INFINITE MATRICES, WAVELET COEFFICIENTS AND FRAMES

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We study the action of $A$ on $f \in L^2(\mathbb{R})$ and on its wavelet coefficients, where $A = (a_{lmjk})_{lmjk}$ is a double infinite matrix. We find the frame condition for $A$-transform of $f \in L^2(\mathbb{R})$ whose wavelet series expansion is known.

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1. Introduction. The notation of frame goes back to Duffin and Schaeffer [7] in the early 1950s to deal with the problems in nonharmonic Fourier series. There has been renewed interest in the subject related to its role in wavelet theory. For a glance of the recent development and work on frames and related topics, see [3, 4, 5, 6, 9]. In this note, we will use the regular double infinite matrices (see [9, 10]) to obtain the frame conditions and wavelet coefficients.

2. Notations and known results. $\mathbb{N}$ is the set of positive integers, $\mathbb{Z}$ is the set of integers, $\mathbb{R}$ is the set of real numbers. The space $L^2(\mathbb{R})$ of measurable function $f$ is defined on the real line $\mathbb{R}$, that satisfies

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty. \quad (2.1)$$

The inner product of two square integrable functions $f, g \in L^2(\mathbb{R})$ is defined as

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx,$$

$$\|f\|^2 = (f, f)^{1/2}. \quad (2.2)$$

Every function $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{j,k \in \mathbb{Z}} C_{j,k} \psi_{j,k}(x). \quad (2.3)$$

This series representation of $f$ is called wavelet series. Analogous to the notation of Fourier coefficients, the wavelet coefficients $C_{j,k}$ are given by

$$C_{j,k} = \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx = \langle f, \psi_{j,k} \rangle,$$

$$\psi_{j,k} = 2^{j/2} \psi(2^j x - k). \quad (2.4)$$
Now, if we define an integral transform
\[(W\psi f)(b,a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x) \psi\left(\frac{x-b}{a}\right) dx, \quad f \in L^2(\mathbb{R}),\]
then the wavelet coefficients become
\[C_{j,k} = (W\psi f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right),\]
\[A sequence \{x_n\} in a Hilbert space H is a frame if there exist constants c_1 and c_2, 0 < c_1 \leq c_2 < \infty, such that\]
\[c_1 \|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, x_n \rangle|^2 \leq c_2 \|f\|^2,\]
for all \(f \in H\). The supremum of all such numbers \(c_1\) and infimum of all such numbers \(c_2\) are called the frame bounds of the frame. The frame is called tight frame when \(c_1 = c_2\) and is called normalized tight frame when \(c_1 = c_2 = 1\). Any orthonormal basis in a Hilbert space \(H\) is a normalized tight frame. The connection between frames and numerically stable reconstruction from discretized wavelet was pointed out by Grossmann et al. [8]. In 1985, they defined that a wavelet function \(\psi \in L^2(\mathbb{R})\), constitutes a frame with frame bounds \(c_1\) and \(c_2\), if for any \(f \in L^2(\mathbb{R})\) such that
\[c_1 \|f\|^2 \leq \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 \leq c_2 \|f\|^2.\]
Again, it is said to be tight if \(c_1 = c_2\) and is said to be exact if it ceases to be frame by removing any of its elements. There are many examples proposed by Daubechies et al. [6]. For further details, one can refer to [1, 5, 6]. Chui and Shi [2] proved that \(\{\psi_{j,k}\}\) is a frame for \(L^2(\mathbb{R})\) with bounds \(c_1\) and \(c_2\), if for some \(a > 1\) and \(b > 0\), the Fourier transform \(\hat{\psi}\) satisfies
\[c_1 \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} |\hat{\psi}(a^j w)|^2 \leq c_2 \text{ a.e.,}\]
for some constants \(c_1\) and \(c_2\). By integrating each term in
\[\frac{c_1}{|w|} \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} \frac{|\hat{\psi}(a^j w)|^2}{|w|} \leq \frac{c_2}{|w|},\]
over \(1 \leq |w| \leq a\), we have
\[2c_1 \log a \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} \int_{1 \leq |w| \leq a} \frac{|\hat{\psi}(a^j w)|^2}{|w|} dw \leq 2c_2 \log a,\]
which immediately yields
\[c_1 \leq \frac{1}{2b \log a} \int_{-\infty}^{\infty} \frac{|\hat{\psi}(a^j w)|^2}{|w|} dw \leq c_2.\]
Let $A = (a_{mnjk})$ be a double infinite matrix of real numbers. Then, $A$-transform of a double sequence $x = (x_{jk})$ is

$$
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{mnjk} x_{jk},
$$

which is called $A$-means or $A$-transform of the sequence $x = (x_{ij})$. This definition is due to Móricz and Rhoades [9].

