COMPACT SPACE-LIKE HYPERSURFACES IN DE SITTER SPACE

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We present some integral formulas for compact space-like hypersurfaces in de Sitter space and some equivalent characterizations for totally umbilical compact space-like hypersurfaces in de Sitter space in terms of mean curvature and higher-order mean curvatures.

1. Introduction

It is well known that the semi-Riemannian (pseudo-Riemannian) manifolds \((M,g)\) of Lorentzian signature play a special role in geometry and physics, and that they are models of space time of general relativity. Let \(M_{n+1}(c)\) be an \((n+1)\)-dimensional complete connected semi-Riemannian manifold with constant sectional curvature \(c\) and index \(p\) (see [13, page 227]). It is called an indefinite space form of index \(p\) and simply a space form when \(p = 0\). According to \(c > 0\), \(c = 0\), and \(c < 0\), \(M_{n+1}(c)\) is called de Sitter space, Minkowski space, and anti-de Sitter space, and is denoted by \(S_{1}^{n+1}(c)\), \(\mathbb{R}_{1}^{n+1}\), and \(H_{1}^{n+1}(c)\), respectively.

In spite of the fact that the geometry of de Sitter space is the simplest model of space time of general relativity, this geometry was not studied thoroughly. Let \(\phi : M^n \rightarrow S_{1}^{n+1}(c)\) be a smooth immersion of an \(n\)-dimensional connected manifold into \(S_{1}^{n+1}(c)\). If the semi-Riemannian metric of \(S_{1}^{n+1}(c)\) induces a Riemannian metric on \(M^n\) via \(\phi\), \(M^n\) is called a space-like hypersurface in de Sitter space.

The study of space-like hypersurfaces in de Sitter space \(S_{1}^{n+1}(c)\) has been of increasing interest in the last years, because of their nice Bernstein-type properties. Since Goddard [7] conjectured in 1977 that complete space-like hyperspaces in \(S_{1}^{n+1}(c)\) with constant mean curvature \(H\) must be totally umbilical, which turned out to be false in this original statement, an important number of authors have considered the problem of characterizing the totally umbilical space-like hypersurfaces in de Sitter space in terms of some appropriate geometric assumptions. Actually, Akutagawa [1] proved that Goddard’s conjecture is true when \(H^2 \leq c\) if \(n = 2\), and \(H^2 < (4(n - 1)/n^2)c\) if \(n \geq 3\). On the other hand, Montiel [11] proved that Goddard’s conjecture is also true under the additional hypothesis of the compactness of the hypersurfaces. We also refer to [14] for an alternative proof of both facts given by Ramanathan in the 2-dimensional case. More recently, Cheng and Ishikawa [5] have shown that compact space-like hyperspaces in \(S_{1}^{n+1}(c)\) with constant...
scalar curvature $S < n(n - 1)c$ must be totally umbilical. Aledo el al. [3] have recently found some other characterizations of the totally umbilical compact space-like hypersurfaces in de Sitter space with constant higher-order mean curvatures, under appropriate hypothesis.

In this paper, we will study various equivalent characterizations of totally umbilical compact space-like hypersurfaces in de Sitter space in terms of mean curvature and higher-order mean curvatures. The whole paper is organized as follows. Section 2 gives some preliminaries, Section 3 gives some inequalities on the normalized symmetric functions, and Section 4 reviews some selfadjoint second-order differential operator. The main results of this paper are contained in Section 5, which gives us a more specific and complete picture of totally umbilical compact space-like hypersurfaces in de Sitter space. For simplicity, we omit the volume form $dV$ in all integrals.

2. Preliminaries

We consider Minkowski space $\mathbb{R}^{n+2}$ as the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric $\langle \cdot, \cdot \rangle$ given by

$$\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{n+1} x_i y_i, \quad (2.1)$$

for $x, y \in \mathbb{R}^{n+2}$. Then de Sitter space $S^{n+1}_1(c)$ can be defined as the following hyperquadric of $\mathbb{R}^{n+2}_1$:

$$S^{n+1}_1(c) = \left\{ x \in \mathbb{R}^{n+2}_1 \mid |x|^2 = \frac{1}{c} \right\}. \quad (2.2)$$

The induced metric from $\langle \cdot, \cdot \rangle$ makes $S^{n+1}_1(c)$ into a Lorentzian manifold with constant sectional curvature $c$. Moreover, if $x \in S^{n+1}_1(c)$, we can put

$$T_x S^{n+1}_1(c) = \left\{ v \in \mathbb{R}^{n+2} \mid \langle v, x \rangle = 0 \right\}. \quad (2.3)$$

We denote by $\nabla^L$ and $\nabla$ the metric connections of $\mathbb{R}^{n+2}_1$ and $S^{n+1}_1(c)$, respectively. Then, we have

$$\nabla^L_v w - \nabla_v w = -c \langle v, w \rangle x \quad (2.4)$$

for all $v, w \in T_x S^{n+1}_1(c)$. Let

$$\phi : M^n \longrightarrow S^{n+1}_1(c) \quad (2.5)$$

be a space-like hypersurface in $S^{n+1}_1(c)$ defined above. First, we want to know whether a compact one is orientable. The following proposition gives us the affirmative answer (see [11] or [2] for a proof).

**Proposition 2.1.** Let $\phi : M^n \rightarrow S^{n+1}_1(c)$ be a space-like hypersurface in $S^{n+1}_1(c)$, $n \geq 2$. If $M^n$ is compact, then $M^n$ is diffeomorphic to $S^n$. In particular, compact totally umbilical space-like hypersurfaces in $S^{n+1}_1(c)$, $n \geq 2$, are round $n$-spheres.
Throughout the following, we will exclusively deal with compact space-like hypersurfaces in $S^{n+1}(c)$, $n \geq 2$. The above proposition ensures that $M^n$ is orientable. Let $N$ be a time-like unit normal vector field for the immersion $\phi$. The field $N$ can be viewed as the Gauss map of $M^n$ into hyperbolic space:

$$N : M^n \rightarrow H^{n+1},$$

(2.6)

where $H^{n+1} = \{ x \in \mathbb{R}^{n+2} \mid |x|^2 = -1, x_0 \geq 1 \}$. We will say that $M^n$ is oriented by $N$. A well-known result is that the Gauss map $N$ is harmonic if and only if the mean curvature $H$ is constant. For a proof, one can refer to [4].

