ON THE PERIODIC NATURE OF SOME MAX-TYPE DIFFERENCE EQUATIONS

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We study some qualitative behavior of solutions of some max-type difference equations with periodic coefficients. Some new results of the periodicity character of solutions of that type of difference equations will be established.

1. Introduction

Recently there has been a lot of interest in studying the global attractivity, the boundedness character, and the periodicity nature of nonlinear difference equations. In [5, 6, 8] some global convergence results were established which can be applied to nonlinear difference equations in proving that every solution of these equations converges to a periodic solution (which need not necessarily be stable). The periodic nature of nonlinear difference equations of the max type has been investigated by many authors. See for example [1, 2, 3, 4].

Our main objective in this paper is to extend the study of boundedness and periodicity to solutions of some max-type difference equations. We deal with the following difference equation:

$$x_{n+1} = \max\left\{ \frac{1}{x_n}, \frac{A_n}{x_n}, \frac{A_n}{x_{n-1}} \right\}, \quad n = 0, 1, \ldots,$$

where \( \{A_n\}_{n=0}^{\infty} = \{\ldots, \alpha, \beta, \alpha, \beta, \ldots\} \) is a periodic sequence of positive numbers of period two with \( \beta > \alpha > 1 \). The case where \( \{A_n\}_{n=0}^{\infty} \) is a periodic sequence of positive numbers of period three and \( A_n \in (0, 1] \) was investigated in [4].

2. Invariant interval and boundedness

In this section, we show that every solution of (1.1) is bounded and persists.

The following lemmas are quite important results in their own; however these lemmas will be used in the subsequent discussion.

**Lemma 2.1.** Every positive solution of (1.1) is bounded and persists.
Proof. Let \( \{x_n\}_{n=-1}^{\infty} \) be a solution of (1.1). It follows from (1.1) for an integer number \( N \geq 0 \) that
\[
x_{n+1}x_n \geq 1, \quad x_{n+1}x_{n-1} \geq \alpha > 1 \quad \forall n \geq N. \tag{2.1}
\]
Thus
\[
\min \{x_{n+1}x_n, x_{n+1}x_{n-1}\} \geq 1 \tag{2.2}
\]
or
\[
x_{n+1} \min \{x_n, x_{n-1}\} \geq 1 \quad \forall n \geq N. \tag{2.3}
\]
That is, there exists a positive real number \( m \) such that
\[
x_n \geq m \quad \forall n \geq N. \tag{2.4}
\]
Thus from (1.1), we see that
\[
x_{n+1} = \max \left\{ \frac{1}{x_n}, \frac{A_n}{x_n x_{n-1}} \right\} \leq \max \left\{ \frac{1}{m}, \frac{A_n}{m} \right\} = M. \tag{2.5}
\]
Hence
\[
x_n \leq M \quad \forall n \geq N. \tag{2.6}
\]
Thus from inequalities (2.4) and (2.6) we get
\[
0 < m \leq x_n \leq M < \infty \quad \forall n \geq N. \tag{2.7}
\]
Therefore every solution of (1.1) is bounded and persists. \( \square \)

Lemma 2.2. Assume that \( \{x_n\}_{n=-1}^{\infty} \) is a positive solution of (1.1). Suppose there exists \( N \geq 0 \) such that
\[
x_{N-1}, x_N \in \left[ \frac{1}{\sqrt{\alpha}}, \beta \sqrt{\alpha} \right] \quad \text{for some } N \geq 0. \tag{2.8}
\]
Then
\[
x_n \in \left[ \frac{1}{\sqrt{\alpha}}, \beta \sqrt{\alpha} \right] \quad \forall n \geq N. \tag{2.9}
\]
Proof. Observe from (1.1) that
\[
x_{n+1} = \max \left\{ \frac{1}{x_n}, \frac{A_n}{x_n x_{n-1}} \right\} \geq \max \left\{ \frac{1}{\sqrt{\alpha}}, \frac{\alpha}{\beta \sqrt{\alpha}} \right\} = \frac{\sqrt{\alpha}}{\beta},
\]
\[
x_{n+1} \leq \max \{\sqrt{\alpha}, \alpha \sqrt{\alpha}\} = \alpha \beta < \beta \sqrt{\alpha}. \tag{2.10}
\]
Then
\[ \frac{\sqrt{\alpha}}{\beta} \leq x_{N+1} < \beta \sqrt{\alpha}. \] (2.11)

Again
\[
x_{N+2} = \max \left\{ \frac{1}{x_{N+1}} \frac{A_{N+1}}{x_N} \right\} \geq \max \left\{ \frac{1}{\beta \sqrt{\alpha}}, \frac{\beta}{\beta \sqrt{\alpha}} \right\} = \frac{1}{\sqrt{\alpha}},
\]
\[
x_{N+2} \leq \max \left\{ \frac{\beta}{\sqrt{\alpha}}, \beta \sqrt{\alpha} \right\} = \beta \sqrt{\alpha}.
\] (2.12)

Then
\[ \frac{1}{\sqrt{\alpha}} \leq x_{N+2} \leq \beta \sqrt{\alpha}. \] (2.13)