A double matrix $A = (a_{mnjk})$ is said to be regular (see [10]) if the following conditions hold:

(i) $\lim_{m,n \to \infty} \sum_{j,k=0}^{\infty} a_{mnjk} = 1$,

(ii) $\lim_{m,n \to \infty} \sum_{j=0}^{\infty} |a_{mnjk}| = 0, \ (k = 0, 1, 2, \ldots)$,

(iii) $\lim_{m,n \to \infty} \sum_{k=0}^{\infty} |a_{mnjk}| = 0, \ (j = 0, 1, 2, \ldots)$,

(iv) $\|A\| = \sup_{m,n>0} \sum_{j,k=0}^{\infty} |a_{mn}| < \infty$.

Either of conditions (ii) and (iii) implies that

$$
\lim_{m,n \to \infty} a_{mnjk} = 0.
$$

In this note, we establish the frame condition by using $A$-transform of nonnegative regular matrix, also we find action of the matrix $A$ on wavelet coefficients.

3. Main results. In this section, we prove the following theorems.

**Theorem 3.1.** Let $A = (a_{iljk})$ be a double nonnegative regular matrix. If

$$
f(x) = \sum_{j,k \in \mathbb{Z}} C_{j,k} \psi_{j,k}(x)
$$

is a wavelet expansion of $f \in L^2(\mathbb{R})$ with wavelet coefficients

$$
C_{j,k} = \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx = \langle f, \psi_{j,k} \rangle,
$$

then the frame condition for $A$-transform of $f \in L^2(\mathbb{R})$ is

$$
c_1 \|f\|^2 \leq \sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \leq c_2 \|f\|^2,
$$

where $Af$ is the $A$-transform of $f$ and $0 < c_1 \leq c_2 < \infty$.

**Theorem 3.2.** If $C_{j,k}$ are the wavelet coefficients of $f \in L^2(\mathbb{R})$, that is, $C_{j,k} = \langle f, \psi_{j,k} \rangle$, then the $d_{l,m}$ are the wavelet coefficients of $Af$, where $\{d_{l,m}\}$ is defined as the $A$-transform of $\{C_{j,k}\}$ by

$$
d_{l,m} = \sum_{j,k=-\infty}^{\infty} a_{lmjk} C_{jk}.
$$
Theorem 3.3. Let $A = (a_{lm,j,k})$ be a double nonnegative matrix whose elements are $(\psi_{j,k}, \psi_{l,m})$. Then, $\{\psi_{j,k}\}$ constitutes a frame of $L^2(\mathbb{R})$ if and only if $\{\psi_{l,m}\}$ constitutes a frame of $L^2(\mathbb{R})$, where $C_{j,k} = \langle f, \psi_{j,k} \rangle$ and $d_{l,m} = \langle f, \psi_{l,m} \rangle$.

Proof of Theorem 3.1. We can write

$$f(x) = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}. \quad (3.5)$$

If we take $A$-transform of $f$, we get

$$Af(x) = \sum_{i,l \in \mathbb{Z}} \langle Af, \psi_{i,l} \rangle \psi_{i,l}, \quad (3.6)$$

and therefore

$$\sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \leq \sum_{i,l \in \mathbb{Z}} \int_{-\infty}^{\infty} |Af(x)|^2 |\psi_{i,l}(x)|^2 dx \leq \|A\|^2 \|f\|_2^2 \sum_{i,l \in \mathbb{Z}} \|\psi_{i,l}\|_2^2. \quad (3.7)$$

Since $A$ is regular matrix and $\|\psi_{i,l}\|_2 = 1$, therefore

$$\sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \leq c_2 \|f\|_2^2, \quad (3.8)$$

where $c_2$ is positive constant.

Now, for any arbitrarily $f \in L^2(\mathbb{R})$, define

$$\tilde{f} = \left[ \sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \right]^{-1/2} f. \quad (3.9)$$

Clearly,

$$\langle A\tilde{f}, \psi_{i,l} \rangle = \left[ \sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \right]^{-1/2} \langle Af, \psi_{i,l} \rangle, \quad (3.10)$$

then

$$\sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \leq 1. \quad (3.11)$$
Hence, if there exists $\alpha$ a positive constant, then
\[
\|A\hat{f}\|_2^2 \leq \alpha,
\]
\[
\left[ \sum_{i,l \in \mathbb{Z}} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \right]^{-1} \|Af\|_2^2 \leq \alpha.
\] (3.12)

Since $A$ is regular, we have
\[
\left[ \sum_{i,l \in \mathbb{Z}} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \right]^{-1} \|f\|_2^2 \leq \alpha_1 \left( \frac{\alpha}{\|A\|^2} \right),
\] (3.13)
where $\alpha_1$ is another positive constant. Therefore,
\[
c_1 \|f\|_2^2 \leq \sum_{i,l \in \mathbb{Z}} \left| \langle Af, \psi_{i,l} \rangle \right|^2,
\] (3.14)
where $c_1 = \alpha > 0$.

