ON HEREDITARY INTERVAL ALGEBRAS

M. ALAMI and D. ZHANI

Received 19 June 2003

We show that each hereditary interval algebra has a countable density and not conversely. Moreover, we show that, for an interval algebra, having countable density and being subalgebra of the interval algebra over the real line are equivalent statements.

2000 Mathematics Subject Classification: 06E05, 06E99, 54F05.

1. Introduction. Boolean algebras that are generated by subchains, that is, subsets that are linearly ordered under the Boolean partial order, were introduced in 1939 by Mostowski and Tarski [7] and have been extensively studied since then. Nowadays they are called interval algebras. All basic facts about these algebras can be found in [6, Section 15]. We remark, at this stage, that a subalgebra of an interval algebra need not be an interval algebra. For instance, one can consider the algebra of finite and co-finite subsets of the first uncountable cardinal. This leads us to the study of hereditary interval algebras, that is, those algebras of which any subalgebra is an interval algebra. The main concern of this note is to shed more light on these algebras. This note is organized as follows. Section 2 deals with definitions. Section 3 is a presentation of the main theorem. In Section 4, some consequences of this theorem are given.

2. Preliminaries. A partial ordered set \((T, \leq)\) is called a tree (resp., a pseudotree) whenever, for each element \(t\) in \(T\), \(\{u \in T : u \leq t\}\) is a well-founded chain (resp., a chain). Throughout this note, we denote by \(B(T)\) the tree algebra (resp., the pseudotree algebra on \(T\)), that is, the subalgebra of the power set \(\mathcal{P}(T)\) generated by \(\{b_t : t \in T\}\), where \(b_t \overset{\text{def}}{=} \{u \in T : t \leq u\}\).

Theorem 16.7 in [6] enables us to look at trees with a single root without losing generality. Furthermore, if \(T'\) is a subtree of \(T\), then \(B(T')\) embeds into \(B(T)\) by [6, Theorem 16.9]. These two facts remain valid for pseudotree algebras; for indications about the modifications of the arguments in [6] and for more on pseudotree algebras, we refer the reader to [2, 4]. If \(T\) is a chain, say \(L\), then we usually denote \(B(T)\) by \(\text{Int}(L)\) and we call \(\text{Int}(L)\) the interval algebra over \(L\).

Now, let \(C\) be a given chain and \(a < b\) in \(C\). We say that \(b\) covers \(a\) in \(C\) whenever \([a, b[ = \{a\}\). We then set \(\text{Gap}(C) \overset{\text{def}}{=} \{x : \exists a, b \in C [b \text{ covers } a \text{ and } x \in \{a, b\}]\}\). Next, recall the interval topology \(\tau_0\) on \(C\) generated by the set of open intervals \((a, b), a, b \in C^+ = C + \{\infty\}\); see [4]. Also, we say that a subset \(X\) of \(C\) is ordinarily dense in \(C\) whenever \(X\) is topologically dense in \((C, \tau_0)\) and \(\text{Gap}(C) \subseteq X\).
Next, consider the following cardinal invariant on a Boolean algebra $B$: $\pi(B)$ is the algebraic density of $B$ = the minimal size of a dense subset of $B$, $dB$ is the topological density of $B$ = the minimal size of a dense subset of the Stone space of $B$, $\text{Ult}(B)$ = the set of ultrafilters of $B$ endowed with Tychonoff’s topology.

Also, for any topological space $X$, put $dX = \text{Min}\{|D| : D \text{ dense in } X\}$. For more details, we refer the reader to [5]. Finally, recall that a separable topological space is a space that has a countable dense topological subspace, and for any subset $A$ of a topological space $X$, we denote by $\text{cl}(A)$ the topological closure of $A$ in $X$; see [4].

3. Main result. We start with a definition.

**Definition 3.1.** An infinite Boolean algebra is a hereditary interval algebra (h.i.-algebra) whenever any subalgebra of it is an interval algebra.

Next, we state the main theorem of this note.

**Theorem 3.2.** Let $B$ be an interval algebra and consider the following statements:

(i) $B$ is an h.i.-algebra,
(ii) $\pi(B) = \aleph_0$ and $B$ is an interval algebra,
(iii) $B = \text{Int}(L)$, where $L$ is an infinite subset of the real line.

Then (ii) and (iii) are equivalent and (i) implies (ii).

The proof of this theorem will be a consequence of the following lemmas and propositions.

