ON WEAK CONVERGENCE OF ITERATES IN QUANTUM $L_p$-SPACES ($p \geq 1$)

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Equivalent conditions are obtained for weak convergence of iterates of positive contractions in the $L_1$-spaces for general von Neumann algebra and general JBW algebras, as well as for Segal-Dixmier $L_p$-spaces ($1 \leq p < \infty$) affiliated to semifinite von Neumann algebras and semifinite JBW algebras without direct summands of type $I_2$.

1. Introduction and preliminaries

This paper is devoted to a presentation of some results concerning ergodic-type properties of weak convergence of iterates of operators acting in $L_1$-space for general von Neumann algebras and JBW algebras, as well as Segal-Dixmier $L_p$-spaces ($1 \leq p < \infty$) of operators affiliated with semifinite von Neumann algebras and semifinite JBW algebras.

The first results in the field of noncommutative ergodic theory were obtained independently by Sinai and Anselvič [21] and Lance [15]. Developments of the subject are reflected in the monographs of Jajte [13] and Krengel [14] (see also [8, 9, 10, 18]).

We will use facts and the terminology from the general theory of von Neumann algebras (see [5, 7, 17, 19, 22]), the general theory of Jordan and real operator algebras (see [2, 3, 11, 16]), and the theory of noncommutative integration (see [20, 23, 24]).

Let $M$ be a von Neumann algebra, acting on a separable Hilbert space $H$, $M_\ast$ is a predual space of $M$, which always exists according to the Sakai theorem [19]. It is well known that $M_\ast$ could be identified with $L_1$-space for $M$.

Spaces $L_1$ and $L_2$ of the operators affiliated with the semifinite von Neumann algebra $M$ with semifinite faithful trace $\tau$ were introduced by Segal (see [20]). This result was extended to $L_p$-space of operators affiliated with von Neumann algebras $M$, $\tau$, and integrated with $p$th power by Dixmier (see [6]). For an alternative exposition of building $L_p$ based on Grothendieck’s idea of using rearrangements of functions, see also [24]. The theory of $L_p$-spaces was extended further to the von Neumann algebras with faithful normal weight $\rho$. However, these spaces lack some of the properties, for example, in general, these spaces do not intersect.

Recall some standard terminology (see [8, 9, 10, 14]).
Definition 1.1. A linear mapping $T$ from $M_*$ in itself is called a *contraction* if its norm is not greater than one.

Definition 1.2. A contraction $T$ is said to be *positive* if

$$TM_* \subseteq M_*.$$  \hspace{1cm} (1.1)

We will consider the two topologies on the space $M_*$: the *weak topology*, or the $\sigma(M_*, M)$ topology, and the *strong topology* of the $M_*$-space norm convergence.

Definition 1.3. A matrix $(a_{n,i})$, $i, n = 1, 2, \ldots$, of real numbers is called *uniformly regular* if

$$\sup_{n} \sum_{i=1}^{\infty} |a_{n,i}| \leq C < \infty; \quad \lim_{n \to \infty} \sup_{i} |a_{n,i}| = 0; \quad \lim_{n \to \infty} \sum_{i} a_{n,i} = 1.$$  \hspace{1cm} (1.2)

2. Main result: the case of quantum $L_1$-spaces

2.1. The case of noncommutative $L_1$-spaces. The following theorem is valid.

**Theorem 2.1.** The following conditions for a positive contraction $T$ in the predual space of a complex von Neumann algebras $M$ are equivalent.

(i) The sequence $\{T^i\}_{i=1,2,\ldots}$ converges weakly.

(ii) For each strictly increasing sequence of natural numbers $\{k_i\}_{i=1,2,\ldots}$,

$$n^{-1} \sum_{i=1}^{n} T^{k_i}$$  \hspace{1cm} (2.1)

converges strongly.

(iii) For any uniformly regular matrix $(a_{n,i})$, the sequence $\{A_n(T)\}_{n=1,2,\ldots}$,

$$A_n(T) = \sum_{i} a_{n,i} T^i,$$  \hspace{1cm} (2.2)

converges strongly.

