p \) (where \( 1 < p < \infty \)) is \( k \)-NUC, \( k \)-R, NUC, and has a drop property.

2000 Mathematics Subject Classification: 46B20, 46B45.

1. Introduction. Let \((X, \| \cdot \|)\) be a real Banach space and let \( B(X) \) and \( S(X) \) be the closed unit ball and the unit sphere of \( X \), respectively. For any subset \( A \) of \( X \), we denote by \( \text{conv}(A) \) (resp., \( \overline{\text{conv}}(A) \)) the convex hull (resp., the closed convex hull) of Clarkson [1] who introduced the concept of uniform convexity, and it is known that uniform convexity implies reflexivity of Banach spaces. There are different uniform geometric properties which have been defined between the uniform convexity and the reflexivity of Banach spaces. Huff [6] introduced the nearly uniform convexity of Banach spaces. He has proved that the class of nearly uniformly convexifiable spaces is strictly between superreflexive and reflexive Banach spaces.

A Banach space \( X \) is called uniformly convex (UC) if for each \( \epsilon > 0 \), there is \( \delta > 0 \) such that for \( x, y \in S(X) \), the inequality \( \|x - y\| > \epsilon \) implies that

\[
\left\| \frac{1}{2}(x + y) \right\| < 1 - \delta.
\]

(1.1)

For any \( x \notin B(X) \), the drop determined by \( x \) is the set

\[
D(x, B(X)) = \text{conv}(\{x\} \cup B(X)).
\]

(1.2)

Rolewicz [12], basing on Daneš drop theorem [4], introduced the notion of drop property for Banach spaces.

A Banach space \( X \) has the drop property (D) if for every closed set \( C \) disjoint with \( B(X) \), there exists an element \( x \in C \) such that

\[
D(x, B(X)) \cap C = \{x\}.
\]

(1.3)
A Banach space $X$ is said to have the Kadec-Klee property (or property $(H)$) if every weakly convergent sequence on the unit sphere is convergent in norm.

In [13], Rolewicz proved that if the Banach space $X$ has the drop property, then $X$ is reflexive. Montesinos [11] extended this result by showing that $X$ has the drop property if and only if $X$ is reflexive and has the property $(H)$.

Recall that a sequence $\{x_n\} \subset X$ is said to be $\epsilon$-separated sequence for some $\epsilon > 0$ if

$$\text{sep}(x_n) = \inf \{\|x_n - x_m\| : n \neq m\} > \epsilon. \quad (1.4)$$

A Banach space $X$ is said to be nearly uniformly convex (NUC) if for every $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that for every sequence $\{x_n\} \subset B(X)$ with $\text{sep}(x_n) > \epsilon$, we have

$$\text{conv}(x_n) \cap (1 - \delta)B(X) \neq \emptyset. \quad (1.5)$$

Huff [6] proved that every NUC Banach space is reflexive and it has property $(H)$.

Kutzarova [7] has defined $k$-nearly uniformly convex Banach spaces. Let $k \geq 2$ be an integer. A Banach space $X$ is said to be $k$-nearly uniformly convex ($k$-NUC) if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any sequence $\{x_n\} \subset B(X)$ with $\text{sep}(x_n) \geq \epsilon$, there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that

$$\left\| \frac{x_{n_1} + x_{n_2} + x_{n_3} + \cdots + x_{n_k}}{k} \right\| < 1 - \delta. \quad (1.6)$$

Clearly, $k$-NUC Banach spaces are NUC but the opposite implication does not hold in general (see [7]).

Fan and Glicksberg [5] have introduced fully $k$-convex Banach spaces. A Banach space $X$ is said to be fully $k$-rotund ($kR$) if for every sequence $\{x_n\} \subset B(X)$, $\|x_{n_1} + x_{n_2} + \cdots + x_{n_k}\| \to k$ as $n_1, n_2, \ldots, n_k \to \infty$ implies that $\{x_n\}$ is convergent.

It is well known that UC implies $kR$ and $kR$ implies $(k + 1)R$, and $kR$ spaces are reflexive and rotund, and it is easy to see that $k$-NUC implies $kR$.

Denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all natural and real numbers, respectively.

