POWERSUM FORMULA FOR DIFFERENTIAL RESOLVENTS

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We will prove that we can specialize the indeterminate $\alpha$ in a linear differential $\alpha$-resolvent of a univariate polynomial over a differential field of characteristic zero to an integer $q$ to obtain a $q$-resolvent. We use this idea to obtain a formula, known as the powersum formula, for the terms of the $\alpha$-resolvent. Finally, we use the powersum formula to rediscover Cockle’s differential resolvent of a cubic trinomial.

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1. Introduction. It was proved in [4, Theorem 37, page 67] that for any integer $q$, a polynomial $P(t) \equiv \sum_{k=0}^N (-1)^{N-k} e_{N-k} t^k$ of a single variable $t$ whose coefficients $\{e_{N-k}\}_{k=0}^N$ lie in an ordinary differential ring $\mathbb{R}$ with derivation $D$ possesses an ordinary linear differential $\alpha$-resolvent and an ordinary linear differential $q$-resolvent, where $\alpha$ is a constant, transcendental over $\mathbb{R}$. With no loss of generality, we assume that $P$ is monic and has no zero roots. Then the coefficient $e_{N-k}$ is the $(N-k)$th elementary symmetric function of the roots of $P$. We assume that $P(t) = \prod_{k=1}^n (t - z_k)^{\pi_k}$ has $n \leq N$ distinct roots and $\pi_k$ is the multiplicity of the root $z_k$ in $P$. It was proved in [2] that for each root $z_k$, there exists a nonzero solution $y_k$ of the logarithmic differential equation $Dy_k/y_k = \alpha \cdot (Dz_k/z_k)$. Obviously, such solutions are unique only up to a constant multiple. We define the notation $z_k^\alpha$ to represent any such solution $y_k$. Hence, we will call $y_k$ an $\alpha$-power of $z_k$. From now on, we will drop the subscript $k$ on $z_k$ and $y_k$. It will be understood that a different $z$ implies a different $y$.

In this paper, we present the powersum formula as a new method for computing resolvents, although it remains a conjecture whether the powersum formula always yields a (nonzero) resolvent rather than an identically zero equation. It was proved in [5, Theorem 4.1, page 726] that if all the distinct roots of a polynomial are differentially independent over constants, then the powersum formula yields a resolvent. It was shown in [6, Section 11, pages 344-345], how the solution of the Riccati nonlinear differential equation is related to the resolvent of a quadratic polynomial.

2. Notation. Let $\mathbb{N}$ denote the set of positive integers. Let $\mathbb{N}_0$ denote the set of non-negative integers. Let $\mathbb{Z}^\#$ denote the set of nonzero integers. The following notation has been slightly modified from Kolchin’s notation in [2] and Macdonald’s notation in [3]. Let $\mathbb{Z}\{e\}$ denote the differential ring generated by the integers $\mathbb{Z}$ and the $N$ coefficients $e \equiv \{e_k\}_{k=1}^N$ of $t$ in $P$. Let $\mathbb{Q}\{e\}$ denote the differential field generated by the rational numbers $\mathbb{Q}$ and $e$. For each $m \in \mathbb{N}$, let $\mathbb{Z}\{e\}_m$ denote the ordinary (nondifferential) ring
generated by \( \mathbb{Z}, e, \) and the first \( m \) derivatives of \( e. \) (A differential ring must contain infinitely many derivatives of any of its elements.) Let \( \mathbb{Q}(e)_{m}(z) = \mathbb{Q}(e)[z] \) denote the field generated by \( \mathbb{Q}, e, \) the first \( m \) derivatives of \( e, \) and the single root \( z. \) From this point on, we will write \( \mathbb{Q}(e)_{m}[z] \) instead of \( \mathbb{Q}(e)(z) \) for this field to emphasize the fact that elements in this field are polynomial in the root \( z. \) If \( \mathbb{R} \) represents any of the rings or fields mentioned so far, then let \( \mathbb{R}[t, \alpha] \) denote the polynomial ring in the indeterminates \( t \) and \( \alpha \) over \( \mathbb{R}. \)

