We introduce a new model of the system of generalized vector quasi-equilibrium problems with upper semicontinuous set-valued maps and present several existence results of a solution for this system of generalized vector quasi-equilibrium problems and its special cases. The results in this paper extend and improve some results in the literature.

1. Introduction and preliminaries

Throughout this paper, we use \( \text{int} A \) and \( \text{Co} A \) to denote the interior and the convex hull of a set \( A \), respectively.

Let \( I \) be an index set. For each \( i \in I \), let \( Y_i, E_i \) be two Hausdorff topological vector spaces. Consider a family of nonempty convex subsets \( \{X_i\}_{i \in I} \) with \( X_i \subseteq E_i \). Let \( X = \prod_{i \in I} X_i \) and \( E = \prod_{i \in I} E_i \). An element of the set \( X^i = \prod_{j \in I \setminus i} X_j \) will be denoted by \( x^i \); therefore, \( x \in X \) will be written as \( x = (x^i, x_i) \in X^i \times X_i \). For each \( i \in I \), let \( D_i : X \to 2^{X_i} \) and \( F_i : X \times X_i \to 2^{Y_i} \) be two set-valued maps, let \( C_i : X \to 2^{Y_i} \) be a set-valued map such that \( C_i(x) \) is a convex, pointed, and closed cone with \( \text{int} C_i(x) \neq \emptyset \) for all \( x \in X \). Then the system of generalized vector quasi-equilibrium problems with set-valued maps (in short, SGVQEP) is to find \( \bar{x} = (\bar{x}^i, \bar{x}_i) \) in \( X \) such that for each \( i \in I \),

\[
\bar{x}_i \in D_i(\bar{x}), \quad F_i(\bar{x}, y_i) \notin \text{int} C_i(\bar{x}), \quad \forall y_i \in D_i(\bar{x}).
\]  

(1.1)

Remark 1.1. In [32], we introduced and studied another type of system of generalized vector quasi-equilibrium problems with lower semicontinuous set-valued maps, which is to find \( \bar{x} = (\bar{x}^i, \bar{x}_i) \) in \( X \) such that for each \( i \in I \),

\[
\bar{x}_i \in D_i(\bar{x}), \quad F_i(\bar{x}, y_i) \subseteq Y_i \setminus (\text{int} C_i(\bar{x})), \quad \forall y_i \in D_i(\bar{x}).
\]  

(1.2)

It is apparent that this problem is different from the SGVQEP.

The following problems are special cases of the SGVQEP.

(1) For each \( i \in I \) and for all \( x \in X_i \), if \( D_i(x) \equiv X_i \), then the SGVQEP reduces to the system of generalized vector equilibrium problems with set-valued maps (in short, SGVEP)
which is to find $\bar{x}$ in $X$ such that for each $i \in I$,

$$F_i(\bar{x}, y_i) \notin - \text{int } C_i(\bar{x}), \quad \forall y_i \in X_i.$$  \hfill (1.3)

This problem was introduced and studied by Ansari et al. in [20].

(2) For each $i \in I$, if the set-valued map $F_i$ is replaced by a vector-valued map $\varphi_i : X \times X_i \to Y_i$, then the SGVQEP reduces to a system of vector quasi-equilibrium problems (in short, SVQEP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in $X$ such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \quad \varphi_i(\bar{x}_i, y_i) \notin - \text{int } C_i(\bar{x}), \quad \forall y_i \in D_i(\bar{x}).$$  \hfill (1.4)

For each $i \in I$, let $\varphi_i : X \to Y_i$ be a vector-valued map, and let $f_i(x, y_i) = \varphi_i(x^i, y_i) - \varphi_i(x)$, then the SVQEP is equivalent to the following Debreu-type equilibrium problem for vector-valued maps (in short, Debreu VEP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in $X$ such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \quad \varphi_i(\bar{x}_i, y_i) - \varphi_i(x) \notin - \text{int } C_i(\bar{x}), \quad \forall y_i \in D_i(\bar{x}).$$  \hfill (1.5)

The SVQEP and the Debreu VEP were introduced and studied by Ansari et al. [21]. And if $D_i(x) \equiv X_i$ for each $i \in I$ and for all $x \in X$, then the Debreu VEP becomes the Nash equilibrium problem for vector-valued maps in [19].

If $Y_i = R$, $C_i(x) = \{y \in R \mid y \leq 0\}$ for each $i \in I$ and for all $x \in X$, and $\varphi_i$ is a scalar real-valued function from $X \times X_i$ to $R$, then the SVQEP reduces to the model of generalized variational inequality problems in [22, page 286] and quasivariational inequalities in [23, page 152-153].

(3) For each $i \in I$, let $\eta_i : X_i \times X_i \to E_i$ be a single-valued map and $T_i : X \to 2^{L(E_i, Y_i)}$ a set-valued map, where $L(E_i, Y_i)$ denotes the space of all continuous linear operators from $E_i$ into $Y_i$. Let $F_i(x, y_i) = \langle T_i(x), \eta_i(y_i, x_i) \rangle = \bigcup_{v_i \in T_i(x)} \langle v_i, \eta_i(y_i, x_i) \rangle$. Then the SGVQEP reduces to the system of generalized vector quasivariational-like inequality problems (in short, SGVQVLIP), which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in $X$ such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \quad \forall y_i \in D_i(\bar{x}), \quad \exists \bar{v}_i \in T_i(\bar{x}) : \langle \bar{v}_i, \eta_i(y_i, x_i) \rangle \notin - \text{int } C_i(\bar{x}).$$  \hfill (1.6)

For each $i \in I$, let $\eta_i(y_i, x_i) = y_i - x_i$ for all $x_i, y_i \in X_i$, then the SGVQVLIP reduces to the system of generalized vector quasivariational inequality problems (in short, SGVQVIP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in $X$ such that for each $i \in I$, $\bar{x}_i \in D_i(\bar{x})$,

$$\forall y_i \in D_i(\bar{x}), \quad \exists \bar{v}_i \in T_i(\bar{x}) : \langle \bar{v}_i, y_i - \bar{x}_i \rangle \notin - \text{int } C_i(\bar{x}).$$  \hfill (1.7)

The SGVQVLIP and the SGVQVIP were introduced by Peng [18].

For each $i \in I$, if $D_i(x) \equiv X_i$ for all $x \in X$, then the SGVQVLIP reduces to the system of generalized vector variational-like inequality problems (in short, SGVVLIP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in $X$ such that for each $i \in I$,

$$\forall y_i \in X_i, \quad \exists \bar{v}_i \in T_i(\bar{x}) : \langle \bar{v}_i, \eta_i(y_i, x_i) \rangle \notin - \text{int } C_i(\bar{x}).$$  \hfill (1.8)

For each $i \in I$, if $D_i(x) \equiv X_i$ for all $x \in X$, and $\eta_i(y_i, x_i) = y_i - x_i$ for all $x_i, y_i \in X_i$, then the SGVQVLIP reduces to the system of generalized vector variational inequality problems...
problems (in short, SGVVIP) which is to find $\tilde{x} = (\tilde{x}_i, \tilde{x}_i)$ in $X$ such that
\[
\forall y_i \in X_i, \exists \tilde{v}_i \in T_i(\tilde{x}) : \langle \tilde{v}_i, y_i - \tilde{x}_i \rangle \notin -\text{int} C_i(\tilde{x}).
\] (1.9)

The SGVVLIP and the SGVVIP were introduced by Ansari et al. in [20].

For each $i \in I$, if $Y_i \equiv Y$ and $C_i(x) \equiv C$ for all $x \in X$, where $C$ is a convex, closed, and pointed cone in $Y$ with $\text{int} C \neq \emptyset$, then the SGVVIP reduces to another kind of system of generalized vector variational inequality problems (in short, II-SGVVIP) which is to find $\tilde{x} = (\tilde{x}_i, \tilde{x}_i)$ in $X$ such that
\[
\forall y_i \in X_i, \exists \tilde{v}_i \in T_i(\tilde{x}) : \langle \tilde{v}_i, y_i - \tilde{x}_i \rangle \notin -\text{int} C.
\] (1.10)

This was studied by Allevi et al. [17].

If $T_i$ is a single-valued function, then the II-SGVVIP reduces to the system of vector variational inequality problems (in short, SVVIP), which is to find $\tilde{x} = (\tilde{x}_i, \tilde{x}_i)$ in $X$ such that
\[
\langle T_i(\tilde{x}), y_i - \tilde{x}_i \rangle \notin -\text{int} C, \quad \forall y_i \in X_i.
\] (1.11)

This was considered by Ansari et al. in [19].

Let $Y = \mathbb{R}$ and $C = \mathbb{R}^+ = \{ r \in \mathbb{R} : r \geq 0 \}$. For each $i \in I$, if $T_i$ is replaced by $f_i : X \rightarrow \mathbb{R}$, then the SVVIP reduces to the system of scalar variational inequality problems which is to find $\tilde{x} = (\tilde{x}_i, \tilde{x}_i)$ in $X$ such that
\[
\langle f_i(\tilde{x}), y_i - \tilde{x}_i \rangle \geq 0, \quad \forall y_i \in X_i.
\] (1.12)

This was considered in [16, 24, 25, 26].

(4) If the index set $I$ is singleton, then the SGVQEP reduces to the generalized vector quasi-equilibrium problem (in short, GVQEP) studied in [12, 15] which contains the generalized vector equilibrium problems studied in [27, 28, 29] as special cases, and the SGVVLIP reduces to the generalized vector variational-like inequality problem studied by Ding and Tarafdar [31].

In this paper, we present some existence results of a solution for the SGVQEP and its special cases with or without $\Phi$-condensing maps.

Now we introduce a new definition as follows.

**Definition 1.2.** Let $I$ be an index set. For each $i \in I$, let $Y_i$ be a topological vector space and $X_i$ a nonempty convex subset of a Hausdorff topological vector space $E_i$, and $F_i : X \times X_i \rightarrow 2^{Y_i}$ be a set-valued map, let $C_i : X \rightarrow 2^{Y_i}$ be a set-valued map such that $C_i(x)$ be a convex and closed cone with $\text{int} C_i(x) \neq \emptyset$ for all $x \in X_i$. $F_i(x, z_i)$ is said to be $C_{i-x}$-0-partially diagonal quasiconvex if for any finite set $\{ y_{i_1}, y_{i_2}, \ldots, y_{i_n} \}$ in $X_i$, and for all $x = (x', x_i) \in X$ with $x_i \in \text{Co} \{ y_{i_1}, y_{i_2}, \ldots, y_{i_n} \}$, there exists some $j (j = 1, 2, \ldots, n)$ such that
\[
F_i(x, y_{i_j}) \notin -\text{int} C_i(x).
\] (1.13)

**Remark 1.3.** (a) If the formula (1.13) is replaced by $F_i(x, y_{i_j}) \subseteq Y_i \setminus (-\text{int} C_i(x))$, then Definition 1.2 becomes [32, Definition 1.1], which is called to be the second type $C_{i-x}$-0-partially diagonal quasiconvexity of $F_i$ in this paper.
Then we have

\[ X \]

It follows that

\[ F \]

Then,

\[ fi \]

Let

\[ ∥•∥ \]

then it is easy to verify that \( fi \) is a single-valued map.

\[ γ \]

Let \( ∥•∥ \) denote the norm on \( Ei \). If we define a norm on \( Ei \) as

\[ ||x|| = \sum_{i=1}^{n} ||x_i||_i, \quad \forall x = (x_1, x_2, ..., x_n) \in E, \tag{1.14} \]

then it is easy to verify that \( ||•|| \) is a norm on \( Ei \). And hence \( Ei \) is also a real normed space. Let \( Ci : X \rightarrow 2^{Y} \) be defined as \( Ci(x) = [||x||, +\infty) \), for all \( x \in X \), and let \( \{e_1, e_2\} \) denote the line segment joining \( e_1 \) and \( e_2 \). Choosing \( p_i \in Ei \), we define a set-valued map \( F : X \times X \rightarrow 2^{E} \) as

\[ F(x, z_i) = \{ \langle u, z_i - x_i \rangle : u \in [−2||x||||z_i||, p_i, −||x||||z_i||, p_i] \}, \quad \forall (x, z_i) \in X \times X, \tag{1.15} \]

Then, \( F \) is \( Ci - 0 \) partially diagonal quasiconvex in the second argument. Otherwise, there exists finite set \( \{z_i, z_{i_2}, ..., z_{i_n}\} \subseteq Xi \), and there is \( x \in X \) with \( x_i = \sum_{j=1}^{n} \alpha_j z_{ij} (\alpha_j \geq 0, \sum_{j=1}^{n} \alpha_j = 1) \) such that for all \( j = 1, 2, ..., n \), \( F(x, z_i) \subseteq \int Ci(x) \). Then for each \( j \), for all \( \lambda_j \in [0, 1] \), we have

\[ \langle \lambda_j (−2||x||||z_i||, p_i), z_i - x_i \rangle < −||x|| \leq 0. \tag{1.16} \]

It follows that

\[ \langle p_i, z_i - x_i \rangle > 0, \quad j = 1, 2, ..., n. \tag{1.17} \]

Then we have

\[ 0 < \sum_{j=1}^{n} \alpha_j \langle p_i, z_i - x_i \rangle = \langle p_i, x_i - x_i \rangle = 0, \tag{1.18} \]

a contradiction.

**Definition 1.4** [18]. For each \( i \in I \), let \( E_i, Y_i \) be two topological vector spaces, \( X_i \) a non-empty and convex subset of \( E_i \), \( Ci : X \rightarrow 2^{Y_i} \) a set-valued map such that \( Ci(x) \) is a closed, pointed, and convex cone for each \( x \in X \). Let \( \eta_i : X_i \times X_i \rightarrow E_i \) be a single-valued map. \( Ti : X \rightarrow 2^{L(Ei, Yi)} \) is said to satisfy the generalized partial \( L-\eta_i \)-condition if and only if for any finite set \( \{y_{i_1}, y_{i_2}, ..., y_{i_n}\} \) in \( X_i \) for all \( \bar{x} = (\bar{x}^i, \bar{x}_i) \) with \( \bar{x}_i = \sum_{j=1}^{n} \alpha_j y_{ij} \), where \( \alpha_j \geq 0 \) and \( \sum_{j=1}^{n} \alpha_j = 1 \), there exists \( \bar{v}_i \in Ti(\bar{x}) \) such that

\[ \langle \bar{v}_i, \sum_{j=1}^{n} \alpha_j \eta_i(y_{ij}, \bar{x}_i) \rangle \notin \int Ci(\bar{x}). \tag{1.19} \]
Then there exists Lemma 1.10 [34].