Also we see from (1.1) that
\[
x_{N+3} = \max \left\{ \frac{1}{x_{N+2}} \frac{A_{N+2}}{x_{N+1}} \right\} \geq \max \left\{ \frac{1}{\beta \sqrt{\alpha}}, \frac{\alpha}{\alpha \sqrt{\alpha}} \right\} = \frac{1}{\sqrt{\alpha}},
\]
\[
x_{N+3} \leq \max \left\{ \sqrt{\beta}, \beta \sqrt{\alpha} \right\} = \beta \sqrt{\alpha}.
\] (2.14)

Then
\[ \frac{1}{\sqrt{\alpha}} \leq x_{N+3} \leq \beta \sqrt{\alpha}. \] (2.15)

Thus following the above procedure we have
\[ \forall n \geq N, \quad x_n \in \left[ \frac{\sqrt{\alpha}}{\beta}, \beta \sqrt{\alpha} \right]. \] (2.16)

The proof is complete. \[ \square \]

**Lemma 2.3.** Every solution of (1.1) which is bounded below by $1/\sqrt{\alpha}$ lies in the interval $[1/\sqrt{\alpha}, \beta \sqrt{\alpha}]$.

**Proof.** Let $\{x_n\}_{n=1}^\infty$ be a positive solution of (1.1) and there exists $N \geq 0$ such that
\[ x_{n-1} \geq \frac{1}{\sqrt{\alpha}} \quad \forall n \geq N. \] (2.17)

It follows from (1.1) that
\[ x_{N+1} = \max \left\{ \frac{1}{x_N} \frac{A_N}{x_{N-1}} \right\} \leq \max \left\{ \sqrt{\alpha}, \sqrt{\alpha} A_N \right\} \leq \beta \sqrt{\alpha}. \] (2.18)

Similarly, we see that
\[ x_{N+2} = \max \left\{ \frac{1}{x_{N+1}} \frac{A_{N+1}}{x_N} \right\} \leq \max \left\{ \sqrt{\alpha}, \sqrt{\alpha} A_{N+1} \right\} \leq \beta \sqrt{\alpha}. \] (2.19)

The rest of the proof follows by Lemma 2.2. \[ \square \]
3. The main result

In this section, we study the periodicity character of solutions of (1.1).

In the following we study the existence of periodic solutions of (1.1) with period four.

**Theorem 3.1.** Assume that \( \{x_n\}_{n=-1}^{\infty} \) is a positive solution of (1.1) with

\[
\frac{1}{\sqrt{\alpha}} < x_{N-1}, \quad x_N < \sqrt{\beta}.
\] (3.1)

Then \( \{x_n\}_{n=-1}^{\infty} \) is a four-cycle solution of (1.1).

**Proof.** Let \( \{x_n\}_{n=-1}^{\infty} \) be a positive solution of (1.1). Suppose there exists \( N \geq 0 \) such that

\[
\frac{1}{\sqrt{\alpha}} < x_{N-1}, \quad x_N < \sqrt{\beta}.
\] (3.2)

Assume that

\[
x_{N-1} = p, \quad x_N = q.
\] (3.3)

Observe from (1.1) that

\[
x_{N+1} = \max \left\{ \frac{1}{x_N}, \frac{A_N}{x_N} \right\}.
\] (3.4)

We consider the following two cases.

1. \( x_{N+1} = 1/x_N = 1/q \). In this case \( 1/x_N > \alpha/x_{N-1} \), (the case \( 1/x_N > \beta/x_{N-1} \) can be treated similarly) and we see that

\[
x_{N+2} = \max \left\{ \frac{1}{x_{N+1}}, \frac{A_{N+1}}{x_N} \right\} = \max \left\{ q, \frac{\beta}{q} \right\} = \frac{\beta}{q},
\]

\[
x_{N+3} = \max \left\{ \frac{1}{x_{N+2}}, \frac{A_{N+2}}{x_{N+1}} \right\} = \max \left\{ \frac{q}{\beta}, \frac{\beta}{q} \right\} = \frac{\beta}{q},
\]

\[
x_{N+4} = \max \left\{ \frac{1}{x_{N+3}}, \frac{A_{N+3}}{x_{N+2}} \right\} = \max \left\{ \frac{1}{aq}, \frac{q}{\beta} \right\} = q,
\]

\[
x_{N+5} = \max \left\{ \frac{1}{x_{N+4}}, \frac{A_{N+4}}{x_{N+3}} \right\} = \max \left\{ \frac{1}{q}, \frac{1}{\beta} \right\} = \frac{1}{q},
\]

\[
x_{N+6} = \max \left\{ \frac{1}{x_{N+5}}, \frac{A_{N+5}}{x_{N+4}} \right\} = \max \left\{ q, \frac{\beta}{q} \right\} = \frac{\beta}{q}.
\] (3.5)
Then clearly the solution becomes in the form
\[
\left\{ \ldots, q, \frac{\beta}{q}, \alpha q, q, \frac{\beta}{q}, \alpha q, \ldots \right\}.
\] (3.6)