Let $\nabla$ be the Levi-Civita connection associated to the Riemannian metric on $M^n$ induced from $\langle \cdot, \cdot \rangle$. Then, we have

$$h(\nu, w) = \nabla_\nu w - \nabla_w \nu = -\langle \mathcal{A} \nu, w \rangle N,$$

$$\mathcal{A} \nu = -\nabla_\nu N = -\nabla^L_\nu N,$$

(2.7)

where $\mathcal{A}$ stands for the shape operator of the immersion $\phi$ with respect to $N$ and $\nu, w$ are vector fields tangent to $M^n$. The operator $L = -\mathcal{A}$ is the Weingarten endomorphism. The eigenvalues of the operator $L$ are called the principal curvatures and will be denoted by $\lambda_1, \ldots, \lambda_n$. The Codazzi equation is expressed by

$$\langle \nabla_\nu \mathcal{A} \rangle w = \langle \nabla_w \mathcal{A} \rangle \nu.$$

(2.8)

For a suitably chosen local field of orthonormal frames $e_1, \ldots, e_n$ on $M^n$, we have

$$\mathcal{A} e_i = -\lambda_i e_i.$$

(2.9)

The $k$th mean curvature of the space-like hypersurface $M^n$ is defined by

$$H_k = \frac{1}{\binom{n}{k}} \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

(2.10)

Note that when $k = 1$, $H_1$ is the mean curvature $H$, and when $k = n$, $H_n$ is the Gauss-Kronecker curvature. We can easily see that the scalar curvature

$$S = n(n-1)c - \left( \sum_i \lambda_i \right)^2 + \sum_i \lambda_i^2 = n(n-1)(c - H_2)$$

(2.11)

and the characteristic polynomial of $\mathcal{A}$ can be written in terms of the $H_k$’s as

$$\det(tI - \mathcal{A}) = \sum_{k=0}^{n} \binom{n}{k} H_k t^{n-k},$$

(2.12)

where $H_0 = 1$. 
Minkowski formulas provide us with a convenient tool in the study of hypersurfaces. One can refer to [12] for the well-known version for space forms. Many interesting results have been got in the study of hypersurfaces by means of Minkowski formulas, for example, [9, 10, 12, 16, 17], and so forth. The proof in [12] followed the idea in [15]. Similar to it, one can easily give the proof of Minkowski formulas for compact space-like hypersurfaces in de Sitter space (see [3]). The following proposition is Minkowski formulas for compact space-like hypersurfaces in de Sitter space.

**Proposition 2.2.** Let $\phi : M^n \to S^{n+1}_1(c)$ be a compact space-like hypersurface in $S^{n+1}_1(c)$, $n \geq 2$, then

$$\int_{M^n} c H_k(\phi, a) - H_{k+1}(N, a) = 0, \quad k = 0, 1, \ldots, n-1,$$

for any $a \in \mathbb{R}^{n+2}$.

3. **Inequalities on the normalized symmetric functions**

Let $x_1, \ldots, x_n \in \mathbb{R}$. The elementary symmetric functions of $n$ variables $x_1, \ldots, x_n$ are defined by

$$\sigma_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad k = 0, 1, \ldots, n,$$

where $\sigma_0 = 1$. For our purpose, it is useful to consider the normalized symmetric functions by dividing each $\sigma_k$ by the number of its summands. We denote the normalized symmetric function by

$$E_k = \frac{1}{n \binom{n}{k}} \sigma_k, \quad k = 0, 1, \ldots, n,$$

where $E_0 = 1$. Since

$$(x-x_1) \cdots (x-x_n) = \sum_{i=0}^n (-1)^i \sigma_{n-i} x^{n-i} = \sum_{i=0}^n (-1)^i \binom{n}{i} E_i x^{n-i},$$

we see that at least $r$ of $x_i$’s are zero if and only if $E_{n-r+1} = \cdots = E_n = 0$.

**Proposition 3.1.** All $x_i \geq 0$ if and only if all $E_i \geq 0$, and all $x_i > 0$ if and only if all $E_i > 0$.

**Proof.** We prove it by induction on $n$. For $n = 1$, the proposition holds clearly. Now assume that $n > 1$ and the proposition holds for $n - 1$. Let $P(x) = (x - x_1) \cdots (x - x_n)$ and $Q(x) = (1/n) P'(x) = (x - y_1) \cdots (x - y_{n-1})$. By Rolle’s theorem, $y_1, \ldots, y_{n-1}$ are all real and $x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq y_{n-1} \leq x_n$. Clearly, the inductive assumption applies to $y_1, \ldots, y_{n-1}$. Thus, it follows easily that the proposition holds for $n$. \qed

There are some well-known inequalities on the normalized symmetric functions, for example, Newton-Maclaurin inequalities. One can refer to [8] for the case of $n$ positive numbers. For the sake of completeness, we include here a proof of Newton’s inequalities for the general case.
Proposition 3.2.

\[ E_k^2 \geq E_{k-1}E_{k+1}, \quad k = 1, \ldots, n-1, \]  \hspace{1cm} (3.4)

and each equality holds if and only if \( x_1 = \cdots = x_n \), or \( E_k = 0 = E_{k-1}E_{k+1} \).

Proof. We prove it by induction on \( n \). For \( n = 2 \), the inequality holds clearly and the equality holds if and only if \( x_1 = x_2 \) since \( E_1 = 0 = E_0E_2 = E_2 \) implies that \( x_1 = x_2 = 0 \). Now assume that \( n > 2 \) and the proposition holds for \( n-1 \). Let \( P(x) = (x - x_1) \cdots (x - x_n) \) and \( Q(x) = (1/n)P'(x) \). Then

\[ P(x) = \sum_{i=0}^{n} (-1)^i \sigma_i x^{n-i} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} E_i x^{n-i}, \]  \hspace{1cm} (3.5)

\[ Q(x) = \frac{1}{n} P'(x) = \sum_{i=0}^{n-1} (-1)^i \frac{n-i}{n} \binom{n}{i} E_i x^{n-i-1} = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} E_i x^{n-1-i}. \]