**Proof of Theorem 2.1.** We first prove the following lemma.

**Lemma 2.2.** Let there exist a uniformly regular matrix $(a_{n,i})$ such that for each strictly increasing sequence $\{k_i\}_{i=1,2,\ldots}$ of natural numbers,

$$B_n = \sum_{i} a_{n,i} T^{k_i}$$  \hspace{1cm} (2.3)

converges strongly. Then the sequence $\{T^i\}_{i=1,2,\ldots}$ converges weakly.

**Proof.** Let $(a_{n,i})$ be a matrix with the aforementioned properties. Then the limit $B_n$ is not dependant upon the choice of the sequence $\{k_i\}_{i=1,2,\ldots}$. In fact, let $\{k_i\}_{i=1,2,\ldots}$ and $\{l_i\}_{i=1,2,\ldots}$ be the sequences for which the limits $B_n$ are different. This means that for some $x \in M_*$,

$$\sum_{i} a_{n,i} T^{k_i}x \rightharpoonup x_1, \quad \sum_{i} a_{n,i} T^{l_i}x \rightharpoonup x_2,$$  \hspace{1cm} (2.4)
for \( n \to \infty \). For a matrix \((a_{n,i})\), we build increasing sequences \(\{i_j\}_{j=1,2,...} \) and \(\{n_j\}_{j=1,2,...} \) such that

\[
\lim_{j \to \infty} \left( \sum_{i<i_j} |a_{n_j,i}| + \sum_{i>n_j} |a_{n_j,i}| \right) = 0.
\]

(2.5)

Let

\[
m_i = k_i \quad \text{for } i \in [i_{2j-1},i_{2j}), \quad m_i = l_i \quad \text{for } i \in [i_{2j},i_{2j+1}), \ j = 1,2,\ldots
\]

(2.6)

Then

\[
\lim_j \left\| \sum_i a_{n_{2j-1,i}} T^{m_i} x - x_1 \right\| = 0; \quad \lim_j \left\| \sum_i a_{n_{2j,i}} T^{m_i} x - x_2 \right\| = 0,
\]

(2.7)

which contradict (2.3), and therefore \( x_1 = x_2 \). Let now \( y \in M \) such that

\[
(T^n x - x_1, y) \to 0,
\]

(2.8)

when \( n \to \infty \). We choose a subsequence \(\{k_i\} \) such that

\[
(T^{k_i} x - x_1, y) \to y \neq 0,
\]

(2.9)

where \( y \) is a real number. Then, from the uniform regularity of the matrix \((a_{n,i})\), it follows that

\[
\lim_n \left( \sum_i a_{n,i} T^{k_i} x - x_1, y \right) = y,
\]

(2.10)

which contradicts the choice of the matrix \((a_{n,i})\). □

The implication (iii) \(\Rightarrow\) (ii) is trivial because the matrix \((a_{n,i})\),

\[
a_{n,i} = \frac{1}{n} \sum_{i<n} \delta_{j,k_i},
\]

(2.11)

is uniformly regular. Applying Lemma 2.2 to the matrix

\[
a_{n,i} = \frac{1}{n},
\]

(2.12)

\( i \leq n \) and \( a_{n,i} = 0 \) for \( i > n \), we get the implication (ii) \(\Rightarrow\) (i).

To prove the implication (i) \(\Rightarrow\) (iii), we would need the following lemma.

**Lemma 2.3.** Let \( Q \) be a contraction in the Hilbert space \( H \). Then the weak convergence of \( Q^n x \) in \( H \), where \( x \in H \), implies the strong convergence of

\[
\sum_i a_{n,i} Q^i x
\]

(2.13)

for any uniformly regular matrix \((a_{n,i})\).
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**Proof.** If the weak limit $Q^n x$ exists and is equal to $x_1$, then

$$Qx_1 = Q\left( \lim_{n \to \infty} Q^n x \right) = x_1,$$

(2.14)

where the limit is considered in the weak topology, that is, $x_1$ is $Q$-invariant. Replacing $x$ on $x - x_1$ (if necessary), we may suppose that $Q^n x$ converges weakly to $0$, and hence

$$(Q^n x, x) \to 0.$$  \quad (2.15)