Definition 1.5 [36]. Let $X$ be a nonempty convex subset of a topological vector space $E$ and $P$ a closed, pointed, and convex cone in a topological vector space $Y$. Let $F : X \to 2^Y$ be a set-valued map. Then $F$ is said to be naturally quasiconvex on $X$, if for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,

\[
F(\lambda x_1 + (1 - \lambda)x_2) \subseteq \text{Co} \{F(x_1) \cup F(x_2)\} - P. \tag{1.20}
\]

Definition 1.6 [8]. Let $E$ be a Hausdorff topological space and $L$ a lattice with least element, denoted by $0$. A map $\Phi : 2^E \to L$ is a measure of noncompactness provided that the following conditions hold for all $M, N \in 2^E$:

(i) $\Phi(M) = 0$ if and only if $M$ is precompact (i.e., it is relatively compact);

(ii) $\Phi(\text{conv}M) = \Phi(M)$; where $\text{conv}M$ denotes the convex closure of $M$;

(iii) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

Definition 1.7 [8]. Let $\Phi : 2^E \to L$ be a measure of noncompactness on $E$ and $D \subseteq E$. A set-valued map $Q : D \to 2^E$ is called $\Phi$-condensing provided that if $M \subseteq D$ with $\Phi(Q(M)) \geq \Phi(M)$, then $M$ is relatively compact.

It is clear that if $Q : D \to 2^E$ is $\Phi$-condensing and $Q^* : D \to 2^E$ satisfies $Q^*(x) \subseteq Q(x)$ for all $x \in D$, then $Q^*$ is also $\Phi$-condensing.

In the next section, we will use the following particular form of a maximal element theorem for a family of set-valued maps due to Deguire et al. [33, Theorem 7] (also see [20, Theorem 1]).

Lemma 1.8. Let $I$ be any index set and $\{X_i\}_{i \in I}$ a family of nonempty convex subsets where each $X_i$ is contained in a Hausdorff topological vector space $E_i$. For each $i \in I$, let $S_i : X_i \to 2^{X_i}$ be a set-valued map such that

(i) $S_i(x)$ is convex,

(ii) for each $x \in X$, $x_i \notin S_i(x),$

(iii) for each $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in $X_i$,

(iv) there exist a nonempty compact subset $N$ of $X$ and a nonempty compact convex subset $B_i$, such that for each $x \in X \cap N$ there exists $i \in I$ satisfying $S_i(x) \cap B_i \neq \emptyset$. Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.

The following lemma is a particular form of [35, Corollary 4].

Lemma 1.9 (maximal element theorem). Let $I$ be any index set and $\{X_i\}_{i \in I}$ a family of nonempty, closed, and convex subsets where each $X_i$ is contained in a locally convex Hausdorff topological vector space $E_i$. For each $i \in I$, let $S_i : X_i \to 2^{X_i}$ be a set-valued map such that

(i) for each $x \in X$, $x_i \notin \text{Co} S_i(x),$

(ii) for each $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in $X_i$,

(iii) the set-valued map $S : X \to 2^X$ defined as $S(x) = \prod_{i \in I} S_i(x)$, for all $x \in X$, is $\Phi$-condensing.

Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.

Lemma 1.10 [34]. Let $X$ and $Y$ be topological spaces and $T : X \to 2^Y$ be an upper semicontinuous set-valued map with compact values. Suppose $\{x_\alpha\}$ is a net in $X$ such that $x_\alpha \to x_0$. 

J. Peng and X. Yang 2413
If \( y_\alpha \in T(x_\alpha) \) for each \( \alpha \), then there is a \( y_0 \in T(x_0) \) and a subset \( \{ y_\beta \} \) of \( \{ y_\alpha \} \) such that \( y_\beta \to y_0 \).

2. Existence results

In this section, we present some existence results of a solution for the SGVQEP and its special cases without or with \( \Phi \)-condensing maps.

**Theorem 2.1.** Let \( I \) be any index set. For each \( i \in I \), let \( Y_i \) be a topological vector space and \( X_i \) a nonempty convex set in a Hausdorff topological vector space \( E_i \), let \( F_i : X \times X_i \to 2^{Y_i} \) be a set-valued map, and let \( D_i : X \to 2^{X_i} \) be a set-valued map such that \( D_i(x) \) is nonempty and convex for all \( x \in X \), \( D_i^{-1}(y_i) \) is open in \( X \) for all \( y_i \in X_i \), and the set \( W_i = \{ x \in X : x_i \in D_i(x) \} \) is closed in \( X \). Let \( C_i : X \to 2^{Y_i} \) be a set-valued map such that \( C_i(x) \) is a convex, pointed, and closed cone with \( \text{int} C_i(x) \neq \emptyset \) for all \( x \in X \). Assume that the following conditions are satisfied:

(i) for each \( i \in I \), \( F_i \) is \( C_{i-x} \)-0-partially diagonal quasiconvex;

(ii) for each \( i \in I \), for each \( y_i \in X_i \), \( F_i(\cdot, y_i) \) is upper semicontinuous on \( X \) with compact values;

(iii) for each \( i \in I \), the set-valued map \( - \text{int} C_i \) has open graph in \( X \times Y_i \);

(iv) there exist a nonempty and compact subset \( N \) of \( X \) and a nonempty, compact, and convex subset \( B_i \) of \( X_i \) for each \( i \in I \) such that for all \( x = (x^i, x_i) \in X \setminus N \), there exist \( i \in I \) and \( y_i \in B_i \) such that \( y_i \in D_i(x) \) and \( F_i(x, y_i) \subseteq - \text{int} C_i(x) \). Then, the solution set of the SGVQEP is nonempty.