On the other hand,

\[ Q(x) = (x - y_1) \cdots (x - y_{n-1}) = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} E_i (y_1, \ldots, y_{n-1}) x^{n-1-i}, \]  \hspace{1cm} (3.6)

where \( y_1, \ldots, y_{n-1} \) are \( n-1 \) roots of the polynomial \( Q(x) \). Comparing the coefficients of the powers of \( x \) in the above two expressions for \( Q(x) \) gives us

\[ E_i (y_1, \ldots, y_{n-1}) = E_i (x_1, \ldots, x_n), \quad i = 0, \ldots, n-1. \]  \hspace{1cm} (3.7)

By Rolle’s theorem \( y_1, \ldots, y_{n-1} \) are all real. Clearly, \( y_1 = \cdots = y_{n-1} \) if and only if \( x_1 = \cdots = x_n \). Thus the inductive assumption applies to \( E_i (y_1, \ldots, y_{n-1}) \), \( i = 0, \ldots, n-1 \), and the proposition holds for \( k = 1, \ldots, n-2 \) by (3.7).

It remains to prove for \( k = n-1 \), that is,

\[ E_{n-1}^2 (x_1, \ldots, x_n) \geq E_{n-2} (x_1, \ldots, x_n) E_n (x_1, \ldots, x_n), \]  \hspace{1cm} (3.8)

with equality if and only if \( x_1 = \cdots = x_n \), or \( E_{n-1} = 0 = E_{n-2}E_n \).

Case 1. If some \( x_i = 0 \), then \( E_n (x_1, \ldots, x_n) = x_1 \cdots x_n = 0 \). Clearly, (3.8) holds with equality if and only if \( E_{n-1} = (1/n) \prod_{j \neq i} x_j = 0 \), and thus if and only if some \( x_j = 0, \ j \neq i \).

Case 2. If all \( x_i \neq 0 \), let \( x'_i = 1/x_i \). Then, we have

\[ E_i (x_1, \ldots, x_n) E_n (x_1, \ldots, x_n) = E_{n-i} (x'_1, \ldots, x'_n). \]  \hspace{1cm} (3.9)

Since \( E_n (x_1, \ldots, x_n) = x_1 \cdots x_n \neq 0 \), we see that (3.8) is equivalent to

\[ E_1^2 (x'_1, \ldots, x'_n) \geq E_2 (x'_1, \ldots, x'_n), \]  \hspace{1cm} (3.10)

which is true since \( n > 2 \).
This completes the proof. □

Remark 3.3. For our future purpose, we concern most when each of the above equalities holds if and only if $x_1 = \cdots = x_n$, that is, to find some restrictions on $x_i$’s to exclude the possibility of $E_k = 0 = E_{k-1}E_{k+1}$ and $x_i$’s are not all zero. We only know that $E_1^2 = E_2$ holds if and only if $x_1 = \cdots = x_n$ since $E_1 = 0 = E_0E_2 = E_2$ implies that $x_1 = \cdots = x_n = 0$, while we cannot expect it for $k \geq 2$ even if all $x_i \geq 0$, for example, when only one of $x_i$’s is positive. In particular, when all $x_i$’s have the same sign, that is, nonnegative or nonpositive simultaneously, and at least $k$ of $x_i$’s are nonzero (equivalently, $E_1 \cdots E_k \neq 0$) or $E_1 = \cdots = E_n = 0$, we have that $E_k^2 = E_{k-1}E_{k+1}$ holds if and only if $x_1 = \cdots = x_n$.

Newton’s inequalities have a very important consequence, Maclaurin’s inequalities, by investigating that

$$E_1^2E_2^4 \cdots E_k^{2k} \geq (E_0E_2)(E_1E_3)^2 \cdots (E_{k-1}E_{k+1})^k,$$

where all $x_i \geq 0$. When all $x_i > 0$ and $2 \leq k \leq n - 1$, we have

$$E_k^{1/k} \geq E_{k+1}^{1/(k+1)},$$

with equality if and only if $x_1 = \cdots = x_n$. If some of $x_i$’s are zero and the rest of them are positive, then for $2 \leq k \leq n - 1$, we still have

$$E_k^{1/k} \geq E_{k+1}^{1/(k+1)},$$

with equality if and only if $x_1 = \cdots = x_n$, or at least $n - k + 1$ of $x_i$’s are zero.

Corollary 3.4. If all $x_i \geq 0$, $1 \leq k \leq n - 1$, then

$$E_k^{1/k} \geq E_{k+1}^{1/(k+1)},$$

with equality if and only if $x_1 = \cdots = x_n$, or $E_{n-k+1} = \cdots = E_n = 0$.

Corollary 3.5. If $E_1 > 0, \ldots, E_k > 0$ and $2 \leq k \leq n$, then

$$E_1 \geq E_2^{1/2} \geq \cdots \geq E_k^{1/k}$$

with each equality if and only if $x_1 = \cdots = x_n$.

Now we can give a result on the positiveness of mean curvature and higher-order mean curvatures of the compact space-like hypersurfaces in de Sitter space.

Theorem 3.6. Let $\phi : M^n \rightarrow S^{n+1}(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space with $H_k > 0$ and $2 \leq k \leq n$. If there exists a point of $M^n$, where $H_1, \ldots, H_{k-1}$ are positive, then $H_1, \ldots, H_{k-1}$ are positive everywhere on $M^n$, that is, $H_1 > 0, \ldots, H_{k-1} > 0$. 
Proof. We prove it by an open-closed argument. Let
\[ U = \{ x \in M^n \mid H_1(x) > 0, \ldots, H_{k-1}(x) > 0 \}. \] (3.16)
Clearly \( U \) is open, and it is nonempty by the assumption. To prove that \( U = M^n \), we only need to prove that \( U \) is also closed by the connectedness of \( M^n \). Since \( H_k > 0 \) and \( M^n \) is compact, we have
\[ a = \min_{x \in M^n} H_k(x) > 0. \] (3.17)
For any \( x \in U \), we have
\[ H_1(x) \geq H_2(x)^{1/2} \geq \cdots \geq H_{k-1}(x)^{1/(k-1)} \geq H_k(x)^{1/k} \geq a^{1/k} > 0, \] (3.18)
by Corollary 3.5. Thus \( U \) is closed. This completes the proof. \(\square\)

Finally, we give another two sets of important inequalities by investigating that
\[
\begin{align*}
E_k E_{k+1}^2 \cdots E_{l}^2 &\geq (E_{k-1} E_{k+1}) (E_k E_{k+2}) \cdots (E_{l-1} E_{l+1}), \\
E_k^{1/k} E_{l-1}^{1/(l-1)} E_{l}^{l/(l+1)/l} &\geq E_{k+1}^{1/(k+1)} \cdots E_{l}^{l/(l+1)},
\end{align*}
\] (3.19)
where all \( x_i \geq 0 \) and \( 1 \leq k < l \leq n-1 \). Using the argument above leading to Corollary 3.4, we can get the following important inequalities.