(2) \(x_{n+1} = \alpha/x_{n-1} = \alpha/p\). In this case we see that
\[
x_{n+2} = \max \left\{ \frac{1}{x_{n+1}}, \frac{A_{n+1}}{x_n} \right\} = \max \left\{ \frac{p}{\alpha}, \frac{\beta}{q} \right\} = \frac{\beta}{q},
\] (3.7)

where \(x_{n-1} > 1/\sqrt{\beta} \Rightarrow \beta x_{n-1} > \sqrt{\beta} > x_n\),
\[
x_{n+4} = \max \left\{ \frac{1}{x_{n+3}}, \frac{A_{n+3}}{x_{n+2}} \right\} = \max \left\{ \frac{1}{p}, q \right\} = q,
\]
\[
x_{n+5} = \max \left\{ \frac{1}{x_{n+4}}, \frac{A_{n+4}}{x_{n+3}} \right\} = \max \left\{ \frac{1}{q}, \frac{\alpha}{p} \right\} = \frac{\alpha}{p},
\] (3.8)

and so the solution becomes in the form
\[
\left\{ \ldots, p, q, \frac{\beta}{p}, \alpha p, q, \frac{\beta}{p}, \alpha p, \ldots \right\}.
\] (3.9)

The proof is complete.

**Theorem 3.2.** Every positive solution of (1.1) which is bounded from below by \(1/\sqrt{\alpha}\) is eventually periodic with period four.

**Proof.** Let \(\{x_n\}_{n=-1}^{\infty}\) be a positive solution of (1.1). By Lemma 2.3, we assume
\[
\frac{1}{\sqrt{\alpha}} < x_{n-1}, \quad x_n < \beta \sqrt{\alpha} \text{ for some integer } N \geq 2.
\] (3.10)

From (1.1), we see that
\[
x_{n+1} = \max \left\{ \frac{1}{x_n}, \frac{\alpha}{x_{n-1}} \right\}.
\] (3.11)

We consider the following two cases.

(A) \(x_{n+1} = 1/x_n\). In this case \(1/x_n > \alpha/x_{n-1}\), and we see that
\[
x_{n+2} = \max \left\{ \frac{1}{x_{n+1}}, \frac{A_{n+1}}{x_n} \right\} = \max \left\{ x_n, \frac{\beta}{x_n} \right\}.
\] (3.12)

We consider the following two cases.
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\((A_{11})\) \(x_{N+2} = x_N\). In this case \(x_N > \beta / x_N\), and we see that

\[ x_{N+3} = \max \left\{ \frac{1}{x_{N+2}}, \frac{A_{N+2}}{x_{N+1}} \right\} = \max \left\{ \frac{1}{x_N}, ax_N \right\} = ax_N, \quad (3.13) \]

where \(x_N > 1/ \sqrt{a} \Rightarrow ax_N > 1/x_N,\)

\[ x_{N+4} = \max \left\{ \frac{1}{x_{N+3}}, \frac{A_{N+3}}{x_{N+2}} \right\} = \max \left\{ \frac{1}{ax_N}, \frac{\beta}{x_N} \right\} = \frac{\beta}{x_N}, \]
\[ x_{N+5} = \max \left\{ \frac{1}{x_{N+4}}, \frac{A_{N+4}}{x_{N+3}} \right\} = \max \left\{ \frac{x_N}{\beta}, \frac{1}{x_N} \right\} = \frac{x_N}{\beta}, \]
\[ x_{N+6} = \max \left\{ \frac{1}{x_{N+5}}, \frac{A_{N+5}}{x_{N+4}} \right\} = \max \left\{ \frac{\beta}{x_N}, x_N \right\} = x_N, \]
\[ x_{N+7} = \max \left\{ \frac{1}{x_{N+6}}, \frac{A_{N+6}}{x_{N+5}} \right\} = \max \left\{ \frac{1}{x_N}, \frac{\alpha \beta}{x_N} \right\} = \frac{\alpha \beta}{x_N}, \]
\[ x_{N+8} = \max \left\{ \frac{1}{x_{N+7}}, \frac{A_{N+7}}{x_{N+6}} \right\} = \max \left\{ \frac{x_N}{\alpha \beta}, \frac{\beta}{x_N} \right\} = \frac{\beta}{x_N}, \quad (3.14) \]
\[ x_{N+9} = \max \left\{ \frac{1}{x_{N+8}}, \frac{A_{N+8}}{x_{N+7}} \right\} = \max \left\{ \frac{x_N}{\beta}, \frac{\beta}{x_N} \right\} = \frac{x_N}{\beta}, \]
\[ x_{N+10} = \max \left\{ \frac{1}{x_{N+9}}, \frac{A_{N+9}}{x_{N+8}} \right\} = \max \left\{ \frac{\beta}{x_N}, x_N \right\} = x_N, \]
\[ x_{N+11} = \max \left\{ \frac{1}{x_{N+10}}, \frac{A_{N+10}}{x_{N+9}} \right\} = \max \left\{ \frac{\alpha \beta}{x_N}, \frac{\beta}{x_N} \right\} = \frac{\beta}{x_N}, \]
\[ x_{N+12} = \max \left\{ \frac{1}{x_{N+11}}, \frac{A_{N+11}}{x_{N+10}} \right\} = \max \left\{ \frac{x_N}{\alpha \beta}, \frac{\beta}{x_N} \right\} = \frac{\beta}{x_N}. \]

We see that the solution is in the form

\[ \{ \ldots, \frac{\beta}{x_N}, \frac{x_N}{\beta}, x_N, \frac{\alpha \beta}{x_N}, \frac{\beta}{x_N}, \frac{x_N}{\beta}, \frac{\alpha \beta}{x_N}, \frac{x_N}{\beta}, \frac{\alpha \beta}{x_N}, \ldots \} \]. \quad (3.15) \]

Therefore \(\{x_n\}_{n=1}^\infty\) is a periodic solution with period four.