Let $X$ be a real vector space. A functional $\varrho : X \to [0, \infty]$ is called a modular if it satisfies the following conditions:

(i) $\varrho(x) = 0$ if and only if $x = 0$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar $\alpha$ with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \leq \alpha \varrho(x) + \beta \varrho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular $\varrho$ is called convex if

(iv) $\varrho(\alpha x + \beta y) \leq \alpha \varrho(x) + \beta \varrho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. 

If \( \varrho \) is a modular in \( X \), we define
\[
X_\varrho = \left\{ x \in X : \lim_{\lambda \to 0^+} \varrho(\lambda x) = 0 \right\},
\]
\[
X_\varrho^* = \{ x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0 \}.
\]
(1.7)

It is clear that \( X_\varrho \subseteq X_\varrho^* \). If \( \varrho \) is a convex modular, for \( x \in X_\varrho \), we define
\[
\| x \| = \inf \left\{ \lambda > 0 : \varrho\left(\frac{X}{\lambda}\right) \leq 1 \right\}.
\]
(1.8)

Orlicz [10] proved that if \( \varrho \) is a convex modular on \( X \), then \( X_\varrho = X_\varrho^* \) and \( \| \cdot \| \) is a norm on \( X_\varrho \) for which \( X_\varrho \) is a Banach space. The norm \( \| \cdot \| \), defined as in (1.8), is called the Luxemburg norm.

A modular \( \varrho \) is said to satisfy the \( \delta_2 \)-condition (\( \varrho \in \delta_2 \)) if for any \( \epsilon > 0 \), there exist constants \( K \geq 2 \) and \( a > 0 \) such that
\[
\varrho(2u) \leq K \varrho(u) + \epsilon
\]
(1.9)

for all \( u \in X_\varrho \) with \( \varrho(u) \leq a \).

If \( \varrho \) satisfies the \( \delta_2 \)-condition for any \( a > 0 \) with \( K \geq 2 \) dependent on \( a \), we say that \( \varrho \) satisfies the strong \( \delta_2 \)-condition (\( \varrho \in \delta_{s2} \)).

The following known results are very important for our consideration.

THEOREM 1.1. If \( \varrho \in \delta_{s2} \), then for any \( L > 0 \) and \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
| \varrho(u + v) - \varrho(u) | < \epsilon
\]
whenever \( u, v \in X_\varrho \) with \( \varrho(u) \leq L \) and \( \varrho(v) \leq \delta \).

PROOF. See [2, Lemma 2.1].

THEOREM 1.2. (1) If \( \varrho \in \delta_{s2} \), then for any \( x \in X_\varrho \), \( \| x \| = 1 \) if and only if \( \varrho(x) = 1 \).

(2) If \( \varrho \in \delta_2 \), then for any sequence \( (x_n) \) in \( X_\varrho \), \( \| x_n \| \rightarrow 0 \) if and only if \( \varrho(x_n) \rightarrow 0 \).

PROOF. See [2, Corollary 2.2 and Lemma 2.3].

THEOREM 1.3. If \( \varrho \in \delta_{s2} \), then for any \( \epsilon \in (0,1) \), there exists \( \delta \in (0,1) \) such that \( \varrho(x) \leq 1 - \epsilon \) implies \( \| x \| \leq 1 - \delta \).

PROOF. Suppose that the theorem does not hold, then there exist \( \epsilon > 0 \) and \( x_n \in X_\varrho \) such that \( \varrho(x_n) < 1 - \epsilon \) and \( 1/2 \leq \| x_n \| \) \( \neq 1 \). Let \( a_n = \frac{1}{\| x_n \| - 1} \). Then \( a_n \rightarrow 0 \) as \( n \rightarrow \infty \). Let \( L = \sup \{ \varrho(2x_n) ; n \in \mathbb{N} \} \). Since \( \varrho \in \delta_{s2} \), there exists \( K \geq 2 \)
such that
\[ \varrho(2u) \leq K \varrho(u) + 1 \] (1.11)
for every \( u \in X_\varrho \) with \( \varrho(u) < 1 \).