Combining (3.8) and (3.14), we have
\[
c_1 \|f\|_2^2 \leq \sum_{i,l \in \mathbb{Z}} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \leq c_2 \|f\|_2^2.
\] (3.15)

This completes the proof. \(\square\)

**Proof of Theorem 3.2.** We can write
\[
\langle Af, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} Af(x) \overline{\psi_{l,m}(x)} dx
\]
\[
= \int_{-\infty}^{\infty} \sum_{j,k=-\infty}^{\infty} a_{lm} \overline{c_{j,k} \psi_{j,k}(x)}} \overline{\psi_{l,m}(x)} dx.
\] (3.16)

Now,
\[
\sum_{l,m=-\infty}^{\infty} \langle Af, \psi_{l,m} \rangle \psi_{l,m} = \sum_{l,m=-\infty}^{\infty} \sum_{j,k=-\infty}^{\infty} a_{lm} \overline{c_{j,k} \psi_{j,k}(x)}} \overline{\psi_{l,m}(x)} dx
\]
\[
= \sum_{l,m=-\infty}^{\infty} d_{l,m} \psi_{l,m} \int_{-\infty}^{\infty} \|\psi_{l,m}(x)\|_2^2
\]
\[
= \sum_{l,m=-\infty}^{\infty} d_{l,m} \psi_{l,m}.
\] (3.17)

Therefore,
\[
\sum_{l,m=-\infty}^{\infty} d_{l,m} \psi_{l,m} = \sum_{l,m=-\infty}^{\infty} \langle Af, \psi_{l,m} \rangle \psi_{l,m}.
\] (3.18)

This implies that $d_{l,m}$ are wavelet coefficients of $Af$. 
Thus,
\[ d_{l,m} = \langle f, \psi_{l,m} \rangle. \]  
(3.19)

This completes the proof.

**Proof of Theorem 3.3.** We observe that

\[
a_{lmjk}C_{j,k} = \langle \psi_{j,k}, \psi_{l,m} \rangle \langle f, \psi_{j,k} \rangle
\]
= \[\int_{-\infty}^{\infty} \psi_{j,k}(x) \overline{\psi_{l,m}(x)} \, dx \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) \, dx\]
= \[\int_{-\infty}^{\infty} f(x) \overline{\psi_{l,m}(x)} \, dx \int_{-\infty}^{\infty} \psi_{j,k}(x) \psi_{j,k}(x) \, dx\]
= \[\int_{-\infty}^{\infty} f(x) \overline{\psi_{l,m}(x)} \, dx\]
= \[\langle f, \psi_{l,m} \rangle,\]
(3.20)

that is, \[a_{lmjk}C_{j,k} = d_{l,m}.\]

Now,
\[
\sum_{l,m} |d_{l,m}|^2 = \sum_{l,m} |a_{lmjk}C_{j,k}|^2 = \sum_{l,m} |\langle f, \psi_{l,m} \rangle|^2
\]
= \[\frac{1}{(2\pi)^2} \sum_{l,m} |\langle \hat{f}, \hat{\psi}_{l,m} \rangle|^2,\]
(3.21)

by Parseval’s formula for trigonometric Fourier series.

Now
\[
\left| \sum_{p=\infty}^{\infty} \hat{f}(w + 2\pi p) \overline{\psi(w + 2\pi p)} e^{ilmw} \right|^2
\]
= \[\frac{1}{2\pi} \int_{0}^{2\pi} \left| \sum_{p=\infty}^{\infty} \hat{f}(w + 2\pi p) \overline{\psi(w + 2\pi p)} \right|^2 \, dw,\]
(3.22)

Let \(f(w) = \sum_{q=\infty}^{\infty} \hat{f}(w + 2\pi q) \overline{\psi(w + 2\pi q)}.\)
Therefore,

\[
p = \frac{1}{2\pi} \left( \int_{0}^{2\pi} \left| \sum_{p=-\infty}^{\infty} \hat{f}(w + 2\pi p) \hat{\psi}(w + 2\pi p) dw \right|^2 \right)
\]

\[
= \frac{1}{2\pi} \left( \int_{0}^{2\pi} \sum_{p=-\infty}^{\infty} \hat{f}(w + 2\pi p) \hat{\psi}(w + 2\pi p) dw F(w) dw \right)
\]

\[
= \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} \hat{f}(w) \hat{\psi}(w) F(w) dw \right)
\]

\[
= \frac{1}{2\pi} \left\{ \sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(w) \hat{\psi}(w) \hat{f}(w + 2\pi q) \hat{\psi}(w + 2\pi q) dw \right\}
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) \hat{\psi}(w) \hat{f}(w + 2\pi q) \hat{\psi}(w + 2\pi q) dw
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(w)|^2 |\hat{\psi}(w)|^2 dw
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw
\]

\[
= \|f\|_2^2,
\]

that is,

\[
\sum_{l,m} |d_{lm}|^2 = \|f\|_{L_2}^2, \quad f \in L_2(\mathbb{R}). \tag{3.24}
\]

Therefore, for a regular matrix \( A = (a_{lmjk}) \), we have

\[
c_1 \|f\|_2^2 \leq \sum_{l,m} |d_{lm}|^2 \leq c_2 \|f\|_2^2 \tag{3.25}
\]

if and only if

\[
c_1' \|f\|_2^2 \leq \sum_{j,k} |c_{jk}|^2 \leq c_2' \|f\|_2^2, \tag{3.26}
\]

where, \( 0 \leq c_1', c_2' < \infty \). This completes the proof. \(\square\)

**REFERENCES**


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