\((A_{12})\) \(x_{N+2} = \beta / x_N\). In this case \(\beta / x_N > x_N\), and we see that

\[ x_{N+3} = \max \left\{ \frac{1}{x_{N+2}}, \frac{A_{N+2}}{x_{N+1}} \right\} = \max \left\{ \frac{x_N}{\beta}, ax_N \right\} = ax_N, \]
\[ x_{N+4} = \max \left\{ \frac{1}{x_{N+3}}, \frac{A_{N+3}}{x_{N+2}} \right\} = \max \left\{ \frac{1}{ax_N}, x_N \right\} = x_N, \quad (3.16) \]
where \( x_N > 1/\sqrt{\alpha} \Rightarrow x_N > 1/\alpha \Rightarrow x_N > 1/\alpha x_N \),

\[
x_{N+5} = \max \left\{ \frac{1}{x_{N+4}}, \frac{A_{N+4}}{x_{N+3}} \right\} = \max \left\{ \frac{1}{x_N}, \frac{1}{x_N} \right\} = \frac{1}{x_N},
\]

\[
x_{N+6} = \max \left\{ \frac{1}{x_{N+5}}, \frac{A_{N+5}}{x_{N+4}} \right\} = \max \left\{ x_N, \frac{\beta}{x_N} \right\} = \frac{\beta}{x_N},
\]

\[
x_{N+7} = \max \left\{ \frac{1}{x_{N+6}}, \frac{A_{N+6}}{x_{N+5}} \right\} = \max \left\{ x_N, \frac{\alpha x_N}{\beta} \right\} = \frac{\alpha x_N}{\beta},
\]

\[
x_{N+8} = \max \left\{ \frac{1}{x_{N+7}}, \frac{A_{N+7}}{x_{N+6}} \right\} = \max \left\{ \frac{1}{\alpha x_N}, x_N \right\} = x_N,
\]

\[
x_{N+9} = \max \left\{ \frac{1}{x_{N+8}}, \frac{A_{N+8}}{x_{N+7}} \right\} = \max \left\{ x_N, \frac{1}{x_N} \right\} = \frac{1}{x_N},
\]

\[
x_{N+10} = \max \left\{ \frac{1}{x_{N+9}}, \frac{A_{N+9}}{x_{N+8}} \right\} = \max \left\{ x_N, \frac{\beta}{x_N} \right\} = \frac{\beta}{x_N},
\]

\[
x_{N+11} = \max \left\{ \frac{1}{x_{N+10}}, \frac{A_{N+10}}{x_{N+9}} \right\} = \max \left\{ \frac{x_N}{\beta}, \alpha x_N \right\} = \alpha x_N,
\]

\[
x_{N+12} = \max \left\{ \frac{1}{x_{N+11}}, \frac{A_{N+11}}{x_{N+10}} \right\} = \max \left\{ \frac{1}{\alpha x_N}, x_N \right\} = x_N.
\]

Therefore \( \{x_n\}_{n=-1}^{\infty} \) is a periodic solution with period four as follows:

\[
\{ \ldots, x_N, \frac{\beta}{x_N}, \alpha x_N, x_N, \frac{\beta}{x_N}, \alpha x_N, \ldots \}.
\]

(A2) \( x_{N+1} = x_{N-1}/\alpha \). In this case \( \alpha / x_{N-1} > 1/x_N \), and we see that

\[
x_{N+2} = \max \left\{ \frac{1}{x_{N+1}}, \frac{A_{N+1}}{x_N} \right\} = \max \left\{ \frac{x_{N-1}}{\alpha}, \frac{\beta}{x_N} \right\}.
\]

We consider the following two cases.

(A21) \( x_{N+2} = x_{N-1}/\alpha \). In this case \( x_{N-1}/\alpha > \beta/x_N \), and we see that

\[
x_{N+3} = \max \left\{ \frac{1}{x_{N+2}}, \frac{A_{N+2}}{x_{N+1}} \right\} = \max \left\{ \frac{\alpha}{x_{N-1}}, x_{N-1} \right\} = x_{N-1},
\]

where \( \beta \sqrt{\alpha x_{N-1}} > x_{N-1} x_N > \alpha \beta \Rightarrow x_{N-1} > \sqrt{\alpha} \),

\[
x_{N+4} = \max \left\{ \frac{1}{x_{N+3}}, \frac{A_{N+3}}{x_{N+2}} \right\} = \max \left\{ \frac{1}{x_{N-1}}, \frac{\alpha \beta}{x_{N-1}} \right\} = \frac{\alpha \beta}{x_{N-1}},
\]

\[
x_{N+5} = \max \left\{ \frac{1}{x_{N+4}}, \frac{A_{N+4}}{x_{N+3}} \right\} = \max \left\{ \frac{x_{N-1}}{\alpha \beta}, \frac{\alpha}{x_{N-1}} \right\} = \frac{x_{N-1}}{\alpha \beta},
\]

We consider the following two cases.

(A211) \( x_{N+5} = x_{N-1}/\alpha \beta \). In this case \( x_{N-1}/\alpha \beta > \alpha / x_{N-1} \), and we see that

\[
x_{N+6} = \max \left\{ \frac{1}{x_{N+5}}, \frac{A_{N+5}}{x_{N+4}} \right\} = \max \left\{ \frac{\alpha \beta}{x_{N-1}}, \frac{x_{N-1}}{\alpha} \right\} = \frac{x_{N-1}}{\alpha},
\]
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where $x_{N-1}/\alpha\beta > \alpha/x_{N-1} \Rightarrow x_{N-1}/\alpha > \alpha\beta/x_{N-1}$,

\[
x_{N+7} = \max \left\{ \frac{1}{x_{N+6}}, \frac{A_{N+6}}{x_{N+5}} \right\} = \max \left\{ \frac{\alpha}{x_{N-1}}, \frac{\alpha^2\beta}{x_{N-1}} \right\} = \frac{\alpha^2\beta}{x_{N-1}},
\]

\[
x_{N+8} = \max \left\{ \frac{1}{x_{N+7}}, \frac{A_{N+7}}{x_{N+6}} \right\} = \max \left\{ \frac{x_{N-1}}{\alpha}, \frac{\alpha\beta}{\alpha^2\beta} \right\} = \frac{\alpha\beta}{x_{N-1}},
\]

where $x_{N-1} < \beta/\alpha \Rightarrow x_{N-1}^2 < \beta^2\alpha < \beta^2\alpha^3 \Rightarrow \alpha\beta/x_{N-1} > x_{N-1}/\alpha^2\beta$,

\[
x_{N+9} = \max \left\{ \frac{1}{x_{N+8}}, \frac{A_{N+8}}{x_{N+7}} \right\} = \max \left\{ \frac{x_{N-1}}{\alpha\beta}, \frac{x_{N-1}}{\alpha} \right\} = x_{N-1},
\]

\[
x_{N+10} = \max \left\{ \frac{1}{x_{N+9}}, \frac{A_{N+9}}{x_{N+8}} \right\} = \max \left\{ \frac{\alpha^2\beta}{x_{N-1}}, \frac{\alpha}{x_{N-1}} \right\} = \frac{\alpha^2\beta}{x_{N-1}},
\]

\[
x_{N+11} = \max \left\{ \frac{1}{x_{N+10}}, \frac{A_{N+10}}{x_{N+9}} \right\} = \max \left\{ \frac{\alpha}{x_{N-1}}, \frac{\alpha\beta}{\alpha^2\beta} \right\} = \frac{\alpha\beta}{x_{N-1}},
\]

\[
x_{N+12} = \max \left\{ \frac{1}{x_{N+11}}, \frac{A_{N+11}}{x_{N+10}} \right\} = \max \left\{ \frac{x_{N-1}}{\alpha}, \frac{x_{N-1}}{\alpha\beta} \right\} = \frac{x_{N-1}}{\alpha\beta}.
\]

Therefore the solution can be written as

\[
\left\{ \ldots, \frac{\alpha\beta}{x_{N-1}}, \frac{x_{N-1}}{\alpha}, \frac{x_{N-1}}{\alpha\beta}, \frac{\alpha\beta}{x_{N-1}}, \frac{x_{N-1}}{\alpha}, \frac{x_{N-1}}{\alpha\beta}, \ldots \right\}.
\]

Then $\{x_n\}_{n=-1}^{\infty}$ is a periodic solution with period four.

We consider the following two cases.

$(A_{212})$ $x_{N+5} = \alpha/x_{N-1}$. In this case $\alpha/x_{N-1} > x_{N-1}/\alpha\beta$, and we see that

\[
x_{N+6} = \max \left\{ \frac{1}{x_{N+5}}, \frac{A_{N+5}}{x_{N+4}} \right\} = \max \left\{ \frac{x_{N-1}}{\alpha}, \frac{x_{N-1}}{\alpha} \right\} = \frac{x_{N-1}}{\alpha},
\]

\[
x_{N+7} = \max \left\{ \frac{1}{x_{N+6}}, \frac{A_{N+6}}{x_{N+5}} \right\} = \max \left\{ \frac{\alpha}{x_{N-1}}, \frac{\alpha}{x_{N-1}} \right\} = x_{N-1},
\]

\[
x_{N+8} = \max \left\{ \frac{1}{x_{N+7}}, \frac{A_{N+7}}{x_{N+6}} \right\} = \max \left\{ \frac{1}{x_{N-1}}, \frac{\alpha\beta}{x_{N-1}} \right\} = \frac{\alpha\beta}{x_{N-1}},
\]

\[
x_{N+9} = \max \left\{ \frac{1}{x_{N+8}}, \frac{A_{N+8}}{x_{N+7}} \right\} = \max \left\{ \frac{x_{N-1}}{\alpha\beta}, \frac{x_{N-1}}{\alpha} \right\} = \frac{x_{N-1}}{\alpha},
\]

\[
x_{N+10} = \max \left\{ \frac{1}{x_{N+9}}, \frac{A_{N+9}}{x_{N+8}} \right\} = \max \left\{ \frac{x_{N-1}}{\alpha}, \frac{x_{N-1}}{\alpha} \right\} = \frac{x_{N-1}}{\alpha},
\]

\[
x_{N+11} = \max \left\{ \frac{1}{x_{N+10}}, \frac{A_{N+10}}{x_{N+9}} \right\} = \max \left\{ \frac{\alpha}{x_{N-1}}, \frac{x_{N-1}}{x_{N-1}} \right\} = x_{N-1},
\]

\[
x_{N+12} = \max \left\{ \frac{1}{x_{N+11}}, \frac{A_{N+11}}{x_{N+10}} \right\} = \max \left\{ \frac{1}{x_{N-1}}, \frac{x_{N-1}}{x_{N-1}} \right\} = \frac{x_{N-1}}{x_{N-1}}.
\]
It is also easy to see that the solution takes the form

\[
\left\{ \ldots, \frac{\alpha}{x_{N-1}}, \frac{x_{N-1}}{\alpha}, \frac{\alpha \beta}{x_{N-1}}, \frac{x_{N-1}}{\alpha}, \frac{\alpha}{x_{N-1}}, \frac{\alpha \beta}{x_{N-1}}, \ldots \right\},
\]

which is periodic with period four.

\((A_{22})\) \(x_{N+2} = \beta/x_N\). In this case \(\beta/x_N > x_{N-1}/\alpha\), and we see that

\[
x_{N+3} = \max \left\{ \frac{1}{x_{N+2}}, \frac{A_{N+2}}{x_{N+1}} \right\} = \max \left\{ \frac{x_N}{\beta}, x_{N-1} \right\}.
\]

We consider the following two cases.

\((A_{221})\) \(x_{N+3} = x_{N-1}\). In this case \(x_{N-1} > x_N/\beta\), and we see that

\[
x_{N+4} = \max \left\{ \frac{1}{x_{N+3}}, \frac{A_{N+3}}{x_{N+2}} \right\} = \max \left\{ \frac{1}{x_{N-1}}, x_N \right\} = x_N,
\]

\[
x_{N+5} = \max \left\{ \frac{1}{x_{N+4}}, \frac{A_{N+4}}{x_{N+3}} \right\} = \max \left\{ \frac{1}{x_{N-1}}, \frac{\alpha}{x_{N-1}} \right\} = \frac{\alpha}{x_{N-1}},
\]

\[
x_{N+6} = \max \left\{ \frac{1}{x_{N+5}}, \frac{A_{N+5}}{x_{N+4}} \right\} = \max \left\{ \frac{x_{N-1}}{\alpha}, \frac{\beta}{x_N} \right\} = \frac{\beta}{x_N},
\]

\[
x_{N+7} = \max \left\{ \frac{1}{x_{N+6}}, \frac{A_{N+6}}{x_{N+5}} \right\} = \max \left\{ \frac{x_N}{\beta}, x_{N-1} \right\} = x_{N-1},
\]

\[
x_{N+8} = \max \left\{ \frac{1}{x_{N+7}}, \frac{A_{N+7}}{x_{N+6}} \right\} = \max \left\{ \frac{1}{x_{N-1}}, x_N \right\} = x_N,
\]

\[
x_{N+9} = \max \left\{ \frac{1}{x_{N+8}}, \frac{A_{N+8}}{x_{N+7}} \right\} = \max \left\{ \frac{1}{x_N}, \frac{\alpha}{x_{N-1}} \right\} = \frac{\alpha}{x_{N-1}},
\]

\[
x_{N+10} = \max \left\{ \frac{1}{x_{N+9}}, \frac{A_{N+9}}{x_{N+8}} \right\} = \max \left\{ \frac{x_{N-1}}{\alpha}, \frac{\beta}{x_N} \right\} = \frac{\beta}{x_N},
\]

\[
x_{N+11} = \max \left\{ \frac{1}{x_{N+10}}, \frac{A_{N+10}}{x_{N+9}} \right\} = \max \left\{ \frac{x_N}{\beta}, x_{N-1} \right\} = x_{N-1},
\]

\[
x_{N+12} = \max \left\{ \frac{1}{x_{N+11}}, \frac{A_{N+11}}{x_{N+10}} \right\} = \max \left\{ \frac{1}{x_{N-1}}, x_N \right\} = x_N.
\]

One can easily see that the solution will be in the form

\[
\left\{ \ldots, x_{N-1}, x_N, \frac{\alpha}{x_{N-1}}, \frac{\beta}{x_N}, x_{N-1}, x_N, \frac{\alpha}{x_{N-1}}, \frac{\beta}{x_N}, \ldots \right\},
\]

and so the solution is periodic with period four.
(A_{222}) x_{N+3} = x_N / \beta. In this case \( x_N / \beta > x_{N-1} \), and we see that

\[
x_{N+4} = \max \left\{ \frac{1}{x_{N+3}}, \frac{A_{N+3}}{x_{N+2}} \right\} = \max \left\{ \frac{\beta}{x_N}, x_N \right\} = x_N,
\]

where \( x_N > \beta x_{N-1} > \beta / \sqrt{\alpha} > \beta / \sqrt{\beta} = \sqrt{\beta} \Rightarrow x_N^2 > \beta \Rightarrow x_N > \beta / x_N \),

\[
x_{N+5} = \max \left\{ \frac{1}{x_{N+4}}, \frac{A_{N+4}}{x_{N+3}} \right\} = \max \left\{ \frac{\alpha \beta}{x_N}, x_N \right\} = \frac{\alpha \beta}{x_N},
\]

where \( x_N < \beta \sqrt{\alpha} \),

\[
x_{N+7} = \max \left\{ \frac{1}{x_{N+6}}, \frac{A_{N+6}}{x_{N+5}} \right\} = \max \left\{ \frac{x_N}{\beta}, \frac{\beta}{\beta} \right\} = \frac{x_N}{\beta},
\]

\[
x_{N+8} = \max \left\{ \frac{1}{x_{N+7}}, \frac{A_{N+7}}{x_{N+6}} \right\} = \max \left\{ \beta, x_N \right\} = x_N,
\]

\[
x_{N+9} = \max \left\{ \frac{1}{x_{N+8}}, \frac{A_{N+8}}{x_{N+7}} \right\} = \max \left\{ \frac{\alpha \beta}{x_N}, x_N \right\} = \frac{\alpha \beta}{x_N},
\]

\[
x_{N+10} = \max \left\{ \frac{1}{x_{N+9}}, \frac{A_{N+9}}{x_{N+8}} \right\} = \max \left\{ x_N, \frac{\beta}{\alpha \beta} \right\} = x_N,
\]

\[
x_{N+11} = \max \left\{ \frac{1}{x_{N+10}}, \frac{A_{N+10}}{x_{N+9}} \right\} = \max \left\{ \frac{x_N}{\beta}, x_N \right\} = \frac{x_N}{\beta},
\]

\[
x_{N+12} = \max \left\{ \frac{1}{x_{N+11}}, \frac{A_{N+11}}{x_{N+10}} \right\} = \max \left\{ \frac{\beta}{x_N}, x_N \right\} = x_N.
\]

Then the solution can be written in the form

\[
\left\{ \ldots, \frac{\beta}{x_N}, x_N, \frac{x_N}{\beta}, \frac{\alpha \beta}{x_N}, \frac{\beta}{x_N}, \frac{x_N}{\beta}, \frac{\alpha \beta}{x_N}, \ldots \right\},
\]

and so the solution is periodic with period four.

This completes the proof. The proof of Theorem 3.2 is thus completed. \( \Box \)

Lemma 3.3. Assume \( \{x_n\}_{n=-1}^{\infty} \) is a positive solution of (1.1) and suppose there exists \( m \geq 2 \) such that

\[
x_m < \frac{1}{\sqrt{\alpha}} < x_{m+1}.
\]

Then either \( \{x_n\}_{n=-1}^{\infty} \) is eventually periodic solution with period four or

\[
\liminf_{n \to -\infty} x_n \geq \frac{1}{\sqrt{\alpha}}.
\]
Proof. Observe that $x_m < 1/\sqrt{\alpha}$ and either $x_{m+1} < \beta/\sqrt{\alpha}$ or $x_{m+1} > \beta/\sqrt{\alpha}$.

(i) Assume that $x_{m+1} < \beta/\sqrt{\alpha}$. It follows from (1.1) that

$$x_{m+2} = \max \left\{ \frac{1}{x_{m+1}}, \frac{A_{m+1}}{x_m} \right\} = \max \left\{ \frac{1}{x_{m+1}}, \frac{\alpha}{x_m} \right\} = \frac{\alpha}{x_m},$$

(3.37)

where $x_m x_{m+1} \geq 1 \Rightarrow x_{m+1} > \sqrt{\alpha} > 1$,

$$x_{m+3} = \max \left\{ \frac{1}{x_{m+2}}, \frac{A_{m+2}}{x_{m+1}} \right\} = \max \left\{ \frac{x_m}{\alpha}, \frac{\beta}{x_{m+1}} \right\} = \frac{\beta}{x_{m+1}},$$

(3.38)

where $x_m x_{m+1} < \beta/\sqrt{\alpha}/\sqrt{\alpha} = \beta < \alpha\beta$, and

$$x_{m+4} = \max \left\{ \frac{1}{x_{m+3}}, \frac{A_{m+3}}{x_{m+2}} \right\} = \max \left\{ \frac{x_{m+1}}{\beta}, x_m \right\}.$$  

(3.39)

Then either

$$x_{m+4} = \frac{x_{m+1}}{\beta} \quad \text{or} \quad x_{m+4} = x_m$$

(3.40)

and by simple computations the solution becomes either

$$\begin{cases} \cdots, \frac{x_{m+1}}{\beta}, x_{m+1}, \frac{\alpha\beta}{x_{m+1}}, \frac{\beta}{x_{m+1}}, x_{m+1}, \frac{\alpha\beta}{x_{m+1}}, \frac{\beta}{x_{m+1}}, \cdots \end{cases},$$

(3.41)

or

$$\begin{cases} \cdots, x_m, x_{m+1}, \frac{\alpha}{x_m}, \frac{\beta}{x_{m+1}}, x_m, x_{m+1}, \frac{\alpha}{x_m}, \frac{\beta}{x_{m+1}}, \cdots \end{cases},$$

(3.42)

and so in either case $\{x_n\}_{n=-1}^\infty$ is a periodic solution with period four.

(ii) Assume that $x_{m+1} > \beta/\sqrt{\alpha}$. In this case we see from (1.1) that

$$x_{m+2} = \max \left\{ \frac{1}{x_{m+1}}, \frac{A_{m+1}}{x_m} \right\} = \max \left\{ \frac{1}{x_{m+1}}, \frac{\alpha}{x_m} \right\} = \frac{\alpha}{x_m},$$

$$x_{m+3} = \max \left\{ \frac{1}{x_{m+2}}, \frac{A_{m+2}}{x_{m+1}} \right\} = \max \left\{ \frac{x_m}{\alpha}, \frac{\beta}{x_{m+1}} \right\}.$$  

(3.43)

We consider the following two cases.
Periodicity of max-type difference equations

\((B_1)\) \(x_{m+3} = x_m/\alpha\). In this case we see that

\[
x_{m+4} = \max \left\{ \frac{1}{x_{m+3}}, \frac{A_{m+3}}{x_{m+2}} \right\} = \max \left\{ \frac{\alpha}{x_m}, x_m \right\} = \frac{\alpha}{x_m},
\]
\[
x_{m+5} = \max \left\{ \frac{1}{x_{m+4}}, \frac{A_{m+4}}{x_{m+3}} \right\} = \max \left\{ \frac{x_m}{\alpha}, \frac{\alpha \beta}{x_m} \right\} = \frac{\alpha \beta}{x_m},
\]
\[
x_{m+6} = \max \left\{ \frac{1}{x_{m+5}}, \frac{A_{m+5}}{x_{m+4}} \right\} = \max \left\{ \frac{x_m}{\alpha \beta}, x_m \right\} = x_m,
\]
\[
x_{m+7} = \max \left\{ \frac{1}{x_{m+6}}, \frac{A_{m+6}}{x_{m+5}} \right\} = \max \left\{ \frac{1}{x_m}, \frac{x_m}{\alpha} \right\} = \frac{1}{x_m},
\]
\[
x_{m+8} = \max \left\{ \frac{1}{x_{m+7}}, \frac{A_{m+7}}{x_{m+6}} \right\} = \max \left\{ x_m, \frac{\alpha}{x_m} \right\} = \frac{\alpha}{x_m},
\]
\[
x_{m+9} = \max \left\{ \frac{1}{x_{m+8}}, \frac{A_{m+8}}{x_{m+7}} \right\} = \max \left\{ \frac{x_m}{\alpha}, \beta x_m \right\} = \beta x_m,
\]
\[
x_{m+10} = \max \left\{ \frac{1}{x_{m+9}}, \frac{A_{m+9}}{x_{m+8}} \right\} = \max \left\{ \frac{1}{\beta x_m}, \frac{x_m}{\alpha} \right\}.
\]

We consider the following two cases.

\((B_{11})\) \(x_{m+10} = x_m\). In this case the solution eventually will be periodic with period four as

\[
\left\{ \ldots, x_m, \frac{1}{x_m}, \frac{\alpha}{x_m}, \beta x_m, x_m, \frac{1}{x_m}, \frac{\alpha}{x_m}, \beta x_m, \ldots \right\}.
\]

\((B_{12})\) \(x_{m+10} = 1/\beta x_m\). In this case straightforward calculations show that the solution will be in the form

\[
\left\{ \ldots, \frac{x_m}{\alpha}, \frac{\alpha \beta}{x_m}, x_m, \frac{1}{x_m}, \frac{\alpha}{x_m}, \beta x_m, \frac{1}{\beta x_m}, x_m, \alpha \beta x_m, \ldots \right\}.
\]

Thus the subsequence \(\{x_{m+3i}\}_{i=0}^{\infty}\) is increasing and so

\[
\lim_{i \to \infty} x_{m+3i} = \frac{1}{\sqrt{\alpha}}.
\]

\((B_2)\) \(x_{m+3} = \beta/x_{m+1}\). This can be treated similarly to the case \(x_{m+3} = x_m/\alpha\) and the solution is either periodic with period four or \(\lim_{i \to \infty} x_{m+3i} \geq 1/\sqrt{\alpha}\).

The proof is complete. \(\square\)

Remark 3.4. Observe by assumption that \(x_m, x_{m+1} < 1/\sqrt{\alpha}\) is not possible as can be seen from (1.1).

Now, we can state the main result in this section.

**Theorem 3.5.** Every solution of (1.1) is periodic with period four.

**Proof.** The proof of this theorem follows from Theorem 3.2 and Lemma 3.3. \(\square\)
References


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