Theorem 3.7. If all \( x_i \geq 0 \) and \( 1 \leq k < l \leq n-1 \), then
\[ E_k E_l \geq E_{k-1} E_{l+1}, \] (3.20)
with equality if and only if \( x_1 = \cdots = x_n \), or \( E_{n-l+1} = \cdots = E_n = 0 \).

Theorem 3.8. If \( E_{k-1} > 0, \ldots, E_{l+1} > 0 \) and \( 1 \leq k < l \leq n-1 \), then
\[ E_k E_l \geq E_{k-1} E_{l+1}, \] (3.21)
with equality if and only if \( x_1 = \cdots = x_n \).

Theorem 3.9. If all \( x_i \geq 0 \) and \( 1 \leq k < l \leq n-1 \), then
\[ E_k^{1/k} E_l \geq E_{l+1}, \] (3.22)
with equality if and only if \( x_1 = \cdots = x_n \), or \( E_{n-l+1} = \cdots = E_n = 0 \).

Theorem 3.10. If \( E_1 > 0, \ldots, E_{l+1} > 0 \) and \( 1 \leq k < l \leq n-1 \), then
\[ E_k^{1/k} E_l \geq E_{l+1}, \] (3.23)
with equality if and only if \( x_1 = \cdots = x_n \).
4. Some selfadjoint second-order differential operators

First, we introduce two known selfadjoint second-order differential operators, the Laplace operator $\triangle$ and the Cheng-Yau operator $\Box$. For any $C^2$-function $f$ defined on $M^n$, we consider the symmetric bilinear form

$$\left( \nabla^2 f, \nu \right) = \nu(w f) - \left( \nabla_w w \right) f. \quad (4.1)$$

The Laplace operator $\triangle$ acting on any $C^2$-function $f$ defined on $M^n$ is given by

$$\triangle f = \sum_i \left( \nabla^2 f \right) (e_i, e_i). \quad (4.2)$$

Since $M^n$ is compact and oriented, the Laplace operator $\triangle$ is selfadjoint relative to the $L^2$-inner product of $M^n$, that is,

$$\int_{M^n} f(\triangle g) = \int_{M^n} (\triangle f) g. \quad (4.3)$$

Following Cheng and Yau [6], we introduce an operator $\Box$ acting on any $C^2$-function $f$ defined on $M^n$ by

$$\Box f = \sum_{i,j} \left[ nH (e_i, e_j) + \langle \mathcal{A} e_i, e_j \rangle \right] \left( \nabla^2 f \right) (e_i, e_j) = \sum_i (nH - \lambda_i) \left( \nabla^2 f \right) (e_i, e_i). \quad (4.4)$$

Note that the following holds at umbilical points:

$$\Box f = \sum_i (n - 1)H \left( \nabla^2 f \right) (e_i, e_i) = (n - 1)H \triangle f. \quad (4.5)$$

By the Codazzi equation and [6, Proposition 1], we can prove that the operator $\Box$ is selfadjoint relative to the $L^2$-inner product of $M^n$, that is,

$$\int_{M^n} f(\Box g) = \int_{M^n} (\Box f) g. \quad (4.6)$$

Naturally, we may ask the following question.

Question 4.1. Can we find other selfadjoint second-order differential operators in terms of the shape operator $\mathcal{A}$, mean curvature, and higher-order mean curvatures?

Fortunately, we do have such a selfadjoint second-order differential operator $\mathcal{L}_k$ for each $k = 0, 1, \ldots, n - 1$. The idea is contained in [15, 17]. Following [3], we introduce the $k$th Newton transformation $T_k$ associated to the shape operator $\mathcal{A}$:

$$T_k = \sum_{i=0}^k \binom{n}{i} H_i \mathcal{A}^{k-i}, \quad (4.7)$$
or inductively,

\[ T_0 = I, \quad T_k = \binom{n}{k} H_k I + \mathcal{A} T_{k-1}. \]  

(4.8)

It follows from (2.12) that \( T_n = 0 \). Since the shape operator \( \mathcal{A} \) is selfadjoint, it follows easily that the Newton transformations \( T_k \)'s are selfadjoint. Clearly, the orthonormal basis \( \{ e_1, \ldots, e_n \} \) diagonalizes the Newton transformations \( T_k \)'s since it diagonalizes the shape operator \( \mathcal{A} \).

**Proposition 4.2.** If the shape operator \( \mathcal{A} \) is negative definite, the Newton transformations \( T_k \)'s, \( k = 0, 1, \ldots, n-1 \), are positive definite.

**Proof.** Since the shape operator \( \mathcal{A} \) is negative definite, all \( \lambda_i > 0 \). Without loss of generality, to prove that \( T_k \) is positive definite, we only need to prove that \( \langle T_k e_1, e_1 \rangle > 0 \). Let \( \lambda'_i = \lambda_i / \lambda_1 \), \( i = 1, \ldots, n \), then we have

\[ \langle T_k e_1, e_1 \rangle = \sum_{i=0}^k \binom{n}{i} H_i (-\lambda_1)^{k-i} \]

\[ = \sum_{i=0}^k \sigma_i(\lambda_1, \ldots, \lambda_n)(-\lambda_1)^{k-i} \]

\[ = \lambda_1^{k} \sum_{i=0}^k (-1)^{k-i} \sigma_i(1, \lambda'_2, \ldots, \lambda'_n). \]

(4.9)

Now we prove that

\[ \sum_{i=0}^k (-1)^{k-i} \sigma_i(1, x_2, \ldots, x_n) > 0, \quad k = 0, 1, \ldots, n-1, \]