**Proof.** For each \( i \in I \), we define a set-valued map \( P_i : X \to 2^{X_i} \) by

\[
P_i(x) = \{ y_i \in X_i : F_i(x, y_i) \subseteq - \text{int} C_i(x) \}, \quad \forall x = (x^i, x_i) \in X.
\]

By hypothesis (i), we have \( x_i \notin \text{Co} P_i(x) \) for each \( i \in I \) and for all \( x \in X \). To see this, suppose, by way of contradiction, that there exist \( i \in I \) and a point \( \tilde{x} = (\tilde{x}^i, \tilde{x}_i) \in X \) such that \( \tilde{x}_i \in \text{Co} P_i(\tilde{x}) \). Then there exist finite points \( y_i, y_{i'}, \ldots, y_{i_n} \in X_i \) and \( \lambda_j \geq 0 \) with \( \sum_{j=1}^n \lambda_j = 1 \) such that \( \tilde{x}_i = \sum_{j=1}^n \lambda_j y_{i_j} \) and \( y_{i_j} \in P_i(\tilde{x}) \) for all \( j = 1, 2, \ldots, n \). That is, \( F_i(\tilde{x}, y_{i_j}) \subseteq - \text{int} C_i(\tilde{x}) \) for \( j = 1, 2, \ldots, n \), which contradicts the hypothesis that \( F_i \) is \( C_{i-x} \)-0-partially diagonal quasiconvex.

By hypothesis (ii), we can prove that for each \( i \in I \), and for each \( y_i \in X_i \), the set \( P_i^{-1}(y_i) = \{ x \in X : F_i(x, y_i) \subseteq - \text{int} C_i(x) \} \) is open, that is, \( P_i \) has open lower sections. Let \( Q_i(y_i) = \{ x \in X : F_i(x, y_i) \not\subseteq - \text{int} C_i(x) \} \) for all \( y_i \in X_i \). We can prove that \( Q_i(y_i) = \{ x \in X : F_i(x, y_i) \not\subseteq - \text{int} C_i(x) \} = X \setminus P_i^{-1}(y_i) \) is closed for all \( y_i \in X_i \). In fact, consider a net \( x_t \in X \) such that \( x_t \to x \in X \). For every \( t \) there exists \( u_t \in F_i(x_t, y_i) \) with \( u_t \notin - \text{int} C_i(x_t) \). Since \( F_i(\cdot, y_i) \) is upper semicontinuous with compact values, by Lemma 1.10, it suffices to find a subset \( \{ u_t \} \) which converges to some \( u \in F_i(x, y_i) \); then \( (x_t, u_t) \to (x, u) \). It follows that \( u \notin - \text{int} C_i(x) \) since the graph of \( - \text{int} C_i(\cdot) \) is open, hence, \( x \in Q_i(y_i) \).

So we have proved that \( P_i \) has open lower sections. Then by [4, Lemma 5], we know that the set-valued map \( \text{Co} P_i : X \to 2^{X_i} \) defined by \( \text{Co} P_i(x) = \text{Co}(P_i(x)) \), for all \( x \in X \) also has open lower sections. For each \( i \in I \), we also define another set-valued map \( S_i : X \to 2^{X_i} \)
Then, for each $i \in I$, it is clear that $S_i(x)$ is convex for all $x \in X$ and $x_i \notin S_i(x)$. Since

$$\forall i \in I, \forall y_i \in X_i, \quad S_i^{-1}(y_i)$$

$$= \{ x \in X : y_i \in S_i(x) \} = \{ x \in W_i : y_i \in D_i(x) \} \cup \{ x \in X \setminus W_i : y_i \in D_i(x) \} = (W_i \cap D_i^{-1}(y_i)) \cup (X \setminus W_i) \cap D_i^{-1}(y_i)$$

$$= [(W_i \cap D_i^{-1}(y_i)) \cap \text{Co} P_i^{-1}(y_i)] \cup ((X \setminus W_i) \cap D_i^{-1}(y_i))$$

$$= [D_i^{-1}(y_i) \cap \text{Co} P_i^{-1}(y_i)] \cup (X \setminus W_i) \cap D_i^{-1}(y_i) = (D_i^{-1}(y_i) \cap \text{Co} P_i^{-1}(y_i) ) \cup (X \setminus W_i) \cap D_i^{-1}(y_i)),$$

(2.3)

and $D_i^{-1}(y_i)$, $\text{Co} P_i^{-1}(y_i)$ and $X \setminus W_i$ are open in $X$, we have $S_i^{-1}(y_i)$ open in $X$.