We are going to show that

$$\sum_n a_{i,n} Q^n x \xrightarrow{\|\cdot\|} 0,$$  \quad (2.16)

where $(a_{i,n})$ is uniformly regular matrix. One can see that

$$\left\| \sum_i a_{N,i} Q^i x \right\|^2 \leq \sum_i \sum_j a_{N,i} a_{N,j} (Q^i x, Q^j x) \leq \sum_i \sum_j |a_{N,i} a_{N,j} (Q^i x, Q^j x)|.$$  \quad (2.17)

We fix $\epsilon > 0$. Because $Q$ is a contraction, the limit $\|Q^n x\|$ does exist. Now, we can find $K > 0$, such that for $k > K$ and $j \geq 0$,

$$\|Q^k x\| - \|Q^{k+j} x\| \leq \epsilon^2, \quad |(Q^k x, x)| \leq \epsilon.$$  \quad (2.18)

Then,

$$\begin{align*}
| (Q^k x, x) - (Q^{k+j} x, Q^j x) | &= | (Q^k x, x) - (Q^{k+j} x, x) | \\
&\leq \|Q^k x - Q^{k+j} x\| \cdot \|x\| = \left( \|Q^k x - Q^{k+j} x\|^2 \right)^{1/2} \cdot \|x\| \\
&= \left( \|Q^k x\|^2 - 2\|Q^{k+j} x\|^2 + \|Q^{k+j} x\|^2 \right)^{1/2} \cdot \|x\| \\
&\leq \left( \|Q^k x\|^2 - \|Q^{k+j} x\|^2 \right) \cdot \|x\| \leq \epsilon \cdot \|x\|,
\end{align*}$$  \quad (2.19)

and therefore

$$| (Q^{k+j} x, Q^j x) | \leq \epsilon \cdot (1 + \|x\|)$$  \quad (2.20)

for all $k > K$ and $j \geq 0$, or for $|i - j| \geq k$, the inequality

$$| (Q^i x, Q^j x) | \leq \epsilon \cdot (1 + \|x\|)$$  \quad (2.21)

is valid. We will fix $\eta > 0$, and let $N$ be a natural number such that

$$\max_i |a_{n,i}| < \eta,$$

(2.22)
for \( n \geq N \). Then the expression (1) for \( n \geq N \) could be estimated in the following way:

\[
\sum_i \sum_j |a_{N,i}a_{N,j}(Q'x, Q'x) |
\]

\[
= \sum_i |a_{n,i}a_{n,j}(Q'x, Q'x) | + \sum_{|i-j| > k} |a_{n,i}a_{n,j}(Q'x, Q'x) |
\]

\[
\leq \sum_i |a_{n,i}| \cdot \eta \cdot \|x\|^2 \cdot (2k - 1) + \sum_{|i-j| > k} |a_{n,i}a_{n,j}| \cdot \epsilon \cdot (1 + \|x\|)
\]

\[
\leq C \cdot \eta \cdot \|x\|^2 \cdot (2k - 1) + C^2 \cdot \epsilon \cdot (1 + \|x\|).
\]

(2.23)

From the arbitrariness of the values of \( \epsilon \) and \( \eta \), it follows that the strong convergence is present and the lemma is proven. \( \square \)

We prove the implication (i) \( \Rightarrow \) (iii). Let \( x \in M_{**} \) and the sequence \( \{T^i x\}_{i=1,2,\ldots} \) converges weakly. Without the loss of generality, we can consider \( \|x\| \leq 1 \), and let

\[
\bar{x} = \lim_{n \to \infty} T^n x,
\]

where the limit is understood in the weak sense. We consider

\[
y = \sum_{n=0}^{\infty} 2^{-n} T^n x.
\]

(2.25)

The series that defines \( y \) is convergent in the norm of the space \( M_* \). From the positivity of \( x \) and the properties of the operator \( T \), it follows that

\[
Ty \leq 2y,
\]

(2.26)

and, therefore, for all \( k = 1,2,\ldots \),

\[
s(T^k y) \leq s(y),
\]

(2.27)

where we denote by \( s(z) \) the support of the normal functional \( z \).