(4.10)

by induction on \( n \), where \( x_2, \ldots, x_n > 0 \). Clearly, (4.10) holds for \( k = 0 \) or \( n = 1 \). Now assume that \( m > 1 \), \( 0 < l \leq m-1 \), and (4.10) holds for all \( n < m \) and all \( k < l \) for \( n = m \). Let \( n = m \) and \( k = l \), then we have

\[ \sum_{i=0}^k (-1)^{k-i} \sigma_i(1, x_2, \ldots, x_n) = \sum_{i=0}^k (-1)^{k-i} \sigma_i(1, x_2, \ldots, x_{n-1}) \]

\[ + x_n \sum_{i=1}^k (-1)^{k-i} \sigma_{i-1}(1, x_2, \ldots, x_n) \]

\[ = \sum_{i=0}^k (-1)^{k-i} \sigma_i(1, x_2, \ldots, x_{n-1}) \]

\[ + x_n \sum_{i=0}^{k-1} (-1)^{k-1-i} \sigma_i(1, x_2, \ldots, x_n) > 0 \]

(4.11)

by the inductive assumption and the fact that \( \sum_{i=0}^k (-1)^{k-i} \sigma_i(1, x_2, \ldots, x_{n-1}) = 0 \) for \( k = n-1 \). This completes the proof. \( \square \)
The following algebraic properties of $T_k$ can be easily established from the definitions.

\[ \text{tr } T_k = (n-k) \binom{n}{k} H_k = n \binom{n-1}{k} H_k, \quad (4.12) \]

\[ \text{tr}(T_k \mathcal{A}) = -(k+1) \binom{n}{k+1} H_{k+1} = -n \binom{n-1}{k} H_{k+1}, \quad (4.13) \]

\[ \text{tr} \left( T_k \mathcal{A}^2 \right) = n \binom{n}{k+1} H H_{k+1} - (k+2) \binom{n}{k+2} H_{k+2} 
= n \binom{n}{k+1} H H_{k+1} - n \binom{n-1}{k+1} H_{k+2}. \quad (4.14) \]

One can also easily derive the identities

\[ \text{tr} \left( T_k \nabla_v \mathcal{A} \right) = - \binom{n}{k+1} \langle \nabla H_{k+1}, v \rangle, \quad (4.15) \]

where $v$ is any vector field tangent to $M^n$. Now for each $k = 0, 1, \ldots, n-1$, we can define a second-order differential operator $\mathcal{L}_k$ acting on any $C^2$-function $f$ defined on $M^n$ by

\[ \mathcal{L}_k f = \text{div} (T_k \nabla f). \quad (4.16) \]

It can be easily seen that the operators $\mathcal{L}_k$'s are selfadjoint. Clearly when $k = 0$, the operator $\mathcal{L}_0$ is the Laplace operator $\Delta = \text{div} \circ \nabla$. Later, we will see that when $k = 1$, the operator $\mathcal{L}_1$ is the Cheng-Yau operator $\Box$.

Finally, we can easily derive the following useful expression for $\mathcal{L}_k$ (see [3]):

\[ \mathcal{L}_k f = \sum_i \langle T_k \nabla e_i \nabla f, e_i \rangle = \sum_i \langle T_k e_i, e_i \rangle \nabla^2 f (e_i, e_i) \quad (4.17) \]

for any $C^2$-function $f$ defined on $M^n$.

**Remark 4.3.** More specifically,

\[ \mathcal{L}_k f = \sum_{i=0}^k \sum_{j=0}^k \binom{n}{j} H_j (-\lambda_i)^{k-j} \nabla^2 f (e_i, e_i). \quad (4.18) \]

Clearly when $k = 1$, the operator $\mathcal{L}_1$ is the Cheng-Yau operator $\Box = \sum_i (nH - \lambda_i) \nabla^2$. Note that the following holds at umbilical points:

\[ \mathcal{L}_k f = \sum_{i=0}^k \sum_{j=0}^k \binom{n}{j} H_j (-\lambda_i)^{k-j} \nabla^2 f (e_i, e_i) = \sum_{i=0}^k (-1)^{k-i} \binom{n}{i} \cdot H_k \Delta f. \quad (4.19) \]
Remark 4.4. When $T_k$ is positive definite, the operator $\mathcal{L}_k$ is elliptic. In particular, when the shape operator $\mathcal{A}$ is negative definite, the operator $\mathcal{L}_k$ is elliptic by proposition 4.2.

5. Main results

Let $\phi : M^n \to S^{n+1}_1(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space, $N$ a time-like unit normal vector field for $\phi$, and $a \in \mathbb{R}^{n+2}_1$ arbitrary. We consider the height function $\langle \phi, a \rangle$ and the function $\langle N, a \rangle$ on $M^n$. Using (2.4), (2.7), we can get the following expressions for the gradient and Hessian of the above two functions:

$$
\langle \nabla \langle \phi, a \rangle, v \rangle = \langle v, a \rangle, \quad \langle \nabla \langle N, a \rangle, v \rangle = -\langle \mathcal{A}v, a \rangle,
$$

$$
(\nabla^2 \langle \phi, a \rangle)(v, w) = wv \langle \phi, a \rangle - (\nabla_w v) \langle \phi, a \rangle = -c \langle v, w \rangle \langle \phi, a \rangle - \langle \mathcal{A}v, w \rangle \langle N, a \rangle, \quad (5.1)
$$

$$(\nabla^2 \langle N, a \rangle)(v, w) = wv \langle N, a \rangle - (\nabla_w v) \langle N, a \rangle = c \langle \mathcal{A}v, w \rangle \langle \phi, a \rangle + \langle \mathcal{A}v, \mathcal{A}w \rangle \langle N, a \rangle - \langle (\nabla_w \mathcal{A})v, a \rangle,$$

where $v, w$ are vector fields tangent to $M^n$. Thus, we have

$$
\mathcal{L}_k \langle \phi, a \rangle = \sum_{i} \sum_{j=0}^{k} \binom{n}{j} H_j (\mathcal{A})^{k-j} \frac{\partial^2}{\partial \phi} \langle \phi, a \rangle (e_i, e_i)
$$

$$
= \sum_{i} \sum_{j=0}^{k} \binom{n}{j} H_j (\mathcal{A})^{k-j} [ - c \langle \phi, a \rangle + \lambda_i \langle N, a \rangle ]
$$

$$
= -c \sum_{j=0}^{k} (-1)^{k-j} \binom{n}{j} H_j \sum_{i} \lambda_i^{k-j} \langle \phi, a \rangle
$$

$$
+ \sum_{j=0}^{k} (-1)^{k-j} \binom{n}{j} H_j \sum_{i} \lambda_i^{k+1-j} \langle N, a \rangle
$$

$$
= -c(n-k) \binom{n}{k} H_k \langle \phi, a \rangle + (k+1) \binom{n}{k+1} H_{k+1} \langle N, a \rangle
$$

$$
= n \binom{n-1}{k} [ - cH_k \langle \phi, a \rangle + H_{k+1} \langle N, a \rangle ].
$$