By assumption (iv), we know that the condition (iv) of Lemma 1.8 holds. Hence, by Lemma 1.8, there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$, for all $i \in I$. Since for all $i \in I$ and for all $x \in X$, $D_i(x)$ is nonempty, we have $\bar{x} \in W_i$, and $D_i(\bar{x}) \cap \text{Co} P_i(\bar{x}) = \emptyset$, for all $i \in I$. This implies $\bar{x} \in D_i(\bar{x})$ and $D_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$, for all $i \in I$. Therefore, for all $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \quad F_i(\bar{x}, y_i) \not\subseteq - \text{int} C_i(\bar{x}), \quad \forall y_i \in D_i(\bar{x}).$$

(2.4)

That is, the solution set of the SGVQEP is nonempty. This completes the proof. \hfill \Box

**Corollary 2.2.** If we replace, in Theorem 2.1, condition (i) by the following conditions:

(a) for each $i \in I$, for all $x \in X$, $y_i \mapsto F_i(x, y_i)$ is natural $P_i$-quasiconvex;

(b) for each $i \in I$, for all $x \in X$, $F_i(x, x_i) \not\subseteq - \text{int} C_i(x)$;

then the conclusion of Theorem 2.1 still holds, that is, the solution set of the SGVQEP is nonempty.

**Proof.** For each $i \in I$, we define a set-valued map $P_i : X \to 2^X$ by

$$P_i(x) = \{ y_i \in X_i : F_i(x, y_i) \not\subseteq - \text{int} C_i(x) \}, \quad \forall x = (x^j, x_i) \in X.$$  

(2.5)

Then $P_i(x)$ is convex for each $i \in I$ and for all $x \in X$.

To prove it, we fix arbitrary $i \in I$ and $x \in X$. Let $y_{i_1}, y_{i_2} \in P_i(x)$ and $\lambda \in [0, 1]$, then we have

$$F_i(x, y_{i_j}) \not\subseteq - \text{int} C_i(x), \quad j = 1, 2.$$

(2.6)
Since $\Pi_{i}$ is natural $P_{i}$-quasiconvex,

$$F_{i}(\lambda y_{i} + (1 - \lambda) y_{i}) \subseteq \text{Co} \{ F_{i}(x, y_{i}) \cup F_{i}(x, y_{i}) \} - P_{i} \subseteq - \text{int} C_{i}(x) - P_{i} \subseteq - \text{int} C_{i}(x).$$

(2.8)

Hence, $\lambda y_{i} + (1 - \lambda) y_{i} \in P_{i}(x)$ and so $P_{i}(x)$ is convex.

We show that $F_{i}$ is $C_{i} - x$-0-partially diagonal quasiconvex for each $i \in I$. If not, then there exists a finite subset $\{ y_{i1}, y_{i2}, \ldots, y_{in} \}$ in $X_{i}$, and a point $x = (x', x_{i}) \in X$ with $x_{i} \in \text{Co} \{ y_{i1}, y_{i2}, \ldots, y_{in} \}$ such that

$$F_{i}(x, y_{ij}) \subseteq - \text{int} C_{i}(x), \quad j = 1, 2, \ldots, n.$$  

(2.9)

Since $P_{i}(x) = \{ y_{i} \in X : F_{i}(x, y_{i}) \subseteq - \text{int} C_{i}(x) \}$ is a convex set, $x_{i} \in P_{i}(x)$, that is, $F_{i}(x, x_{i}) \subseteq - \text{int} C_{i}(x)$, which contradicts the condition (b).

By Theorem 2.1, we know the conclusion of Corollary 2.2 holds. This completes the proof.

**Corollary 2.3.** If we replace, in Theorem 2.1, condition (i) by the following conditions:

(a) for each $i \in I$, $F_{i}$ is $C_{i}(x)$-quasiconvex-like;

(b) for each $i \in I$, for all $x \in X$, $F_{i}(x, x_{i}) \notin - \text{int} C_{i}(x)$;

then the conclusion of Theorem 2.1 still holds, that is, the solution set of the (SGVQEP) is nonempty.

**Proof.** For each $i \in I$, let the set-valued map $P_{i} : X \rightarrow 2^{X_{i}}$ be defined the same as that in the proof of Corollary 2.2. Then by the assumption (a) and the proof of [20, Theorem 3], it is easy to see that for all $i \in I$ and for all $x \in X$, $P_{i}(x)$ is convex. The conclusion of Corollary 2.3 follows from the Corollary 2.2. This completes the proof.

**Remark 2.4.** If $D_{i}(x) = X_{i}$ for all $x \in X$ and for all $i \in I$, then by Corollary 2.3, we recover [20, Theorem 3]. Hence, Theorem 2.1 generalizes [20, Theorem 3] from the system of generalized vector equilibrium problems to the system of generalized vector quasi-equilibrium problems with more general convex conditions.

**Remark 2.5.** If $F_{i}$ is replaced by a single-valued map $f_{i} : X \times X_{i} \rightarrow Y_{i}$, then by [21, Remark 5(1) and Corollary 2.2], we can obtain [21, Theorem 2]. Hence, Theorem 2.1 generalizes [21, Theorem 2] from single-valued case to set-valued case with more general convex conditions.