**Lemma 2.4.** Let \( u \in M_{**} \) and \( s(u) \leq s(y) \). Then \( s(\bar{u}) \leq s(\bar{x}) \), where

\[
\bar{u} = \lim_{n \to \infty} T^n u.
\]

(2.28)

**Proof.** In fact, We fix \( \epsilon > 0 \). From the density of the set

\[
\mathcal{L}_y = \{ w \in M_{**}, w \leq \lambda y, \text{ for some } \lambda > 0 \}
\]

in the set

\[
\mathcal{S} = \{ w \in M_{**}, s(w) \leq s(y) \}
\]

in the norm of the space \( M_* \), it follows that there are \( \lambda > 0 \) and \( w \in \mathcal{L}_y \) such that

\[
\|w - u\| \leq \epsilon, \quad w \leq \lambda y.
\]

(2.29)
Let
\[ w = \lim_{n \to \infty} T^n w. \]  \hspace{1cm} (2.32)
Then
\[ w(1-s(x)) = \lim_{n \to \infty} (T^n(w))(1-s(x)) \]
\[ \leq \lambda \cdot \lim_{n \to \infty} (T^n y)(1-s(x)) \]
\[ \leq \lambda \cdot \lim_{n \to \infty} \left( \sum_{k=0}^{\infty} 2^{-k} \cdot (T^{n+k}x)(1-s(x)) \right) \]
\[ = \lambda \cdot \sum_{k=0}^{\infty} 2^{-k} \lim_{n \to \infty} (T^{n+k}x)(1-s(x)) = 0. \]  \hspace{1cm} (2.33)
Because the operator \( T \) does not increase the norm of the functionals from \( M_* \), we get that
\[ \overline{u}(1-s(x)) = \lim_{n \to \infty} (T^n u)(1-s(x)) \leq \lim_{n \to \infty} (T^n w)(1-s(x)) + \lim_{n \to \infty} \|T^n(w-u)\| \leq \varepsilon. \]  \hspace{1cm} (2.34)
The needed inequality follows from the arbitrarity of \( \varepsilon \).

We introduce the following notion. For \( \mu \in M_* \), we will denote by \( \mu \cdot E \), where \( E \) is a projection from the algebra \( M \), the functional
\[ (\mu \cdot E)(A) = \mu(EAE), \]  \hspace{1cm} (2.35)
where \( A \in M \).

We fix \( \varepsilon > 0 \). We will find a number \( N \), such that
\[ (T^n x)(1-s(x)) < \varepsilon^2 \]  \hspace{1cm} (2.36)
for \( n > N \).
Then,
\[ \|T^N x \cdot s(x) - T^N x\| \]
\[ = \sup_{A \in M, \|A\| \leq 1} \| (T^N x)((1-s(x))A(1-s(x))) \]
\[ + (T^N x)((s(x))A(1-s(x))) + (T^N x)((1-s(x))A(s(x))) \| \]
\[ \leq \varepsilon \cdot (\varepsilon + 2\|x\|^{1/2}), \]
because
\[ |\mu(AB)|^2 \leq \mu(A^* A) \cdot \mu(B^* B), \]  \hspace{1cm} (2.38)
where \( \mu \in M_{**} \) and \( A, B \in M \).
Let \( w \in \mathcal{L}_y \) be such that
\[
 w \leq \lambda \bar{x}
\] (2.39)
for some \( \lambda > 0 \) and
\[
 \| T^N x \cdot s(\bar{x}) - w \| \leq \varepsilon.
\] (2.40)

Then, for \( n > N \), the following is valid:
\[
 \| T^n x - T^{n-N} w \| \leq \| T^{n-N} (T^N x - T^N x \cdot s(\bar{x})) \| + \| T^{n-N} (T^N x \cdot s(\bar{x}) - w) \| \leq 4 \cdot \varepsilon.
\] (2.41)

By taking the weak limit in the inequality (2.37) and because the unit ball of \( M_* \) is closed weakly, we will get
\[
 \| x - w \| \leq 4 \cdot \varepsilon,
\] (2.42)
where
\[
 \bar{w} = \lim_{n \to \infty} T^n w.
\] (2.43)