**Definition 2.1.** Define \( G_{m}(t, \alpha) \) to be the polynomial in \( t \) and \( \alpha \) such that \( G_{m}(z, \alpha) = D^{m}y/(\alpha \cdot y) \) for each root \( z \) of \( P \) and \( D^{m} \cdot G_{m}(t, \alpha) \in \mathbb{Z}[e]_{m}[t, \alpha]. \)

A specialization \( \phi \) is a ring homomorphism \( \phi : \mathbb{R} \to \hat{\mathbb{R}} \) from a ring \( \mathbb{R} \) into an integral domain \( \hat{\mathbb{R}}. \) For any polynomial \( P(t) = \sum_{N=0}^{\infty} (-1)^{N}e_{N-k} \cdot t \in \mathbb{R}[t], \) \( \phi(P) \) is defined to be the polynomial \( (\phi P)(t) = \sum_{N=0}^{\infty} (-1)^{N}k \phi(e_{N-k}) \cdot t \in \mathbb{R}[t]. \) A differential specialization \( \phi \) is a specialization \( \phi : \mathbb{R} \to \hat{\mathbb{R}} \) from a differential ring \( \mathbb{R} \) with derivation \( D \) into a differential integral domain \( \hat{\mathbb{R}} \) with derivation \( \hat{D} \) such that \( \phi D = \hat{D} \phi \) on \( \mathbb{R}. \)

3. Specializing \( \alpha. \) Let \( q \in \mathbb{N}. \) Let \( \phi_{q} : \mathbb{Q}(e)[z, \alpha] \to \mathbb{Q}(e)[z] \) be the ring specialization such that \( \phi_{q} \) is the identity on \( \mathbb{Q}(e)[z] \) and \( \phi_{q}(\alpha) = q. \) We may compute \( Dz^{q} / (q \cdot z^{q}). \) Since \( \phi_{q} \) is not defined to act on \( y, \) we are not able to specialize \( y \) to \( z^{q} \) in Theorem 3.1. However, \( \phi_{q} \) is defined to act on \( D^{m}y/(\alpha \cdot y) \) since \( D^{m}y/(\alpha \cdot y) = G_{m}(z, \alpha) / 0^{m} \cdot \mathbb{Z}[e]_{m}[z, \alpha]. \) Theorem 3.1 asserts that \( G_{m}(z, q) = D^{m}z^{q} / (q \cdot z^{q}). \) Theorem 3.2 asserts that \( \phi_{q}(D^{m}y/(\alpha \cdot y)) = D^{m}z^{q} / (q \cdot z^{q}). \)

**Theorem 3.1.** Assume all the same definitions and notations as in the introduction. Then the \( m \)th derivative of \( z^{q} \) can be expressed as a product of \( q \cdot z^{q} \) and an element in \( \mathbb{Q}(e)_{m}[z]. \) More specifically, \( G_{m}(z, q) = D^{m}z^{q} / (q \cdot z^{q}), \) where \( G_{m}(z, q) \in \mathbb{Q}(e)_{m}[z] \) and \( G_{m}(t, \alpha) \) was given in Definition 2.1.

**Proof.** For brevity, write \( G_{m} = G_{m}(z, \alpha) \) for the particular root \( z. \) We emphasize that \( G_{m} \) is \( G_{m}(t, \alpha) \) with \( t \) specialized to the particular root \( z. \) We find that \( \theta \cdot (Dz^{q} / (q \cdot z^{q})) = \theta \cdot (Dz/z) = \theta \cdot G_{1} \in \mathbb{Z}[e]_{1}[z]. \) Therefore,

\[
Dz^{q} = q \cdot z^{q} \cdot G_{1} \Rightarrow D^{2}z^{q} = q \cdot (q \cdot z^{q-1}Dz \cdot G_{1} + z^{q} \cdot DG_{1})
\]

\[
= q \cdot z^{q} \left( q \cdot \left( \frac{Dz}{z} \right) \cdot G_{1} + DG_{1} \right) = q \cdot z^{q} \cdot (q \cdot G_{1}^{2} + DG_{1}) \quad (3.1)
\]

\[
= q \cdot z^{q} \cdot \phi_{q}(\alpha \cdot G_{1}^{2} + DG_{1}) = q \cdot z^{q} \cdot \phi_{q}(G_{1}).
\]

So, \( D^{m}z^{q} = q \cdot z^{q} \cdot \phi_{q}(G_{m}) \) is true for \( m = 1. \) Now assume that it is true for \( m \geq 2. \) Then

\[
D^{m+1}z^{q} = q \cdot (q \cdot z^{q-1}(Dz) \cdot \phi_{q}(G_{m}) + z^{q} \cdot D(\phi_{q}(G_{m}))).
\]

But \( \phi_{q} \) specializes \( \alpha, \) whose derivative is 0, to an integer whose derivative is 0. Thus, \( D(\phi_{q}(G_{m})) = \phi_{q}(D(G_{m})). \)
Hence,
\[
D^{m+1}z^q = q \cdot (q \cdot z^{q-1} \cdot (Dz) \cdot \phi_q(G_m) + z^q \cdot \phi_q(D(G_m)))
\]
\[
= q \cdot z^q \cdot \left( q \cdot \frac{Dz}{z} \cdot \phi_q(G_m) + \phi_q(D(G_m)) \right)
\]
\[
= q \cdot z^q \cdot (\phi_q(\alpha) \cdot G_1 + \phi_q(D(G_m)))
\]
\[
= q \cdot z^q \cdot \phi_q(\alpha \cdot G_1 \cdot G_m + D(G_m))
\]
\[
= q \cdot z^q \cdot \phi_q(G_{m+1}).
\]

Therefore, \( D^{m+1}z^q = q \cdot z^q \cdot G_{m+1}(z,q) \) since \( \phi_q \) affects only \( \alpha \). By the principle of mathematical induction, this equation is true for all positive integers \( m \).

Just because \( D_y/(\alpha \cdot y) = D_z/z = Dz^q/(q \cdot z^q) \) implies that \( D_y/(\alpha \cdot y) \) is independent of \( \alpha \), it does not follow that \( D^m y/(\alpha \cdot y) \) is independent of \( \alpha \) for \( m \geq 2 \). We can see this by observing that \( D^m y/(\alpha \cdot y) \neq D^m z/z \neq D^m z^q/(q \cdot z^q) \neq D^m y/(\alpha \cdot y) \) for \( m \geq 2 \).

**Theorem 3.2.** Assume all the same definitions and notations as in Theorem 3.1 and Section 2. Then, for each \( m \in \mathbb{N} \), the specialization under \( \phi_q \) of \( D^m y/(\alpha \cdot y) \) is \( D^m z^q/(q \cdot z^q) \). That is, \( \phi_q(D^m y/(\alpha \cdot y)) = D^m z^q/(q \cdot z^q) \).

**Proof.** By Definition 2.1, \( D^m y/(\alpha \cdot y) = G_m(z,\alpha) \). By Theorem 3.1, \( D^m z^q/(q \cdot z^q) = G_m(z,q) \). Putting these results together yields

\[
\phi_q \left( \frac{D^m y}{\alpha \cdot y} \right) = \phi_q(G_m(z,\alpha)) = G_m(z,q) = \frac{D^m z^q}{q \cdot z^q}. 
\]  

4. Powersum satisfaction theorem and formula. An \( \alpha \)-resolvent of a polynomial \( P(t) = \sum_{i=0}^{N} (-1)^{n-i} e_{N-i} t^i \in \mathbb{F}[t] \) over a differential field \( \mathbb{F} \) with derivation \( D \) is a linear ordinary differential equation \( \sum_{m=0}^{\alpha} B_m(\alpha) \cdot D^m y = 0 \) of finite order \( \alpha \) such that each of the coefficient functions \( B_m(\alpha) \) lies in the field \( \mathbb{Q}(e)(\alpha) \) (or preferably in the ring \( \mathbb{Z}[e][\alpha] \)) such that not all \( B_m(\alpha) \) are identically zero, and which is satisfied by the \( \alpha \)-power of every root \( z \) of \( P \). In other words, the coefficient functions of the resolvent are independent of the choice of root and are not all zero. By [4, Theorem 37, page 67], resolvents for any polynomial are guaranteed to exist. We state this assertion in Theorem 4.1.

**Theorem 4.1.** Let \( P(t) = \sum_{i=0}^{N} (-1)^{n-i} e_{N-i} t^i \in \mathbb{F}[t] \) be a polynomial of degree \( N \) in \( t \) over a \( d \)-field \( \mathbb{F} \) with \( n \) distinct roots \( \{z_i\}_{i=1}^{n} \). Then there exists an \( \alpha \)th order differential resolvent \( \sum_{m=0}^{\alpha} B_m(\alpha) \cdot D^m y = 0 \) with \( B_m(\alpha) \in \mathbb{Z}[e][\alpha] \), \( B_0(0) = 0 \), and \( \deg_{\alpha} B_m(\alpha) \leq \frac{o(o-1)(o-2)}{2} - m + 1 \) for some \( o \in [n] \). Furthermore, \( o \) may be chosen to equal the number of \( \{y_j\}_{j=1}^{n} \) linearly independent over constants, and all solutions of this resolvent are linear combinations over constants of these \( o \) \( y_j \)’s.

Theorem 4.1 gives us an upper bound on the degree in \( \alpha \) in an \( \alpha \)-resolvent of \( P \). Theorem 4.2 allows us to specialize the indeterminate \( \alpha \) to an integer \( q \) (or any number) to obtain a \( q \)-resolvent.
**Theorem 4.2 (powersum satisfaction theorem).** Let $P \in \mathbb{F}[t]$ be a monic polynomial with $n$ distinct roots $z = \{z_i\}_{i=1}^n$, none of which is zero and not all of which are constants. Let $q \in \mathbb{Z}$. If $R_\alpha = \sum_{m=0}^o B_m(\alpha) \cdot D^m y$ is an $\alpha$-resolvent for $P$ of arbitrary order $o$, where $B_m(\alpha) = \sum_{i \geq 0} b_{i,m} \alpha^i \in \mathbb{Z}[\alpha]$, with $b_{i,m} \in \mathbb{Z}[e]$, then $R_\alpha$ specializes to the $q$-resolvent $R_q = \sum_{m=0}^o B_m(q) \cdot D^m y$ for $q \in \mathbb{Z}^\#$ under $\phi_q(\alpha) = q$ and $\phi_q(u) = u$ for each $u \in \mathbb{Z}[e]$. Furthermore, the $q$th powersum formula $p_q$ satisfies $\sum_{m=0}^o B_m(q) \cdot D^m p_q = 0$ for each $q \in \mathbb{Z}^\#$.

**Proof.** By Definition 2.1 of $G_m(t, \alpha)$, we have

$$\sum_{m=0}^o B_m(\alpha) \cdot D^m y = 0 \iff \sum_{m=0}^o B_m(\alpha) \cdot \frac{D^m y}{\alpha^y} = 0 \iff \sum_{m=0}^o B_m(\alpha) \cdot G_m(z, \alpha) = 0. \quad (4.1)$$

Now for each $q \in \mathbb{Z}^\#$, specialize this equation under $\phi_q$ to get $\sum_{m=0}^o B_m(q) \cdot G_m(z, q) = 0$ by Theorem 3.2, since $\phi_q|f = I$. For any of the roots of $P$, we have $G_m(z, q) = D^m z^q / (q \cdot z^q)$ by Theorem 3.1. Thus,

$$\sum_{m=0}^o B_m(q) \cdot D^m z^q / q \cdot z^q = 0 \iff \sum_{m=0}^o B_m(q) \cdot D^m z^q = 0 \quad (4.2)$$

for each $q \in \mathbb{Z}^\#$. Therefore, an $\alpha$-resolvent specializes to a $q$-resolvent for each $q \in \mathbb{Z}^\#$ under $\phi_q$. Now sum over the $N$ roots of $P$ including their multiplicities to get $\sum_{m=0}^o B_m(q) \cdot D^m p_q = 0$ for each $q \in \mathbb{Z}^\#$. \hfill \square

The powersum satisfaction theorem states that for any monic polynomial $P$, the coefficients $b_{i,m}$ of $\alpha$ in any $\alpha$-resolvent $R_\alpha = \sum_{(i,m) \in S} b_{i,m} \cdot \alpha^i D^m y$ of $P$ satisfy an infinite system of homogeneous equations

$$[q^i D^m p_q]_{1 \leq q < \infty (i,m) \in S} \cdot [b_{i,m}]_{(i,m) \in S} = [0_q]_{1 \leq q < \infty}. \quad (4.3)$$

Here, $S$ denotes the set of pairs $(i,m)$ consisting of a power of $\alpha$, denoted by $i$, and an order of a derivative, denoted by $m$, such that $b_{i,m} \neq 0$. Let $|S|$ denote the size of $S$. We will be interested in proving that the rank

$$\text{rk}[q^i D^m p_q]_{1 \leq q < \infty (i,m) \in S} \quad (4.4)$$

of the matrix $[q^i D^m p_q]_{1 \leq q < \infty (i,m) \in S}$ equals $|S| - 1$ under certain circumstances. Under those circumstances, one can solve this system of equations to get a nonzero solution for $b_{i,m}$. The solution is given by $b_{i,m} = F_{i,m} \equiv (-1)^{\text{sgn}(i,m)} \cdot |q^i D^m p_q|_{(i,m) \in S} = 0_{q \in \mathbb{Z}^\#}$, where $\text{sgn}(i,m)$ indicates the ordering of the term $b_{i,m}$ in the resolvent, and we take $\Gamma$ to be the smallest possible set of positive integers that will guarantee a nonzero solution. In numerous examples, it has been found that $\Gamma \equiv \{k \in \mathbb{N} : 1 \leq k \leq |S| - 1\}$. We call this the powersum formula for a resolvent of $P$. We use the notation $F_{i,m}$ to denote the terms of the resolvent obtained by this method to suggest the word formula. We will denote the resolvent obtained by this formula by $R_\alpha$. So, $R_\alpha = \{F_{i,m}\}$. 


If \( \text{rk}[q^iD^mp_q]_{1 \leq q < \infty \ (i,m) \in S} = |S| \), then the only solution would be \( b_{i,m} = 0 \) for all \((i,m) \in S\), contradicting the hypothesis that \( R_\alpha \) is nonzero. Unfortunately, for a given polynomial \( P \), one does not know a priori what the set \( S \) of nonzero \( b_{i,m} \) is or how large it is. Nevertheless, we may summarize the results obtained so far in a corollary to the powersum satisfaction theorem.

**Corollary 4.3** (the powersum formula). Let \( R_\alpha \equiv \sum_{(i,m) \in S} b_{i,m} \cdot \alpha^iD^m \gamma \) be an \( \alpha \)-resolvent of \( P \), where \( S \subset \mathbb{N}_0 \times \mathbb{N}_0 \) is a finite set. If there exists a set of \( |S| - 1 \) integers \( \Gamma \subset \mathbb{N} \) such that not all the \( F_{i,m} \) given by the powersum formula \( F_{i,m} \equiv (-1)^{\text{sgn}(i,m)} \cdot |q^iD^mp_q|_{(i',m')=\in (i,m) \in \Gamma} \) are zero, then the linear ordinary differential equation (ODE), \( \mathfrak{R}_\alpha \equiv \sum_{(i,m) \in S} F_{i,m} \cdot \alpha^iD^m \gamma \), is an integral \( \alpha \)-resolvent of \( P \). If no such set of integers \( \Gamma \) exists, then the powersum formula yields all zeroes for \( F_{i,m} \).

The author believes that the resolvent \( \mathfrak{R}_\alpha \) given by the powersum formula will be a \( \mathbb{Q}(\alpha) \)-multiple, not just a \( \mathbb{Q} \langle \alpha \rangle \)-multiple of \( R_\alpha \), but this requires proof. For example, let \( \alpha^N \cdot D^H \gamma \) denote the highest power of \( \alpha \) on the highest derivative of \( \gamma \) in \( R_\alpha \). Even though \( F_{M,L}/b_{M,L} \cdot R_\alpha \) and \( \mathfrak{R}_\alpha \) are both resolvents (provided that \( F_{M,L} \neq 0 \)) with the same coefficient function of \( \alpha^N \cdot D^H \gamma \), one must eliminate the possibility that their other terms may differ due to the possibility that \( P \) has resolvents of lower order.

**5. Example.** We will now apply the powersum formula to compute a particular \( \alpha \)-resolvent of a particular trinomial. It has not yet been proved that this formula yields a nonzero differential equation for every polynomial. However, in every polynomial the author has tested, it has been possible to set up an \( \alpha \)-resolvent, itself a polynomial in the power \( \alpha \), and choose the proper set of powersums such that the powersum formula yields a nonzero answer. If the powersum formula yields a nonzero answer, then it is guaranteed by Corollary 4.3 that the answer is a (nonzero) resolvent of the polynomial. By a very long and difficult proof in [4, Theorem 41, page 74] and [5, Theorem 4.1, page 726], it has been shown that in case the distinct roots of the polynomial are differentially independent over constants (i.e., they satisfy no polynomial differential equations over \( \mathbb{Q} \)), then the powersum formula yields a nonzero resolvent.

The powersum formula has the advantage of giving a resolvent in an integral form. In the next example, this means the powersum formula gives a resolvent all of whose terms lie in the ring \( \mathbb{Z}[x, \alpha] \).

**Example 5.1** (Sir James Cockle’s resolvent of a trinomial). Cockle [1] gave a formula for a linear differential \( \alpha \)-resolvent (although he did not call it that) for any trinomial of the form \( t^n + x \cdot t^p - 1 \), where \( Dx \equiv 1 \). Consider the particular trinomial \( P(t) \equiv t^3 + x \cdot t^2 - 1 \), where \( n = 3 \) and \( p = 2 \). Then, Cockle’s resolvent specializes to \( 27 \cdot D^3 \gamma = 4 \cdot (x \cdot D + \alpha/2)(x \cdot D + 3/2 + \alpha/2)(x \cdot D - \alpha/2)(x \cdot D - \alpha)(x \cdot D - 3/2 + \alpha/2)(x \cdot D - 3/2 - \alpha/2)(x \cdot D - \alpha \cdot (1 + \alpha/2) \cdot (x \cdot D - \alpha^2 \cdot (3 + \alpha/2)) \). Replacing \((x \cdot D)^3\) with \(x^3 \cdot D^3 + 3 \cdot x^2 \cdot D^2 + x \cdot D\) and \((x \cdot D)^2\) with \(x^2 \cdot D^2 + x \cdot D\) yields \((4x^3 - 27) \cdot D^3 \gamma + 18 \cdot x^2 \cdot D^2 \gamma + (10 - 3 \cdot \alpha - 3 \cdot \alpha^2) \cdot x \cdot D \gamma - \alpha^2 \cdot (3 + \alpha) \cdot \gamma = 0\), which has the form \( f_1 \cdot D^3 \gamma + f_2 \cdot D^2 \gamma + (f_3 + f_4 \cdot \alpha + f_5 \cdot \alpha^2) \cdot D \gamma + (f_6 \cdot \alpha^2 + f_7 \cdot \alpha^3) \cdot \gamma = 0\). The powersum formula requires one to know a priori the various powers of \( \alpha \) appearing in a resolvent. Specialize \( \alpha \) to one of the six integers \( q \in \{1, 2, 3, 4, 5, 6\} \), then sum the resulting equation over each of the three
roots. Doing this for each \(q \in \{1, 2, 3, 4, 5, 6\}\), one gets a system of six linear equations in the undetermined coefficient functions \(\{f_k\}_{k=1}^{7}\) of the form \(\mathfrak{M} \cdot \vec{f} = \vec{0}\), where \(\mathfrak{M}\) is the \(6 \times 7\) matrix defined by

\[
\mathfrak{M} \equiv \begin{bmatrix}
D^3 p_1 & D^2 p_1 & Dp_1 & 1 \cdot Dp_1 & 1^2 \cdot Dp_1 & 1^2 \cdot p_1 & 1^3 \cdot p_1 \\
D^3 p_2 & D^2 p_2 & 2 \cdot Dp_2 & 2^2 \cdot Dp_2 & 2^2 \cdot p_2 & 2^3 \cdot p_2 \\
D^3 p_3 & D^2 p_3 & 3 \cdot Dp_3 & 3^2 \cdot Dp_3 & 3^2 \cdot p_3 & 3^3 \cdot p_3 \\
D^3 p_4 & D^2 p_4 & 4 \cdot Dp_4 & 4^2 \cdot Dp_4 & 4^2 \cdot p_4 & 4^3 \cdot p_4 \\
D^3 p_5 & D^2 p_5 & 5 \cdot Dp_5 & 5^2 \cdot Dp_5 & 5^2 \cdot p_5 & 5^3 \cdot p_5 \\
D^3 p_6 & D^2 p_6 & 6 \cdot Dp_6 & 6^2 \cdot Dp_6 & 6^2 \cdot p_6 & 6^3 \cdot p_6
\end{bmatrix},
\]

(5.1)

\(\vec{f}\) is the \(7 \times 1\) column vector defined by

\[
\vec{f} \equiv \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6 \\
f_7
\end{bmatrix},
\]

(5.2)

and \(\vec{0}\) is the \(6 \times 1\) column vector defined by

\[
\vec{0} \equiv \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

(5.3)

The following program, written in Mathematica 4.0 for Students and run on a Dell Dimension XPS R400 computer using Windows 98 operating system, computes the seven terms \(\{f_k\}_{k=1}^{7}\) by setting each \(f_k\) to the appropriate cofactor of \(\mathfrak{M}\). This matrix is denoted by \(T\) in the program. The symbol \(s[k]\) stands for the \(k\)th powersum \(p_k\) of the roots of \(P\). The output is denoted by \(f\), which is defined as the transpose of \(\vec{f}\). The result is

\[
\begin{bmatrix}
-27 + 4x^3 \\
18x^2 \\
10x \\
-3x \\
-3x \\
-3 \\
-1
\end{bmatrix},
\]

(5.4)

which is the Cockle resolvent. The computation time is less than 5 seconds.

\[
x=.; \ s[0]=3; \ s[1]=-x; \ s[2]=x^2; \\
T=Table[D[s[k],{x,3}],D[s[k],{x,2}],D[s[k],x],k*D[s[k],x],k^2*D[s[k],x],k^3*D[s[k],x],{k,1,6}];
\]

f=Table[Simplify[m[[1,k]]*(-1)^(7-k)/(466560*x)],{k,1,7}].
To see the output in Mathematica for other variables, remove the semicolon after its formula. For the record, the first six powersums are (written in the form Mathematica gives) $p_1 = -x$, $p_2 = x^2$, $p_3 = 3 - x^3$, $p_4 = -4x + x^4$, $p_5 = 5x^2 - x^5$, and $p_6 = 3 - 6x^3 + x^6$. The cofactors of the matrix $M$ had to be divided by $466560 \cdot x = 2^7 \cdot 3^6 \cdot 5^1 \cdot x$ to get the resolvent in Cohnian form, that is, such that the only divisors in $\mathbb{Z}[x, \alpha]$ among all the terms of the resolvent are $\pm 1$.

References


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