**Proof of (i) implies (ii)**

**Proposition 3.3.** If $A$ is an h.i.-algebra, then neither $\omega_1$ nor $\omega_1^*$ embeds in $A$.

**Proof.** Assume the contrary. Without loss of generality, we may assume $\omega_1 \leq A \overset{\text{def}}{=} \text{Int}(L)$ (since $A$ is an h.i.-algebra). Let $\langle a_\alpha : \alpha < \omega_1 \rangle$ be an increasing continuous enumeration of $\omega_1$ and put $T = \{a_\alpha : \alpha < \omega_1\}$.

Next, define $b_\alpha = a_{\alpha+1} - a_\alpha$ for $\alpha < \omega_1$ and put $T_0 = \{b_\alpha : \alpha < \omega_1\} (\subseteq A)$. Hence $B_0 = \langle T_0 \rangle$ is an uncountable subalgebra of $A$ which is isomorphic to an interval algebra since $A$ is an h.i.-algebra, but this is a contradiction since all chains in $B_0$ are countable.

**Proposition 3.4.** Let $B(T)$ be a tree algebra not embedding $\omega_1$ or $\omega_1^*$. Then every chain in $B(T)$ is at most countable.

**Proof.** Assume the proposition does not hold and let $C$ be a chain in $B(T)$ of size $\aleph_1$. Then, by [6, Theorem 16.20], $\omega_1$ or $\omega_1^*$ embeds in $B(T)$, which leads to a contradiction.

**Proposition 3.5** (S. Koppelberg, J. D. Monk). Every interval algebra has a dense tree algebra.

**Proof.** Let $B = \text{Int}(L)$. Hence $B$ is minimally generated by [3] and, by [3, Theorem 4.3] and the remark after [4, Corollary 2.4], $B$ has a dense subalgebra which is isomorphic to a tree algebra.
Next, to end up the proof of (i) implies (ii), let \( B \) be an h.i.-algebra. By Proposition 3.5, \( B \) has a dense tree algebra, say \( B(T) \). So it would be sufficient to show that \( B(T) \) is countable. Indeed, by Proposition 3.3, \( \omega_1 \) and \( \omega_1^* \) do not embed in \( B \). Hence \( \omega_1, \omega_1^* \) do not embed in \( B(T) \). Now, by Proposition 3.4, every chain in \( B(T) \) is countable. Again, since \( B \) is an h.i.-algebra, \( B(T) \) is in fact an interval algebra in which every chain is countable; that means \( B(T) \) is countable. So, \( \pi(B) = \aleph_0 \). This completes the proof of (i) implies (ii).

**Proof of (iii) implies (ii).** Let \( L \) be isomorphic to a subchain \( C \) of \( \mathbb{R} \). Hence \( B = \text{Int}(L) \equiv \text{Int}(C) \leq \text{Int}(\mathbb{R}) \). But \( \pi(\text{Int}(C)) = \aleph_0 \) by [1, Proposition 1.5]. So, \( \pi(B) = \aleph_0 \).

**Proof of (ii) implies (iii).** Next, we state the well-known result about the set of real line. See, for example, [12, Corollary 3.2].

**Lemma 3.6.** The following statements are equivalent for any infinite set \( A \):

(i) \( A \) is a subchain of \( \mathbb{R} \),

(ii) there is \( B \subseteq A \) countable and ordinarily dense in \( A \).

Let \( B \) be an interval algebra such that \( \pi(B) = \aleph_0, B = \text{Int}(L) \) for some chain \( L \). Now pick \( B_0 \) a countable dense subalgebra of \( B \), and for any \( a \in B_0 \), written under its normal form,

\[
a = \bigcup_{i=1}^{n} [x_i, y_i], \quad x_i, y_i \in L^+ = L + \{\infty\}. \tag{3.1}
\]

We set \( \text{rel}(a) \overset{\text{def}}{=} \{x_i, y_i : i = 1, \ldots, n\} \). Now, if \( b \) covers \( a \) in \( L \), by denseness of \( B_0 \) in \( \text{Int}(L) \), there is a nonzero element \( x_0 \in B_0 \) such that \( x_0 \leq [a, b] = \{a\} \). Hence \( x_0 = [a, b] \in B_0 \). So the set \( L_0 = \{\text{rel}(x) : x \in B_0\} \) is at most countable (since \( |B_0| = \aleph_0 \)) and \( \text{Gap}(L) \subseteq L_0 \).

Now, by the above lemma, it is sufficient to show that \( L_0 \) is topologically dense in \( L \). To this end, let \( x_0 \in L \) and let \( (a, b) \) be an open interval in \( L \) containing \( x_0 \).

We need to show that \( (a, b) \cap L_0 \neq \emptyset \).

**Case 1.** \( x_0 \) covers \( a \) or \( b \) covers \( x_0 \).

In this case, \( x_0 \in L_0 \), and hence \( (a, b) \cap L_0 \neq \emptyset \). This takes care of Case 1.