By (1.11), we have
\[ \varrho(2x_n) \leq K \varrho(x_n) + 1 \]
for all \( n \in \mathbb{N} \). Hence, \( 0 < L < \infty \). By Theorem 1.2(1), we have
\begin{align*}
1 &= \varrho \left( \frac{x_n}{\|x_n\|} \right) = \varrho(2a_n x_n + (1 - a_n)x_n) \\
&\leq a_n \varrho(2x_n) + (1 - a_n) \varrho(x_n) \\
&\leq a_n L + (1 - \epsilon) \rightarrow 1 - \epsilon,
\end{align*}
which is a contradiction. \( \square \)

Let \( l^1 \) be the space of all real sequences. For \( 1 < p < \infty \), the Cesàro sequence space \( (\text{ces}_p) \) is defined by
\[ \text{ces}_p = \left\{ x \in l^0 : \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} \| x(i) \| \right)^p < \infty \right\} \] (1.13)
equipped with the norm
\[ \|x\| = \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} \| x(i) \| \right)^p \right)^{1/p}. \] (1.14)

This space was first introduced by Shiue [14], which is useful in the theory of Matrix operator and others (see [8, 9]). Some geometric properties of the Cesàro sequence space \( \text{ces}_p \) were studied by many authors. It is known that \( (\text{ces}_p, \| \cdot \|) \) is locally uniformly rotund (LUR) and has property (H) (see [9]). Cui and Meng [3] proved that \( (\text{ces}_p, \| \cdot \|) \) has property (\( \beta \)).

Let \( p = (p_n) \) be a sequences of positive real numbers with \( p_n \geq 1 \) for all \( n \in \mathbb{N} \). The generalized Cesàro sequence space \( \text{ces}(p) \) is defined by
\[ \text{ces}(p) = \{ x \in l^0 : \rho(\lambda x) < \infty \text{ for some } \lambda > 0 \}, \] (1.15)
where
\[ \rho(x) = \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} \| x(i) \| \right)^{p_n} \] (1.16)
is a convex modular on \( \text{ces}(p) \).

We consider \( \text{ces}(p) \) equipped with the Luxemburg norm:
\[ \|x\| = \inf \left\{ \epsilon > 0 : \rho \left( \frac{x}{\epsilon} \right) \leq 1 \right\}. \] (1.17)
When \( p_n = q \) for all \( n \in \mathbb{N} \), we see that \( \text{ces}(p) = \text{ces}_q \) and the Luxemburg norm on \( \text{ces}(p) \) given in (1.17) is equal to the norm \( \| \cdot \| \) given in (1.14). In this paper, we show that \( \text{ces}(p) \) equipped with the Luxemburg norm is \( k \)-NUC for \( k \geq 2 \), so it is \( kR \) and (NUC).

Throughout this paper, we assume that \( p = (p_n) \) is bounded with
\[
\liminf_{n \to \infty} p_n > 1
\]
and that \( M = \sup_n p_n \).

### 2. Main results

**Proposition 2.1.** For \( x \in \text{ces}(p) \), the modular \( \rho \) on \( \text{ces}(p) \) satisfies the following properties:

1. If \( 0 < a < 1 \), then \( a^M \rho(x/a) \leq \rho(x) \) and \( \rho(ax) \leq a \rho(x) \),
2. If \( a \geq 1 \), then \( \rho(x) \leq a^M \rho(x/a) \),
3. If \( a \geq 1 \), then \( \rho(x) \leq a \rho(x) \leq \rho(ax) \).

**Proof.** All assertions are clearly obtained by the definition and convexity of \( \rho \). \( \square \)

**Proposition 2.2.** For any \( x \in \text{ces}(p) \),

1. If \( \|x\| \leq 1 \), then \( \rho(x) \leq \|x\| \),
2. If \( \|x\| > 1 \), then \( \rho(x) \geq \|x\| \),
3. \( \|x\| = 1 \) if and only if \( \rho(x) = 1 \).

**Proof.** (1) Suppose that \( \|x\| \leq 1 \). If \( x = 0 \), then \( \rho(x) = \|x\| = 0 \). Suppose \( x \neq 0 \). By the definition of \( \| \cdot \| \), there is a sequence \( (\epsilon_n) \) with \( \epsilon_n \downarrow \|x\| \) such that \( \rho(x/\epsilon_n) \leq 1 \). This implies that \( \rho(x/\|x\|) \leq 1 \). By Proposition 2.1(1), we have
\[
\rho(x) = \rho\left( \frac{\|x\| \cdot x}{\|x\|} \right) \leq \|x\| \rho\left( \frac{x}{\|x\|} \right) \leq \|x\|. \quad (2.1)
\]
(2) Suppose that \( \|x\| > 1 \). Then for \( \epsilon \in (0, (\|x\| - 1)/\|x\|) \), we have \( (1 - \epsilon) \|x\| > 1 \). By Proposition 2.1(1), we have
\[
1 < \rho\left( \frac{x}{(1-\epsilon)\|x\|} \right) \leq \frac{\rho(x)}{(1-\epsilon)\|x\|}, \quad (2.2)
\]
so that \( (1 - \epsilon)\|x\| < \rho(x) \). By taking \( \epsilon \to 0 \), we have \( \rho(x) \geq \|x\| \).