Note that the Minkowski formulas in Proposition 2.2 are regained by the selfadjointness of the operators $\mathcal{L}_k$'s.
For any vector field \( v \) tangent to \( M^n \), we have
\[
\nabla_v \nabla \langle N, a \rangle = c \mathcal{A} v \langle \phi, a \rangle + \mathcal{A}^2 v \langle N, a \rangle - (\nabla_v \mathcal{A}) a^T,
\]
by the selfadjointness of the operator \( \nabla_v \mathcal{A} \), where \( a^T \) is the tangent component of \( a \) to \( M^n \). Thus by (2.8), (4.13), (4.14), and (4.15), we have
\[
\mathcal{L}_k \langle N, a \rangle = \sum_i \langle T_k \nabla_{e_i} \nabla \langle N, a \rangle, e_i \rangle
\]
\[
= \sum_i \langle c T_k \mathcal{A} e_i \langle \phi, a \rangle + T_k \mathcal{A}^2 e_i \langle N, a \rangle - T_k (\nabla_{e_i} \mathcal{A}) a^T, e_i \rangle
\]
\[
= c \text{tr} (T_k \mathcal{A}) \langle \phi, a \rangle + \text{tr} (T_k \mathcal{A}^2) \langle N, a \rangle - \sum_i \langle T_k (\nabla_{e_i} \mathcal{A}) a^T, e_i \rangle
\]
\[
= c \text{tr} (T_k \mathcal{A}) \langle \phi, a \rangle + \text{tr} (T_k \mathcal{A}^2) \langle N, a \rangle - \sum_i \langle T_k (\nabla_{a^T} \mathcal{A}) e_i, e_i \rangle
\]
\[
= c \text{tr} (T_k \mathcal{A}) \langle \phi, a \rangle + \text{tr} (T_k \mathcal{A}^2) \langle N, a \rangle - \text{tr} (T_k \nabla_{a^T} \mathcal{A})
\]
\[
= \sum_i \langle c T_k \mathcal{A} e_i \langle \phi, a \rangle + T_k \mathcal{A}^2 e_i \langle N, a \rangle - T_k (\nabla_{e_i} \mathcal{A}) a^T, e_i \rangle
\]
\[
= \sum_i \langle c T_k \mathcal{A} e_i \langle \phi, a \rangle + T_k \mathcal{A}^2 e_i \langle N, a \rangle - T_k (\nabla_{e_i} \mathcal{A}) a^T, e_i \rangle
\]
\[
= \sum_i \langle c T_k \mathcal{A} e_i \langle \phi, a \rangle + T_k \mathcal{A}^2 e_i \langle N, a \rangle - T_k (\nabla_{e_i} \mathcal{A}) a^T, e_i \rangle
\]
\[
= c \text{tr} (T_k \mathcal{A}) \langle \phi, a \rangle + \text{tr} (T_k \mathcal{A}^2) \langle N, a \rangle - \text{tr} (T_k \nabla_{a^T} \mathcal{A})
\]
\[
= -cn \left( \begin{array}{c} n-1 \\ k \end{array} \right) H_{k+1} \langle \phi, a \rangle + n \left[ \left( \begin{array}{c} n \\ k+1 \end{array} \right) H H_{k+1} - \left( \begin{array}{c} n-1 \\ k+1 \end{array} \right) H_{k+2} \right] \langle N, a \rangle
\]
\[
+ \left( \begin{array}{c} n \\ k+1 \end{array} \right) \langle \nabla H_{k+1}, a^T \rangle
\]
\[
= -cn \left( \begin{array}{c} n-1 \\ k \end{array} \right) H_{k+1} \langle \phi, a \rangle + n \left[ \left( \begin{array}{c} n \\ k+1 \end{array} \right) H H_{k+1} - \left( \begin{array}{c} n-1 \\ k+1 \end{array} \right) H_{k+2} \right] \langle N, a \rangle
\]
\[
+ \left( \begin{array}{c} n \\ k+1 \end{array} \right) \langle \nabla H_{k+1}, a \rangle.
\]

Remark 5.1. In particular, when \( k = 0 \), we have
\[
\triangle \langle N, a \rangle = \mathcal{L}_0 \langle N, a \rangle = -cnH \langle \phi, a \rangle + [n^2 H^2 - n(n-1)H_2] \langle N, a \rangle + n \langle \nabla H, a \rangle.
\]

Proposition 5.2.
\[
\mathcal{L}_k \langle N, a \rangle = -\mathcal{L}_{k+1} \langle \phi, a \rangle + \left( \begin{array}{c} n \\ k+1 \end{array} \right) \left[ H_{k+1} \triangle \langle \phi, a \rangle + \langle \nabla H_{k+1}, a \rangle \right]
\]
for \( k = 0, 1, \ldots, n - 1 \).