**Case 2** (Case 1 fails). There are \( a', b' \in L \) such that \( a < a' < x_0 < b' < b \). Thus \( [a', b'] \in \text{Int}(L) \), and then, by denseness of \( B_0 \) in \( \text{Int}(L) \), pick a nonzero element \( y_0 \in B_0 \) such that \( y_0 \leq [a', b'] \). So \( [a', b'] \cap L_0 \neq \emptyset \). Thus \( (a, b) \cap L_0 \neq \emptyset \). This takes care of Case 2.

Hence the proof of (ii) implies (iii) is finished.

**4. Consequences.** In this section, some corollaries of the main theorem are given.

**Corollary 4.1.** If \( B(T) \) is an uncountable h.i.-algebra, then \( \eta \leq B(T) \), where \( \eta \) is the order type of the set of rationals under their natural ordering.

**Proof.** \( B(T) \) is an h.i.-algebra, say \( \text{Int}(L) \). Hence, by Proposition 2.3, \( \omega_1 \) and \( \omega_1^* \) do not embed in \( L \). Hence, by [11, Corollary 5.30], \( \eta \leq L \), and therefore \( \eta \leq B(T) \). □
Corollary 4.2. If $B$ is an interval algebra of uncountable density, that is, $\pi(B) \neq \aleph_0$, then there is a subalgebra $B_0$ of $B$ which is not an interval algebra.

Corollary 4.3. There is an interval algebra satisfying the condition in Corollary 4.2.

To see this, take, for example, $\text{Int}(L)$ with $L = \omega_1$. Notice that $\text{Int}(\omega_1)$ is a superatomic algebra. For an atomless interval algebra, one can take $\text{Int}(L)$ with $L = \eta \cdot \omega_1$.

Remark 4.4. First notice that in the main theorem, we showed that (i) implies (ii) and by a result of Nikiel (see, e.g., [8, Theorem 3.1]), (i) and (ii) are not equivalent statements in the main theorem. For instance, an erratum appeared in [9] showing that the proof of [8, Theorem 3.1] is wrong. To see this, we consider the following counterexample that appeared in [10] and that was communicated to us by Professor L. Heindorf.

Let $\mathbb{Q}$ denote the subset of all rational numbers and $\mathbb{P}$ the subset of all irrational numbers of $[0,1]$. Let $Y = ((0,1] \times \{0\}) \cup (\mathbb{P} \times \{1\})$, let $\leq$ denote the lexicographic ordering on $Y$, and take $Y$ with its order topology. Then $Y$ is a linearly ordered compact space and $\mathbb{Q} \times \{0\}$ is a dense subset.

Let $X = Y \cup Q$ with the following topology: the points of $\mathbb{Q}$ are isolated and basic neighborhoods of each $(t,i) \in Y$ are of the form $U \cup \{s \in \mathbb{Q} : s \neq t$ and $(s,i) \in U\}$, where $U$ is an open neighborhood of $(t,i)$ in $Y$. One can see that $X$ is a boolean separable space, a continuous image of an orderable space, and yet not orderable.

Acknowledgments. The authors would like to thank the referee and also Professor L. Heindorf for their helpful comments and suggestions.

References


M. Alami: Department of Mathematics, Faculty of Sciences and Technology, Route D'Immouzer, B. P. 2202, Fez, Morocco
*E-mail address:* alami@fstf.ma

D. Zhani: Department of Mathematics, Faculty of Sciences Dhar El Mehrez, B. P. 1796, Fez, Morocco
*E-mail address:* dzhani@fsdmfes.ac.ma
Special Issue on
Boundary Value Problems on Time Scales

Call for Papers

The study of dynamic equations on a time scale goes back to its founder Stefan Hilger (1988), and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics; moreover, it often reveals the reasons for the discrepancies between two theories.

In recent years, the study of dynamic equations has led to several important applications, for example, in the study of insect population models, neural network, heat transfer, and epidemic models. This special issue will contain new researches and survey articles on Boundary Value Problems on Time Scales. In particular, it will focus on the following topics:

- Existence, uniqueness, and multiplicity of solutions
- Comparison principles
- Variational methods
- Mathematical models
- Biological and medical applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/ade/guidelines.html. Authors should follow the Advances in Difference Equations manuscript format described at the journal site http://www.hindawi.com/journals/ade/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>April 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>July 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>October 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor
Alberto Cabada, Departamento de Análise Matemática, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain; alberto.cabada@usc.es

Guest Editor
Victoria Otero-Espinar, Departamento de Análise Matemática, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain; mvictoria.otero@usc.es