We now consider the algebra \( M_{s(\bar{x})} \). The functional \( \bar{x} \) is faithful on the algebra \( M_{s(\bar{x})} \). We will consider the representation \( \pi_x \) of the algebra \( M_{s(\bar{x})} \) constructed using the functional \( x \) [7]. Because the functional \( \bar{x} \) is faithful, we can conclude that the representation \( \pi_x \) is faithful on the algebra \( M_{s(\bar{x})} \), and therefore \( \pi_x \) is an isomorphism of the algebra \( M_{s(\bar{x})} \) and some algebra \( \mathcal{A} \). The algebra \( \mathcal{A} \) is a von Neumann algebra, and its preconjugate space \( \mathcal{A}^* \) is isomorphic to the space \( M_* \cdot s(\bar{x}) \) ([19]). We note now that
\[
 TM_* \cdot s(\bar{x}) \subset M_* \cdot s(\bar{x}).
\] (2.44)
In fact,
\[
 T\mathcal{L}_y \subset \mathcal{L}_y,
\] (2.45)
and therefore, by taking the norm closure, we get
\[
 TS \subset S;
\] (2.46)
by taking now the linear span, we get
\[
 TM_* \cdot s(\bar{x}) \subset M_* \cdot s(\bar{x}).
\] (2.47)

We denote by \( T \) the isomorphic image of the operator \( T \), acting on the space \( \mathcal{A}_* \). Let
\[
 u \in \mathcal{A}_{*+}, \quad u \leq \lambda \bar{x},
\] (2.48)
for some $\lambda > 0$. Then there exists the operator $B \in \mathfrak{A}'$, where $\mathfrak{A}'$ is a commutant of $\mathfrak{A}$, such that

\[(AB\Omega, \Omega) = u(A)\]  \hspace{1cm} (2.49)

for all $A \in \mathfrak{A}$. Note, that from Lemma 2.3,

\[(Tu)(A) = u((T^*) A) = ((T^*) B\Omega, \Omega) = (A((T^*) \lambda B)\Omega, \Omega).\]  \hspace{1cm} (2.50)

Also, from

\[T\mathfrak{A}_{s+} \subset \mathfrak{A}_{s+}, \quad \|Tu\| \leq \|u\|, \quad T\bar{x} = \bar{x},\]  \hspace{1cm} (2.51)

it follows that

\[\|T^* \mathfrak{A}_{s}\| \leq 1, \quad \|T^* A\|_\infty \leq \|A\|_\infty,\]  \hspace{1cm} (2.52)

for all $A \in \mathfrak{A}$. Based on the lemma, we now conclude that

\[\|T^* B\|_\infty \leq \|B\|_\infty; \quad T^* \mathfrak{A}_{+} \subset \mathfrak{A}_{+}; \quad T^* 1 \leq 1,\]  \hspace{1cm} (2.53)

for all $B \in \mathfrak{A}'$.

The space $\mathfrak{A}_{sa}'$ is a pre-Hilbert space of the self-adjoint operators from $\mathfrak{A}'$ with the scalar product

\[(B, C)_\pi = (CB\Omega, \Omega),\]  \hspace{1cm} (2.54)

and using the Kadison inequality [5], we have

\[((T^* B)(T^* B)\Omega, \Omega) \leq (T^* B^2)\Omega, \Omega) \leq (B\Omega, B\Omega),\]  \hspace{1cm} (2.55)

that is, the operator $T^* B$ is a contraction in the pre-Hilbert space $(\mathfrak{A}_{sa}', (\cdot, \cdot)_\pi)$.

We will identify $M_s \cdot s(\bar{x})$ and $\mathfrak{A}_s$. Because $w \in \mathcal{L}$, that is,

\[w \leq \lambda \bar{x}\]  \hspace{1cm} (2.56)

for some $\lambda > 0$, then

\[\bar{w} \leq \lambda \bar{x}\]  \hspace{1cm} (2.57)

as well. Let

\[w(A) = (BA\Omega, \Omega), \quad \bar{w}(A) = (\bar{B}A\Omega, \Omega),\]  \hspace{1cm} (2.58)