Proof. By (5.2), we have
\[
\frac{1}{n} H_{k+1} \triangle \langle \phi, a \rangle = \frac{1}{n \left( \begin{array}{c} n-1 \\ k+1 \end{array} \right)} \mathcal{L}_{k+1} \langle \phi, a \rangle
\]
\[
= (H_1 H_{k+1} - H_{k+2}) \langle N, a \rangle.
\]
Thus by (5.2) and (5.4), we have
\[
\mathcal{L}_k \langle N, a \rangle = \frac{(n-1)}{(n-1)^{k-1}} \mathcal{L}_{k+1} \langle \phi, a \rangle + \binom{n}{k+1} \left[ H_{k+1} \Delta \langle \phi, a \rangle - \frac{1}{(n-1)^{k+1}} \mathcal{L}_{k+1} \langle \phi, a \rangle \right] + \binom{n}{k+1} \langle \nabla H_{k+1}, a \rangle 
\]
(5.8)

Thus, 
\[
\int_{\mathcal{M}^n} \left[ \frac{1}{n} \frac{\mathcal{L}_j H_i}{(n-1)^i} - \frac{1}{n} \frac{\mathcal{L}_i H_j}{(n-1)^j} \right] \langle \phi, a \rangle + (H_{i+1} H_j - H_j H_{i+1}) \langle N, a \rangle = 0, 
\]
(5.9)
or equivalently,
\[
\int_{\mathcal{M}^n} \left[ \frac{1}{n} \frac{T_i \nabla H_j}{(n-1)^i} - \frac{1}{n} \frac{T_j \nabla H_i}{(n-1)^j} \right] + (H_{i+1} H_j - H_j H_{i+1}) \langle N, a \rangle = 0, 
\]
(5.10)
for any vector \( a \in \mathbb{R}_1^{n+2} \).

**Proof.** By (5.2), we have
\[
\frac{1}{n} \frac{H_j \mathcal{L}_i \langle \phi, a \rangle}{(n-1)^i} - \frac{1}{n} \frac{H_i \mathcal{L}_j \langle \phi, a \rangle}{(n-1)^j} = (H_{i+1} H_j - H_j H_{i+1}) \langle N, a \rangle.
\]
(5.11)

Thus,
\[
\int_{\mathcal{M}^n} \left[ \frac{1}{n} \frac{H_j \mathcal{L}_i \langle \phi, a \rangle}{(n-1)^i} - \frac{1}{n} \frac{H_i \mathcal{L}_j \langle \phi, a \rangle}{(n-1)^j} \right] \langle \phi, a \rangle + (H_{i+1} H_j - H_j H_{i+1}) \langle N, a \rangle = \int_{\mathcal{M}^n} (H_{i+1} H_j - H_j H_{i+1}) \langle N, a \rangle.
\]
(5.12)

Since the operators \( \mathcal{L}_k \)'s are selfadjoint, we have
\[
\int_{\mathcal{M}^n} \left[ \frac{1}{n} \frac{\mathcal{L}_j H_i}{(n-1)^i} - \frac{1}{n} \frac{\mathcal{L}_i H_j}{(n-1)^j} \right] \langle \phi, a \rangle + (H_{i+1} H_j - H_j H_{i+1}) \langle N, a \rangle = 0, 
\]
(5.13)
or equivalently,
\[
\int_{\mathcal{M}^n} \left[ \frac{1}{n} \frac{T_i \nabla H_j}{(n-1)^i} - \frac{1}{n} \frac{T_j \nabla H_i}{(n-1)^j} \right] + (H_{i+1} H_j - H_j H_{i+1}) \langle N, a \rangle = 0 
\]
(5.14)
since the operators \( \mathcal{L}_k = \text{div} \circ T_k \nabla \), for any vector \( a \in \mathbb{R}_1^{n+2} \).
Theorem 5.4. Let $\phi: M^n \to S^{n+1}_1(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space, $a \in \mathbb{R}^{n+2}_1$ any unit time-like vector with the same time-orientation as $N$, and $0 \leq k \leq n-2$, then

$$
\int_{M^n} \left< \left( \frac{n-1}{k+1} \right) T_k \nabla H_{k+1} - \left( \frac{n-1}{k} \right) T_{k+1} \nabla H_k, a \right> \geq 0,
$$

(5.15)

and the equality holds if and only if $M^n$ is totally umbilical when $k = 0$, or additionally if $H_{k+1}^2 + H_k^2 H_{k+2}^2 \neq 0$ when $1 \leq k \leq n-2$.

Proof. For any unit time-like vector $a \in \mathbb{R}^{n+2}_1$ with the same time orientation as $N$, that is, $|x|^2 = -1$ and $x_0 \geq 1$, we have $\langle N, a \rangle \leq -1$. Thus by taking $i = k$, $j = k+1$ in Theorem 5.3 and Proposition 3.2, we can deduce that

$$
\int_{M^n} \left< \left( \frac{n-1}{k+1} \right) T_k \nabla H_{k+1} - \left( \frac{n-1}{k} \right) T_{k+1} \nabla H_k, a \right> \geq 0,
$$

(5.16)

and the equality holds if and only if $M^n$ is totally umbilical when $k = 0$ or additionally if $H_{k+1}^2 + H_k^2 H_{k+2}^2 \neq 0$ when $1 \leq k \leq n-2$. \qed

Remark 5.5. In particular, when $k = 0$, we have

$$
\int_{M^n} \langle \nabla H, a \rangle \geq 0,
$$

(5.17)

and the equality holds if and only if $M^n$ is totally umbilical for any unit time-like vector $a \in \mathbb{R}^{n+2}_1$ with the same time orientation as $N$.

Remark 5.6. In particular, if $H_k$ and $H_{k+1}$ are constant, $0 \leq k \leq n-2$, then $M^n$ is totally umbilical when $k = 0$, or additionally if $H_{k+1}^2 + H_k^2 H_{k+2}^2 \neq 0$ when $1 \leq k \leq n-2$. See also [3].

Theorem 5.7. Let $\phi: M^n \to S^{n+1}_1(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space with $H_1 \geq 0, \ldots, H_n \geq 0$, $a \in \mathbb{R}^{n+2}_1$ any unit time-like vector with the same time orientation as $N$, and $0 \leq i < j \leq n-1$, $j \geq i+2$, then

$$
\int_{M^n} \left< \left( \frac{n-1}{j} \right) T_i \nabla H_j - \left( \frac{n-1}{i} \right) T_j \nabla H_i, a \right> \geq 0.
$$

(5.18)

Moreover, if $\sum_{k=n-j+1}^n H_k^2 \neq 0$, then the equality holds if and only if $M^n$ is totally umbilical.