**Remark 2.6.** For each $i \in I$, let the set-valued map $F_{i}$ be replaced by a single-valued map $\phi_{i} : X \times X_{i} \rightarrow Y_{i}$, and let $C_{i}(x) = C_{i}$ and $D_{i}(x) = X_{i}$ for all $x \in X$. By Theorem 2.1, we have the existence result of the SVEP as follows.

Let $I$ be any index set. For each $i \in I$, let $Y_{i}$ be a topological vector space, $X_{i}$ a nonempty compact convex set in a Hausdorff topological vector space $E_{i}$, and $\phi_{i} : X \times X_{i} \rightarrow Y_{i}$ a single-valued map. Let $C_{i} \subseteq Y_{i}$ be a convex, pointed, and closed cone with $\text{int} C_{i}(x) \neq \emptyset$.
for all \( x \in X \). Assume that the following conditions are satisfied
(i) for each \( i \in I \), \( \varphi_i \) is \( C_0 \)-partially diagonal quasiconvex;
(ii) for each \( i \in I \), for each \( y_i \in X_i \), \( \varphi_i(\cdot, y_i) \) is continuous on \( X \).

Then, there exists \( \bar{x} = (\bar{x}_i, \bar{x}_i) \) in \( X \) such that for each \( i \in I \), \( \bar{x}_i \in X_i \) and \( \varphi_i(\bar{x}, y_i) \notin -\text{int} C_i \), for all \( y_i \in X_i \). That is, the solution set of the SVEP is nonempty.

And if the condition (i) of the above result is replaced by \( \varphi_i(x, x_i) = 0 \) for all \( x \in X \) and the map \( y_i \mapsto \varphi_i(x, y_i) \) is \( C_1 \)-quasiconvex, then we recover [19, Theorem 2.1]. If \( \varphi_i(x, x_i) = 0 \) for all \( x \in X \) and the map \( y_i \mapsto \varphi_i(x, y_i) \) is \( C_1 \)-quasiconvex, then \( \varphi_i \) must be \( C_1 \)-partially diagonal quasiconvex. Hence, Theorem 2.1 generalizes [19, Theorem 2.1] in several aspects.

Now we establish an existence result for a solution to the SGVQEP involving \( \Phi \)-condensing maps.

**Theorem 2.7.** Let \( I \) be any index set. For each \( i \in I \), let \( Y_i \) be a topological vector space and \( X_i \) a nonempty, closed, and convex set in a locally convex Hausdorff topological vector space \( E_i \). Let \( F_i : X \times X_i \to 2^{Y_i} \) be a set-valued map and \( D_i : X \to 2^{X_i} \) a set-valued map such that for all \( x \in X \), \( D_i(x) \) is a nonempty convex set, \( \Phi \) is nonempty, and the conditions (i), (ii), and (iii) of Theorem 2.1 hold. Then the solution set of SGVQEP is nonempty.

**Proof.** In view of Lemma 1.9, it is sufficient to show that the set-valued map \( S : X \to 2^X \) defined as \( S(x) = \prod_{i \in I} S_i(x) \) for all \( x \in K \) is \( \Phi \)-condensing, where \( S_i \)'s are the same as defined in the proof of Theorem 2.1. By the definition of \( S_i \), \( S_i(x) \subseteq D_i(x) \) for all \( x \in X \) and for each \( i \in I \), and therefore \( S(x) \subseteq D(x) \) for all \( x \in X \). Since \( D \) is \( \Phi \)-condensing, so is \( S \). This completes the proof.

**Remark 2.8.** Theorem 2.7 generalizes [21, Theorem 3] from single-valued case to set-valued case with more general convex conditions.

**Remark 2.9.** If we replace, in Theorem 2.7, condition (i) and (ii) of Theorem 2.1, respectively, by the following conditions:
(a) for each \( i \in I \), \( F_i \) is the second-type \( C_{1-\infty} \)-partially diagonal quasiconvex;
(b) for each \( i \in I \), for each \( z_i \in X_i \), \( F_i(\cdot, z_i) \) is lower semicontinuous on \( X \);
then there exists \( \bar{x} = (\bar{x}_i, \bar{x}_i) \) in \( X \) such that for each \( i \in I \),
\[
\bar{x}_i \in D_i(\bar{x}), \quad F_i(\bar{x}, z_i) \subseteq Y_i \setminus (-\text{int} C_i(\bar{x})), \quad \forall z_i \in D_i(\bar{x}).
\] (2.10)

This is [32, Theorem 2.1].

We will prove the existence of solutions for the SGVQVLIP as follows.

**Corollary 2.10.** Let \( I \) be any index set. For each \( i \in I \), let \( Y_i \) be a real Hausdorff topological vector space and \( X_i \) a nonempty convex set in a real Hausdorff topological vector space \( E_i \). Let the set-valued maps \( D_i : X \to 2^{X_i} \), \( C_i : X \to 2^{Y_i} \), and the set \( W_i = \{x \in X : x_i \in D_i(x)\} \) be as
in Theorem 2.1, and let $L(E_i, Y_i)$ be equipped with the $\sigma$-topology. Assume that the following conditions are satisfied:

(i) for each $i \in I$, $T_i : X \to 2^{L(E_i, Y_i)}$ is an upper semicontinuous set-valued map with nonempty compact values, $\eta_i : X_i \times X_i \to E_i$ is continuous with respect to the second argument, such that $T_i$ satisfies the generalized partial $L, \eta_i$-condition;

(ii) there exist a nonempty and compact subset $N$ of $X$ and a nonempty, compact, and convex subset $B_i$ of $X_i$ for each $i \in I$ such that for all $x = (x^i, x_i) \in X \setminus N$, there exist $i \in I$ and $\tilde{y}_i \in B_i$, such that $\tilde{y}_i \in D_i(x)$ and $(v_i, \eta_i(\tilde{y}_i, x_i)) \in -\text{int} C_i(x)$, for all $v_i \in T_i(x)$.

Then the SGVQVLI has a solution $\bar{x} \in X$.

Proof. For each $i \in I$, define a set-valued map $P_i : X \to 2^{X_i}$ by

$$P_i(x) = \{y_i \in X_i : \langle T_i(x), \eta_i(y_i, x_i) \rangle \subseteq -\text{int} C_i(x)\}$$

$$= \{y_i \in X_i : \langle v_i, \eta_i(y_i, x_i) \rangle \subseteq -\text{int} C_i(x), \forall v_i \in T_i(x)\}, \quad \forall x \in X.$$  \hspace{1cm} (2.11)

From the proof of [18, Theorem 3.1], we know that $x_i \notin \text{Co}(P_i(x))$ for all $x = (x^i, x_i) \in X$ and the set $P_i^{-1}(y_i) = \{x \in X : \langle T_i(x), \eta_i(y_i, x_i) \rangle \subseteq -\text{int} C_i(x)\}$ is open for each $i \in I$ and for each $y_i \in X_i$. That is, $P_i$ has open lower sections in $X$.

The remainder of the proof is same as that in the proof of Theorem 2.1. \hfill $\Box$

In view of Lemma 1.9 and the proof of Corollary 2.10, it is easy to obtain an existence result of a solution to SGVQVLI as follows.

**Corollary 2.11.** Let $I$ be any index set. For each $i \in I$, let $Y_i$ be a real Hausdorff topological vector space and $X_i$ a nonempty, closed, and convex set in a real locally convex Hausdorff topological vector space $E_i$, let the set-valued maps $D_i : X \to 2^{X_i}$, $D : X \to 2^X$, $C_i : X \to 2^{Y_i}$ and the set $W_i = \{x \in X : x_i \in D_i(x)\}$ be the same as those in Theorem 2.7. Assume that condition (i) of Corollary 2.10 holds. Then the solution set of SGVQVLI is nonempty.

**Remark 2.12.** Let $I$ be an index set and let $I$ be countable. For each $i \in I$, let $Y_i$ be a real Hausdorff topological vector space, let $X_i$ be a nonempty, compact, convex, and metrizable set in a real locally convex Hausdorff topological vector space $E_i$, let $D_i : X \to 2^{X_i}$ be an upper semicontinuous set-valued mapping with nonempty convex closed values and open lower sections, let $C_i : X \to 2^{Y_i}$ be a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone with $\text{int} C_i(x) \neq \emptyset$ for each $x \in X$, and the set-valued map $M_i = Y_i \setminus (\text{-int} C_i) : X \to 2^{Y_i}$ is upper semicontinuous, and let $L(E_i, Y_i)$ be equipped with the $\sigma$-topology. Suppose that the condition (i) of Corollary 2.10 satisfied, then the SGVQVLI has a solution $\bar{x} \in X$.

The above is [18, Theorem 3.1]. It is easy to see that Corollaries 2.10 and 2.11 generalize [18, Theorem 3.1] without compactness and metrizability of $X_i$ and with weaker conditions of $D_i$. Corollaries 2.10 and 2.11 also generalize [20, Corollaries 2 and 3] and [19, Theorems 3.1 and 3.2] in several aspects.

**Remark 2.13.** By the above results, it is easy to obtain the existence results of a solution for the other special cases of the SGVQEP and they are omitted here.
Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant no. 10171118), the Education Committee Project Research Foundation of Chongqing (Grant no. 030801), and the Natural Science Foundation of Chongqing (no. 8409). The authors would like to express their thanks to the referees for their comments and suggestions that improved the presentation of this manuscript.

References

System of generalized vector quasi-equilibrium problems


Jianwen Peng: College of Mathematics and Computer Science, Chongqing Normal University, Chongqing 400047, China
E-mail address: jwpeng6@yahoo.com.cn

Xinmin Yang: College of Mathematics and Computer Science, Chongqing Normal University, Chongqing 400047, China
E-mail address: maxmyang@polyu.edu.cn
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Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

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<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>June 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>September 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>December 1, 2009</td>
</tr>
</tbody>
</table>

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