(3) It follows from Theorem 1.2(1) because \( \rho \in \delta^2_L \).

**Proposition 2.3.** For any \( L > 0 \) and \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
|\rho(u + v) - \rho(u)| < \epsilon \quad (2.3)
\]
whenever \( u, v \in \text{ces}(p) \) with \( \rho(u) \leq L \) and \( \rho(v) \leq \delta \).

**Proof.** Since \( p = (p_n) \) is bounded, it is easy to see that \( \rho \in \delta^2_L \). Hence, the proposition is obtained directly from Theorem 1.1. \( \square \)
Proposition 2.4. For every sequence \((x_n) \in \text{ces}(p)\), \(\|x_n\| \to 0\) if and only if \(\rho(x_n) \to 0\).

Proof. It follows directly from Theorem 1.2(2) because \(\rho \in \delta_s^2\).

Theorem 2.5. For any \(x \in \text{ces}(p)\) and \(\epsilon \in (0,1)\), there exists \(\delta \in (0,1)\) such that \(\rho(x) \leq 1 - \epsilon\) implies \(\|x\| \leq 1 - \delta\).

Proof. Since \(\rho \in \delta_s^2\), the theorem is obtained directly from Theorem 1.3.

Theorem 2.6. The space \(\text{ces}(p)\) is \(k\)-NUC for any integer \(k \geq 2\).

Proof. Let \(\epsilon > 0\) and \((x_n) \subset B(\text{ces}(p))\) with \(\text{sep}(x_n) \geq \epsilon\). For each \(m \in \mathbb{N}\), let
\[
x^m_n = \left(0,0,\ldots,0,x_n(m),x_n(m+1),\ldots\right).
\]
(2.4)
Since for each \(i \in \mathbb{N}\), \((x_n(i))_{n=1}^\infty\) is bounded, by using the diagonal method, we have that for each \(m \in \mathbb{N}\), we can find a subsequence \((x_{mj})\) of \((x_n)\) such that \((x_{mj}(i))\) converges for each \(i \in \mathbb{N}\), \(1 \leq i \leq m\). Therefore, there exists an increasing sequence of positive integers \((t_m)\) such that \(\text{sep}((x^m_{mj})_{j>t_m}) \geq \epsilon\). Hence, there is a sequence of positive integers \((r_m)_{m=1}^\infty\) with \(r_1 < r_2 < r_3 < \cdots\) such that \(\|x^m_{rm}\| \geq \epsilon/2\) for all \(m \in \mathbb{N}\). Then by Proposition 2.4, we may assume that there exists \(\eta > 0\) such that
\[
\rho(x^m_{rm}) \geq \eta \quad \forall m \in \mathbb{N}.
\]
(2.5)
Let \(\alpha > 0\) be such that \(1 < \alpha < \lim_{n \to \infty} \inf p_n\). For fixed integer \(k \geq 2\), let \(\epsilon_1 = ((k^{\alpha-1} - 1)/(k - 1)k^\alpha)(\eta/2)\). Then by Proposition 2.3, there is a \(\delta > 0\) such that
\[
|\rho(u + v) - \rho(u)| < \epsilon_1
\]
(2.6)
whenever \(\rho(u) \leq 1\) and \(\rho(v) \leq \delta\).

Since by Proposition 2.2(1) \(\rho(x_n) \leq 1\) for all \(n \in \mathbb{N}\), there exist positive integers \(m_i\) \((i = 1,2,\ldots,k-1)\) with \(m_1 < m_2 < \cdots < m_{k-1}\) such that \(\rho(x^m_{mi}) \leq \delta\) and \(\alpha \leq p_j\) for all \(j \geq m_{k-1}\). Define \(m_k = m_{k-1} + 1\). By (2.5), we have \(\rho(x^m_{rm_k}) \geq \eta\). Let \(s_i = i\) for \(1 \leq i \leq k - 1\) and \(s_k = r_{mk}\).