Proof. For any unit time-like vector $a \in \mathbb{R}^{n+2}_1$ with the same time orientation as $N$, we have $\langle N, a \rangle \leq -1$. Thus by Theorems 5.3 and 3.7, we can deduce that

$$
\int_{M^n} \left< \left( \frac{n-1}{j} \right) T_i \nabla H_j - \left( \frac{n-1}{i} \right) T_j \nabla H_i, a \right> \geq 0,
$$

(5.19)

and when $\sum_{k=n-j+1}^n H_k^2 \neq 0$, the equality holds if and only if $M^n$ is totally umbilical. \qed
**Corollary 5.8.** Let \( \phi : M^n \to S_{n+1}^n(c), \ n \geq 2, \) be a compact space-like hypersurface in de Sitter space with \( H_1 \geq 0, \ldots, H_n \geq 0 \) and constant \( \sum_{i=1}^{k-1} a_i H_i + H_k, \ a_i \geq 0, \ 2 \leq k \leq n-1. \) If \( \sum_{i=n-k+1}^{n} H_i^2 \neq 0, \) then \( M^n \) is totally umbilical.

**Proof.** Fix a unit time-like vector \( a \in \mathbb{R}_{1}^{n+2} \) with the same time orientation as \( N. \) By Theorems 5.4 and 5.7, we have

\[
\int_{M^n} \langle \nabla H_i, a \rangle \geq 0, \quad i = 1, \ldots, k. \tag{5.20}
\]

Since

\[
0 = \int_{M^n} \left\langle \nabla \left( \sum_{i=1}^{k-1} a_i H_i + H_k \right), a \right\rangle = \sum_{i=1}^{k-1} a_i \int_{M^n} \langle \nabla H_i, a \rangle + \int_{M^n} \langle \nabla H_k, a \rangle \geq 0, \tag{5.21}
\]

we have

\[
\int_{M^n} \langle \nabla H_k, a \rangle = 0. \tag{5.22}
\]

Thus, \( M^n \) is totally umbilical by Theorem 5.7. \( \square \)

**Theorem 5.9.** Let \( \phi : M^n \to S_{n+1}^n(c), \ n \geq 2, \) be a compact space-like hypersurface in de Sitter space with \( H_{k+1} > 0, \ a \in \mathbb{R}_{1}^{n+2} \) any unit time-like vector with the same time orientation as \( N, \) and \( 0 \leq i < j \leq k \leq n-1, \ j \geq i + 2. \) If there exists a point of \( M^n, \) where \( H_1, \ldots, H_k \) are positive, then

\[
\int_{M^n} \left\langle \binom{n-1}{j} T_i \nabla H_j - \binom{n-1}{i} T_j \nabla H_i, a \right\rangle \geq 0, \tag{5.23}
\]

with equality if and only if \( M^n \) is totally umbilical.

**Proof.** For any unit time-like vector \( a \in \mathbb{R}_{1}^{n+2} \) with the same time orientation as \( N, \) we have \( \langle N, a \rangle \leq -1. \) Thus by Theorems 5.3, 3.6, and 3.8, we can deduce that

\[
\int_{M^n} \left\langle \binom{n-1}{j} T_i \nabla H_j - \binom{n-1}{i} T_j \nabla H_i, a \right\rangle \geq 0, \tag{5.24}
\]

and the equality holds if and only if \( M^n \) is totally umbilical. \( \square \)

Let \( a \in \mathbb{R}_{1}^{n+2} \) be a unit time-like vector. The intersection of \( S_{n+1}^n(c) \subset \mathbb{R}_{1}^{n+2} \) and the space-like hyperplane \( \{ x \in \mathbb{R}_{1}^{n+2} \mid \langle x, a \rangle = 0 \} \) defines an \( n \)-sphere which is a totally geodesic hypersurface in \( S_{n+1}^n(c). \) We will refer to that sphere as the equator of \( S_{n+1}^n(c) \) determined by \( a. \) This equator divides the de Sitter space into two connected components; the future which is given by

\[
\{ x \in \mathbb{R}_{1}^{n+2} \mid \langle x, a \rangle < 0 \}, \tag{5.25}
\]
and the past given by
\[ \{ x \in \mathbb{R}_1^{n+2} \mid \langle x, a \rangle > 0 \}. \quad (5.26) \]

Following [3], we can easily get the following corollary.

**Corollary 5.10.** Let \( \phi : M^n \to S_1^{n+1} (c) \), \( n \geq 2 \), be a compact space-like hypersurface in de Sitter space and \( 2 \leq k \leq n-1 \). If \( M^n \) is contained in the chronological future (or past) relative to the equator of \( S_1^{n+1} (c) \) determined by a unit time-like vector \( a \in \mathbb{R}_1^{n+2} \) with the same time orientation as \( N \) and \( H_{k+1} > 0 \) (or \( (-1)^{k+1} H_{k+1} > 0 \)), then
\[ \int_{M^n} \langle \nabla H_i, a \rangle \geq 0 \quad \text{or} \quad (-1)^i \lambda_i \langle N(x_0), a \rangle > 0, \quad 2 \leq i \leq k, \quad (5.27) \]
with each equality if and only if \( M^n \) is totally umbilical.

**Proof.** First we prove the future case. By Theorem 5.9, it is sufficient to prove that there exists a point of \( M^n \), where all \( H_i > 0 \). Since \( M^n \) is contained in the chronological future relative to the equator determined by \( a \) and \( M^n \) is compact, there exists a point \( x_0 \in M^n \) such that
\[ \max_{x \in M^n} \langle \phi(x), a \rangle = \langle \phi(x_0), a \rangle < 0. \quad (5.28) \]
Thus by maximum principle, we have
\[ -c \langle \phi(x_0), a \rangle + \lambda_i \langle N(x_0), a \rangle = -c \langle e_i, e_i \rangle \langle \phi(x_0), a \rangle - \langle s d e_i, e_i \rangle \langle N(x_0), a \rangle \]
\[ = \nabla^2 \langle \phi, a \rangle (e_i, e_i) \leq 0. \quad (5.29) \]
Since \( a \in \mathbb{R}_1^{n+2} \) is a unit time-like vector with the same time orientation as \( N \), we have \( \langle N, a \rangle \leq -1 \). So
\[ \lambda_i \geq c \frac{\langle \phi(x_0), a \rangle}{\langle N(x_0), a \rangle} > 0, \quad i = 1, \ldots, n. \quad (5.30) \]
Thus all \( H_i > 0 \). For the past case, we only need to replace \( N \) and \( a \) by \( -N \) and \( -a \), respectively, and the proof for the future case applies. This completes the proof. \( \square \)

**References**


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Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

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