for all $A \in \mathfrak{A}$, where $B, \bar{B} \in \mathfrak{A}'$. 
Let now \((a_{n,i})\) be a uniformly regular matrix. Using Lemma 2.3, we will find \(k \in \mathbb{N}\) so that
\[
\left\| \sum_{i} a'_{k,i} T^i w - \bar{w} \right\| \\
= \sup_{A \in \mathfrak{A}} \left\| \left( \sum_{i=1}^{\infty} a'_{k,i} (T^{*'})^i (B - \overline{B}) A \Omega, \Omega \right) \right\| \\
\leq \left( \sum_{i=1}^{\infty} a'_{k,i} (T^{*'})^i (B - \overline{B}) \right) \left( \sum_{i=1}^{\infty} a'_{k,i} (T^{*'})^i (B - \overline{B}) \right)^{1/2} \cdot \sup_{A \in \mathfrak{A}} \left\| A \Omega, A \Omega \right\|^{1/2} \\
\leq (\bar{x}(1))^{1/2} \cdot \left\| \sum_{i=1}^{\infty} a'_{k,i} (T^{*'})^i (B - \overline{B}) \right\|_{(\cdot, \cdot)} < \varepsilon 
\]
(2.59)
for \(k > K\), where by \((a'_{n,i})\), we will denote a matrix with the elements
\[
a'_{n,i} = \left( \sum_{j=N}^{\infty} a_{n,j} \right)^{-1} a_{n,j+N}. \tag{2.60}
\]

It is easy to see that the matrix \((a'_{n,i})\) will be uniformly regular as well.

Then, for a big enough \(k > K\), we will have
\[
\left\| \sum_{i} a_{k,i} T^i x - \bar{x} \right\| \leq \sum_{i \leq N} \left| a_{k,i} \right| \left\| T^i x - \bar{x} \right\| + \sum_{i > N} \left| a_{k,i} \right| \left\| T^i x - T^{i-N} w \right\| \\
+ \sum_{i > N} \left| a_{k,i} \right| \left| 1 - \left( \sum_{j>N} a_{k,j} \right)^{-1} \right| \left\| T^{i-N} w \right\| \\
+ \left\| \sum_{j=1}^{\infty} a_{k,j+N} \cdot \left( \sum_{j>N} a_{k,j} \right)^{-1} T^j w - \bar{w} \right\| \\
+ \left\| \left( \sum_{j\leq N} a_{k,j} \right) \cdot \bar{w} \right\| + \left| \sum_{j\geq N} a_{k,j} \right| \left\| \bar{w} - \bar{x} \right\| \\
\leq \sum_{i \leq N} 2 \cdot \frac{\varepsilon}{N} + \sum_{i > N} \left| a_{k,i} \right| \cdot 4 \varepsilon + \sum_{i > N} \left| a_{k,i} \right| \left( 1 - (1 + \varepsilon)^{-1} \right) \cdot 2 \\
+ \sum_{i \geq N} 2 \cdot \frac{\varepsilon}{N} + (1 + \varepsilon) \cdot 4 \varepsilon \\
\leq 2 \varepsilon + (1 + \varepsilon) \cdot 4 \varepsilon + \varepsilon \cdot 2 \cdot (1 + \varepsilon) + \varepsilon + 2 \varepsilon + (1 + \varepsilon) \cdot 4 \varepsilon \leq 25 \varepsilon.
\]

The arbitrariness of \(\varepsilon\) proves the needed statement. The proof of the theorem is now completed. 

\[
\square
\]

2.2. The case of \(L_1\)-spaces for JBW algebras. The \(L_1\)-spaces for semifinite JBW algebras were considered by [4] (see also [1, 12]), where it has been proven that they do coincide
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with predual spaces. A semifinite JBW algebra $A$ is always represented as

$$A = A_{sp} + A_{ex},$$

(2.62)

where $A_{sp}$ is isometrically isomorphic to operator JW algebra, and $A_{ex}$ is isometrically isomorphic to the space $C(X, M_8^3)$ of all continuous mappings from a Hyperstonean compact topological space $X$ onto the exceptional Jordan algebra $M_8^3$ (see [11]). In this case, when $A$ does not have direct summands of type $I_2$, it is going to be a selfadjoint part of a real von Neumann algebra $R(A_{sp})$, whose complexification

$$R(A_{sp}) + iR(A_{sp}) = M,$$

(2.63)

where $M$ is the enveloping von Neumann algebra of $A_{sp}$, and the predual space of $A$, and the space

$$A_* = (A_{sp})_* + (A_{ex})_*,$$

(2.64)

where $(A_{sp})_*$ is the predual space of $A_{sp}$, and $(A_{ex})_*$ is the predual space of $A_{ex}$ (see, e.g., [2, 11]). The main result for the summand $A_{ex}$ follows immediately from the result for $C(X)$, and the fact that the algebra $M_8^3$ is finite dimensional. So, without the loss of generality, we are interested in the operator case only. But in the operator case, the space $(A_{sp})_*$ is a selfadjoint part of $R_* = (R(A_{sp}))_*$, and

$$M_* = R_* + iR_*$$

(2.65)

(see [2, 16] for details). So, the main result for $R_*$ thus follows from the complex case by restriction of scalars, and we obtain the main result for $L_1$-spaces affiliated to semifinite JBW algebras without direct type $I_2$ summand.

3. Main result: the case of quantum $L_p$-spaces ($1 < p < \infty$)

In the case of a noncommutative $L_p$-space for a semifinite von Neumann algebra, the main result is discussed in [25].

We will discuss here the nonassociative case.

In this section, $A$ denotes a semifinite JBW algebra without direct summands of type $I_2$, with a faithful normal trace $\tau$. By $L_p$, we denote the space of operators affiliated to $A$, and integrated with $p$th power ($p > 1$, see, e.g., [1, 2, 12]). Space $L_q$ (here $q = p/(p - 1)$) is a dual as Banach space to $L_p$ (see [1, 12]). The following theorem is valid.

**Theorem 3.1.** The following conditions for a positive contraction $T$ in the $L_p$ are equivalent.

(i) The sequence $\{T^ix\}_{i=1,2,...}$ converges in $\sigma(L_p,L_q)$ topology for $x \in L_p$.

(ii) For each strictly increasing sequence of natural numbers $\{k_i\}_{i=1,2,...}$

$$n^{-1} \sum_{i<n} T^{k_i}x$$

(3.1)

converges in norm of $L_p$ for all $x \in L_p$. 
(iii) For any uniformly regular matrix \((a_{n,i})\), the sequence \(\{A_n(T)x\}_{n=1,2,...}\)

\[
A_n(T)x = \sum_i a_{n,i}T^i x,
\]

(3.2)

converges in norm of \(L_p\) for all \(x \in L_p\).

For the sake of completeness, we give the following definitions (see, e.g., [25]) and sketch of the proof. Let \(\phi\) be a gauge function

\[
\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+,
\]

(3.3)

with

\[
\phi(0) = 0, \quad \lim_{t \to \infty} \phi(t) = \infty.
\]

(3.4)

Hahn-Banach theorem implies for strictly convex Banach spaces \(E\) with conjugate \(E'\) that there exists a duality map

\[
\Phi : E \rightarrow E',
\]

(3.5)

associated with \(\phi\) such that

\[
\langle x, \Phi(x) \rangle = \|x\| \|\Phi(x)\|, \quad \|\Phi(x)\| = \phi(x).
\]

(3.6)

**Definition 3.2.** Map \(\Phi\) is said to satisfy property \((S)\) uniformly if for every \(\epsilon > 0\), there exists \(\delta(\epsilon) > 0\), such that for any \(x, y \in E\),

\[
| \langle x, \Phi(y) \rangle | < \delta(\epsilon)
\]

(3.7)

implies that

\[
| \langle y, \Phi(x) \rangle | < \epsilon.
\]

(3.8)

**Proof.** From [12, Section 4], it follows that the duality map defined as

\[
\Phi(a) = s|a|^{p-1},
\]

(3.9)

for

\[
a = s|a| \in A
\]

(3.10)

(where \(a = s|a|\) is a polar decomposition of element \(a\)) satisfies the property \((S)\) uniformly. Hence, the statement of the theorem follows from [25, Theorem 3.1].

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Weak convergence of iterates in quantum $L_p$-spaces

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