Then in virtue of (2.5), (2.6), and convexity of function \(f_i(u) = |u|^{p_i}\) \((i \in \mathbb{N})\), we have
\[
\rho\left(\frac{x_{s_1} + x_{s_2} + \cdots + x_{s_k}}{k}\right) = \sum_{n=1}^\infty \left(\frac{1}{n} \sum_{i=1}^n \left|\frac{x_{s_1}(i) + x_{s_2}(i) + \cdots + x_{s_k}(i)}{k}\right|^{p_n}\right)
\]
\[ \begin{aligned}
&= \sum_{n=1}^{m_1} \left( \frac{1}{n} \sum_{i=1}^{n} \left| \frac{x_{s_1}(i) + \cdots + x_{s_k}(i)}{k} \right| \right)^{p_n} \\
&+ \sum_{n=m_1+1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} \left| \frac{x_{s_1}(i) + x_{s_2}(i) + \cdots + x_{s_k}(i)}{k} \right| \right)^{p_n} \\
&\leq \sum_{n=1}^{m_1} \left( \frac{1}{n} \sum_{i=1}^{n} \left| x_{s_1}(i) \right| \right)^{p_n} \\
&+ \sum_{n=m_1+1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} \left| x_{s_2}(i) + x_{s_3}(i) + \cdots + x_{s_k}(i) \right| \right)^{p_n} + \epsilon_1 \\
&\leq \sum_{n=1}^{m_1} \frac{1}{k} \sum_{j=1}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} \left| x_{s_j}(i) \right| \right)^{p_n} \\
&+ \sum_{n=m_1+1}^{m_2} \left( \frac{1}{n} \sum_{i=1}^{n} \left| x_{s_2}(i) + x_{s_3}(i) + \cdots + x_{s_k}(i) \right| \right)^{p_n} + \epsilon_1 \\
&\leq \sum_{n=1}^{m_1} \frac{1}{k} \sum_{j=1}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} \left| x_{s_j}(i) \right| \right)^{p_n} \\
&+ \sum_{n=m_1+1}^{m_2} \frac{1}{k} \sum_{j=2}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} \left| x_{s_j}(i) \right| \right)^{p_n} \\
&+ \sum_{n=m_2+1}^{m_3} \frac{1}{k} \sum_{j=3}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} \left| x_{s_j}(i) \right| \right)^{p_n} \\
&+ \cdots + \sum_{n=m_{k-1}+1}^{m_k} \frac{1}{k} \sum_{j=k-1}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} \left| x_{s_j}(i) \right| \right)^{p_n} \\
&+ \sum_{n=m_k+1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} \left| x_{s_k}(i) \right| \right)^{p_n} + (k-1)\epsilon_1
\end{aligned} \]
By Theorem 2.5, there exist $\gamma > 0$ such that $\| (x_{s_1} + x_{s_2} + \cdots + x_{s_k})/k \| < 1 - \gamma$. Therefore, $\text{ces}(p)$ is $k$-NUC.

Since $k$-NUC implies $kR$ and $kR$ implies $R$ and reflexivity holds, and $k$-NUC implies NUC and NUC implies property (H) and reflexivity holds, by Theorem 2.6, the following results are obtained.

**Corollary 2.7.** The space $\text{ces}(p)$ is $kR$, NUC, and has a drop property.

**Corollary 2.8.** For $1 < p < \infty$, the space $\text{ces}_p$ is $k$-NUC.

**Corollary 2.9.** For $1 < p < \infty$, the space $\text{ces}_p$ is $kR$ and NUC.

**Corollary 2.10.** For $1 < p < \infty$, the space $\text{ces}_p$ has the drop property.

**Acknowledgments.** Suthep Suantai would like to thank the Thailand Research Fund for the financial support and the referee for pointing out the work of Cui and Hudzik [2]. Winate Sanhan was supported by The Royal Golden Jubilee Project.
ON k-NUC PROPERTY IN GENERALIZED CESÁRO ...

REFERENCES


Winate Sanhan: Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

E-mail address: winate_s@yahoo.com

Suthep Suantai: Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

E-mail address: scmti005@chiangmai.ac.th
Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal’s Author Guidelines, which are located at http://www.hindawi.com/journals/bvp/guidelines.html. Authors should follow the Boundary Value Problems manuscript format described at the journal site http://www.hindawi.com/journals/bvp/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>May 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>August 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Donal O’Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie