Chan (2004) considered a certain continued fraction expansion and the corresponding Gauss-Kuzmin-Lévy problem. A Wirsing-type approach to the Perron-Frobenius operator of the associated transformation under its invariant measure allows us to obtain a near-optimal solution to this problem.

1. Introduction

The Gauss 1812 problem gave rise to an extended literature. In modern times, the so-called Gauss-Kuzmin-Lévy theorem is still one of the most important results in the metrical theory of regular continued fractions (RCFs). A recent survey of this topic is to be found in [10]. From the time of Gauss, a great number of such theorems followed. See, for example, [2, 6, 7, 8, 18].

Apart from the RCF expansion there are many other continued fraction expansions: the continued fraction expansion to the nearest integer, grotesque expansion, Nakada’s $\alpha$-expansions, Rosen expansions; in fact, there are too many to mention (see [4, 5, 11, 12, 13, 16, 17] for some background information). The Gauss-Kuzmin-Lévy problem has been generalized to the above continued fraction expansions (see [3, 14, 15, 19, 20, 21]).

Taking up a problem raised in [1], we consider another expansion of reals in the unit interval, different from the RCF expansion. In fact, in [1] Chan has studied the transformation related to this new continued fraction expansion and the asymptotic behaviour of its limit distribution function. Giving a solution to the Gauss-Kuzmin-Lévy problem, he showed in [1, Theorem 1] that the convergence rate involved is $O(q^n)$ as $n \to \infty$ with $0 < q < 1$. This unsurprising result can be easily obtained from well-known general results (see [9, pages 202 and 262–266] and [10, Section 2.1.2]) concerning the Perron-Frobenius operator of the transformation under the invariant measure induced by the limit distribution function.

Our aim here is to give a better estimation of the convergence rate discussed. First, in Section 2 we introduce equivalent, but much more concise and rigorous expressions than in [1] of the transformation involved and of the related incomplete quotients. Next, in Section 3, our strategy is to derive the Perron-Frobenius operator of this transformation.
under its invariant measure. In Section 4, we use a Wirsing-type approach (see [22]) to study the optimality of the convergence rate. Actually, in Theorem 4.3 of Section 4 we obtain upper and lower bounds of the convergence rate which provide a near-optimal solution to the Gauss-Kuzmin-Lévy problem.

2. Another expansion of reals in the unit interval

In this section we describe another continued fraction expansion different from the regular continued fraction expansion for a number \( x \) in the unit interval \( I = [0,1] \), which has been actually considered in [1].

Define for any \( x \in I \) the transformation

\[
\tau(x) = 2^{((\log x^{-1})/\log 2)} - 1, \quad x \neq 0; \quad \tau(0) = 0,
\]

where \( \{u\} \) denotes the fractional part of a real \( u \) while \( \log \) stands for natural logarithm. (Nevertheless, the definition of \( \tau \) is independent of the base of the logarithm used.) Putting

\[
a_n(x) = a_1(\tau^{n-1}(x)), \quad n \in \mathbb{N}_+ = \{1,2,...\},
\]

with \( \tau^0(x) = x \) the identity map and

\[
a_1(x) = \left\lfloor \frac{(\log x^{-1})}{\log 2} \right\rfloor,
\]

where \( \lfloor u \rfloor \) denotes the integer part of a real \( u \), one easily sees that every irrational \( x \in (0,1) \) has a unique infinite expansion

\[
x = \frac{2^{-a_1}}{1 + \frac{2^{-a_2}}{1 + \cdots}} = \left[ a_1,a_2,... \right].
\]

Here, the incomplete quotients or digits \( a_n(x) \), \( n \in \mathbb{N}_+ \) of \( x \in (0,1) \) are natural numbers.

Let \( \mathcal{B}_I \) be the \( \sigma \)-algebra of Borel subsets of \( I \). There is a probability measure \( \nu \) on \( \mathcal{B}_I \) defined by

\[
\nu(A) = \frac{1}{\log(4/3)} \int_A \frac{dx}{(x+1)(x+2)}, \quad A \in \mathcal{B}_I,
\]

such that \( \nu(\tau^{-1}(A)) = \nu(A) \) for any \( A \in \mathcal{B}_I \), that is, \( \nu \) is \( \tau \)-invariant.

3. An operator treatment

In the sequel we will derive the Perron-Frobenius operator of \( \tau \) under the invariant measure \( \nu \).

Let \( \mu \) be a probability measure on \( \mathcal{B}_I \) such that \( \mu(\tau^{-1}(A)) = 0 \) whenever \( \mu(A) = 0 \), \( A \in \mathcal{B}_I \), where \( \tau \) is the continued fraction transformation defined in Section 2. In particular,
this condition is satisfied if $\tau$ is $\mu$-preserving, that is, $\mu \tau^{-1} = \mu$. It is known from [10, Section 2.1] that the Perron-Frobenius operator $P_{\mu}$ of $\tau$ under $\mu$ is defined as the bounded linear operator on $L^1_{\mu} = \{ f : I \to \mathbb{C} \mid \int_I |f| d\mu < \infty \}$ which takes $f \in L^1_{\mu}$ into $P_{\mu} f \in L^1_{\mu}$ with

$$
\int_A P_{\mu} f d\mu = \int_{\tau^{-1}(A)} f d\mu, \quad A \in \mathcal{B}_I.
$$

(3.1)

In particular the Perron-Frobenius operator $P_{\lambda}$ of $\tau$ under the Lebesgue measure $\lambda$ is

$$
P_{\lambda}(x) = \frac{d}{dx} \int_{\tau^{-1}([0,x])} f d\lambda \quad \text{a.e. in } I.
$$

(3.2)

**Proposition 3.1.** The Perron-Frobenius operator $P_{\nu} = U$ of $\tau$ under $\nu$ is given a.e. in $I$ by the equation

$$
Uf(x) = \sum_{k \in \mathbb{N}} p_k(x) f(u_k(x)), \quad f \in L^1_{\nu},
$$

(3.3)

where

$$
p_k(x) = \frac{\gamma^{k+1}(x+1)(x+2)}{(\gamma^k+1)(\gamma^{k+1}+x+1)}, \quad x \in I,
$$

$$
\quad u_k(x) = \frac{\gamma^k}{x+1}, \quad x \in I,
$$

(3.4)

with $\gamma = 1/2$.

The proof is entirely similar to that of [10, Proposition 2.1.2].

An analogous result to [10, Proposition 2.1.5] is shown as follows.

**Proposition 3.2.** Let $\mu$ be a probability measure on $\mathcal{B}_I$. Assume that $\mu \ll \lambda$ and let $h = d\mu/d\lambda$. Then

$$
\mu(\tau^{-n}(A)) = \int_A \frac{U^n f(x)}{(x+1)(x+2)} dx
$$

(3.5)

for any $n \in \mathbb{N}$ and $A \in \mathcal{B}_I$, where $f(x) = (x+1)(x+2)h(x)$, $x \in I$.

**4. A Wirsing-type approach**

Let $\mu$ be a probability measure on $\mathcal{B}_I$ such that $\mu \ll \lambda$. For any $n \in \mathbb{N}$, put

$$
F_n(x) = \mu(\tau^n < x), \quad x \in I,
$$

(4.1)

where $\tau^0$ is the identity map. As $(\tau^n < x) = \tau^{-n}((0,x))$, by Proposition 3.2 we have

$$
F_n(x) = \int_0^x \frac{U^n f_0(u)}{(u+1)(u+2)} du, \quad n \in \mathbb{N}, \, x \in I,
$$

(4.2)

with $f_0(x) = (x+1)(x+2)F_0'(x)$, $x \in I$, where $F_0' = d\mu/d\lambda$. 


In this section we will assume that $F'_0 \in C^1(I)$. So, we study the behaviour of $U^n$ as $n \to \infty$, assuming that the domain of $U$ is $C^1(I)$, the collection of all functions $f : I \to \mathbb{C}$ which have a continuous derivative.

Let $f \in C^1(I)$. Then the series (3.3) can be differentiated term-by-term, since the series of derivatives is uniformly convergent. Putting $\Delta_k = y^k - y^{2k}, k \in \mathbb{N}$ we get

$$p_k(x) = y^{k+1} + \frac{\Delta_k}{y^{k+1}+x+1} - \frac{\Delta_{k+1}}{y^{k+1}+x+1},$$

$$(Uf)'(x) = \sum_{k \in \mathbb{N}} \left[ p_k'(x)f\left(\frac{y^k}{x+1}\right) - p_k(x)\frac{y^k}{(x+1)^2}f'\left(\frac{y^k}{x+1}\right) \right]$$

$$= \sum_{k \in \mathbb{N}} \left[ \left( \frac{\Delta_{k+1}}{(y^{k+1}+x+1)^2} - \frac{\Delta_k}{(y^{k+1}+x+1)^2} \right) f\left(\frac{y^k}{x+1}\right) - p_k(x)\frac{y^k}{(x+1)^2}f'\left(\frac{y^k}{x+1}\right) \right]$$

$$= -\sum_{k \in \mathbb{N}} \left[ \frac{\Delta_{k+1}}{(y^{k+1}+x+1)^2} \left( f\left(\frac{y^{k+1}}{x+1}\right) - f\left(\frac{y^k}{x+1}\right) \right) + p_k(x)\frac{y^k}{(x+1)^2}f'\left(\frac{y^k}{x+1}\right) \right],$$

$x \in I$. Thus, we can write

$$(Uf)' = -Vf', \quad f \in C^1(I),$$

where $V : C(I) \to C(I)$ is defined by

$$Vg(x) = \sum_{k \in \mathbb{N}} \left( \frac{\Delta_{k+1}}{(y^{k+1}+x+1)^2} \int_{y^k/(x+1)}^{y^{k+1}/(x+1)} g(u)\,du + p_k(x)\frac{y^k}{(x+1)^2}g\left(\frac{y^k}{x+1}\right) \right),$$

$g \in C(I), x \in I$. Clearly,

$$(U^n f)' = (-1)^n V^n f', \quad n \in \mathbb{N}_+, f \in C^1(I).$$

We are going to show that $V^n$ takes certain functions into functions with very small values when $n \in \mathbb{N}_+$ is large.

**Proposition 4.1.** There are positive constants $v > 0.206968896$ and $w < 0.209364308$, and a real-valued function $\phi \in C(I)$ such that $v\phi \leq V\phi \leq w\phi$.

**Proof.** Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a continuous bounded function such that $\lim_{x \to \infty} h(x) < \infty$. We look for a function $g : (0,1] \to \mathbb{R}$ such that $Ug = h$, assuming that the equation

$$Ug(x) = \sum_{k \in \mathbb{N}} p_k(x)g\left(\frac{y^k}{x+1}\right) = h(x)$$

holds for $x \in \mathbb{R}_+$. Then (4.7) yields

$$\frac{h(x)}{x+2} - \frac{h(2x+1)}{2x+3} = \frac{x+1}{(x+2)(2x+3)}g\left(\frac{1}{x+1}\right), \quad x \in \mathbb{R}_+.$$
Hence
\[ g(u) = (u + 2)h\left(\frac{1}{u} - 1\right) - (u + 1)h\left(\frac{2}{u} - 1\right), \quad u \in (0, 1], \quad (4.9) \]
and we indeed have \( U_g = h \) since
\[
U_g(x) = \sum_{k \in \mathbb{N}} p_k(x) \left[ \left( \frac{y^k}{x^k+1} + 1 \right) h\left( \frac{x+1}{y^k} - 1 \right) - \left( \frac{y^k}{x^k} + 1 \right) h\left( \frac{x+1}{y^k+1} - 1 \right) \right] = h(x), \quad x \in \mathbb{R}_+.
\] (4.10)

In particular, for any fixed \( a \in I \) we consider the function \( h_a : \mathbb{R}_+ \to \mathbb{R} \) defined by \( h_a(x) = 1/(x + a + 1) \), \( x \in \mathbb{R}_+ \). By the above, the function \( g_a : (0, 1] \to \mathbb{R} \) defined as
\[
g_a(x) = (x + 2)h_a\left(\frac{1}{x} - 1\right) - (x + 1)h_a\left(\frac{2}{x} - 1\right) = \frac{x(x+2)}{ax+1} - \frac{x(x+1)}{ax+2}, \quad x \in (0, 1],
\] (4.11)
satisfies \( U_{g_a}(x) = h_a(x), x \in I \). Setting
\[
\varphi_a(x) = g_a'(x) = \frac{3ax^2 + 4(a + 1)x + 6}{(ax + 2)^2(ax + 1)^2},
\] (4.12)
we have
\[
V \varphi_a(x) = -(U_{g_a})'(x) = \frac{1}{(x + a + 1)^2}, \quad x \in I.
\] (4.13)

We choose \( a \) by asking that \((\varphi_a/V \varphi_a)(0) = (\varphi_a/V \varphi_a)(1)\). This amounts to \(3a^4 + 12a^3 + 18a^2 - 2a - 17 = 0\) which yields as unique acceptable solution \( a = 0.794741181\ldots\). For this value of \( a \), the function \( \varphi_a/V \varphi_a \) attains its maximum equal to \((3/2)(a + 1)^2 = 4.83164386\ldots\) at \( x = 0 \) and \( x = 1 \), and has a minimum \( m(a) \approx (\varphi_a/V \varphi_a)(0.39) = 4.776363306\ldots\). It follows that for \( \varphi = \varphi_a \) with \( a = 0.794741181\ldots\), we have
\[
\frac{2\varphi}{3(a+1)^2} \leq V \varphi \leq \frac{\varphi}{m(a)},
\] (4.14)
that is, \( v \varphi \leq V \varphi \leq w \varphi \), where \( v = 2/(a + 1)^2 > 0.206968896 \), and \( w = 1/m(a) < 0.209364308 \). \( \square \)

**Corollary 4.2.** Let \( f_0 \in C^1(I) \) such that \( f_0' > 0 \). Put \( \alpha = \min_{x \in I} \varphi(x)/f_0'(x) \) and \( \beta = \max_{x \in I} \varphi(x)/f_0'(x) \). Then
\[
\frac{\alpha}{\beta} v^n f_0' \leq V^n f_0' \leq \frac{\beta}{\alpha} w^n f_0', \quad n \in \mathbb{N}_+.
\] (4.15)
Proof. Since $V$ is a positive operator, we have
\[ v^n \varphi \leq V^n \varphi \leq w^n \varphi, \quad n \in \mathbb{N}_+. \] (4.16)

Noting that $\alpha f_0' \leq \varphi \leq \beta f_0'$, we can write
\[ \frac{\alpha}{\beta} v^n f_0' \leq \frac{1}{\beta} v^n \varphi \leq \frac{1}{\alpha} V^n f_0' \leq \frac{1}{\alpha} w^n \varphi \leq \frac{\beta}{\alpha} w^n f_0', \quad n \in \mathbb{N}_+, \] (4.17)

which shows that (4.15) holds. \( \square \)

Theorem 4.3 (near-optimal solution to Gauss-Kuzmin-Lévy problem). Let $f_0 \in C^1(I)$ such that $f_0' > 0$. For any $n \in \mathbb{N}_+$ and $x \in I$,
\[ \frac{(\log(4/3))^2}{2\beta} \alpha \min_{x \in I} f_0'(x) v^n F(x)(1 - F(x)) \leq |\mu(\tau^n < x) - F(x)| \leq \frac{(\log(4/3))^2}{\alpha} \beta \max_{x \in I} f_0'(x) w^n F(x)(1 - F(x)), \] (4.18)

where $\alpha$, $\beta$, $v$ and $w$ are defined in Proposition 4.1 and Corollary 4.2 and $F(x) = (1/\log(4/3))(2(x+1))/x+2$. In particular, for any $n \in \mathbb{N}_+$ and $x \in I$,
\[ 0.01023923 v^n F(x)(1 - F(x)) \leq |\lambda(\tau^n < x) - F(x)| \leq 0.334467468 w^n F(x)(1 - F(x)). \] (4.19)

Proof. For any $n \in \mathbb{N}$ and $x \in I$, set $d_n(F(x)) = \mu(\tau^n < x) - F(x)$. Then by (4.2) we have
\[ d_n(F(x)) = \int_0^x \frac{U^n f_0(u)}{(u+1)(u+2)} du - F(x). \] (4.20)

Differentiating twice with respect to $x$ yields
\[ d_n'(F(x)) \left( \frac{1}{(\log(4/3))(x+1)(x+2)} = \frac{U^n f_0(x)}{(x+1)(x+2)} - \frac{1}{(\log(4/3))(x+1)(x+2)}, \right. \]
\[ \left. (U^n f_0(x))' = \frac{1}{(\log(4/3))^2} \frac{d_n''(F(x))}{(x+1)(x+2)}, \quad n \in \mathbb{N}, \ x \in I. \] (4.21)

Hence by (4.6) we have
\[ d_n''(F(x)) = (-1)^n \left( \log \left( \frac{4}{3} \right) \right)^2 (x+1)(x+2) V^n f_0'(x), \quad n \in \mathbb{N}, \ x \in I. \] (4.22)

Since $d_n(0) = d_n(1) = 0$, it follows from a well-known interpolation formula that
\[ d_n(x) = -\frac{x(1-x)}{2} d_n''(\theta), \quad n \in \mathbb{N}, \ x \in I \] (4.23)
for a suitable $\theta = \theta(n,x) \in I$. Therefore

$$\mu(\tau^n < x) - F(x) = (-1)^{n+1} \left( \log \left( \frac{4}{3} \right) \right)^2 \frac{\theta + 1}{2} V^n f_0' (\theta) F(x) (1 - F(x)) \quad (4.24)$$

for any $n \in \mathbb{N}$ and $x \in I$, and another suitable $\theta = \theta(n,x) \in I$. The result stated follows now from Corollary 4.2. In the special case $\mu = \lambda$, we have $f_0(x) = (x + 1)(x + 2)$, $x \in I$. Then with $a = 0.794741181\ldots$, we have

$$\alpha = \min_{x \in I} \frac{\phi(x)}{f_0(x)} = \frac{7a + 10}{5(a + 2)^2(a + 1)^2} = 0.123720515\ldots,$$

$$\beta = \max_{x \in I} \frac{\phi(x)}{f_0(x)} = 0.5,$$

so that $(\log(4/3))^2 \alpha/2\beta = 0.01023923\ldots$ and $(\log 4/3)^2 \beta/\alpha = 0.334467468\ldots$. The proof is complete. \qed

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References

A Wirsing-type approach to some continued fraction


Gabriela Ileana Sebe: Department of Mathematics I, “Politehnica” University of Bucharest, Splaiul Independentei 313, 060042 Bucharest, Romania

E-mail address: gisebe@mathem.pub.ro
Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

Celso Grebogi, Department of Physics, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk