ANOTHER SIMPLE PROOF OF THE QUINTUPLE PRODUCT IDENTITY

HEI-CHI Chan

Received 14 December 2004 and in revised form 15 May 2005

We give a simple proof of the well-known quintuple product identity. The strategy of our proof is similar to a proof of Jacobi (ascribed to him by Enneper) for the triple product identity.

1. Introduction

The well-known quintuple product identity can be stated as follows. For \( z \neq 0 \) and \( |q| < 1 \),

\[
 f(z,q) := \prod_{n=0}^{\infty} \left( 1 - q^{2n+2} \right) \left( 1 - zq^{2n+1} \right) \left( 1 - \frac{1}{z} q^{2n+1} \right) \left( 1 - z^2 q^{4n} \right) \left( 1 - \frac{1}{z^2} q^{4n+4} \right)
\]

\[
 = \sum_{n=-\infty}^{\infty} q^{3n^2+n} \left( z^{3n} q^{-3n} - z^{-3n-1} q^{3n+1} \right).
\]

(1.1)

The quintuple identity has a long history and, as Berndt [5] points out, it is difficult to assign priority to it. It seems that a proof of the identity was first published in H. A. Schwartz’s book in 1893 [19]. Watson gave a proof in 1929 in his work on the Rogers-Ramanujan continued fractions [20]. Since then, various proofs have appeared. To name a few, Carlitz and Subbarao gave a simple proof in [8]; Andrews [2] gave a proof involving basic hypergeometric functions; Blecksmith, Brillhart, and Gerst [7] pointed out that the quintuple identity is a special case of their theorem; and Evans [11] gave a short and elegant proof by using complex function theory. For updated history up to the late 80s and early 90s, see Hirschhorn [15] (in which the author also gave a beautiful generalization of the quintuple identity) and Berndt [5] (in which the author also gave a proof that ties the quintuple identity to the larger framework of the work of Ramanujan on \( q \)-series and theta functions; see also [1]). Since the early 90s, several authors gave different new proofs of the quintuple identity; see [6, 13, 12, 17]. See also Cooper’s papers [9, 10] for the connections between the quintuple product identity and Macdonald identities [18]. Quite recently, Kongsiriwong and Liu [16] gave an interesting proof that makes use of the cube root of unity.
Another simple proof of the quintuple product identity

Our proof below is similar to the proof of the triple product identity by Jacobi (ascribed to him by Enneper; see the book by Hardy and Wright [14]). First, we set \( f(z, q) = \sum a_n z^n \). Then, by considering the symmetry of \( f(z, q) \) as an infinite product, we relate all \( a_n \) to a single coefficient \( a_0 \). All we need to do is to evaluate \( a_0 \). This is achieved by comparing \( f(i, q) \) and \( f(-q^4, q^4) \).

### 2. Proof of the identity

The first step of our proof is pretty standard, for example, see [16] or [4]. Set

\[
f(z, q) = \sum_{n=\infty}^{\infty} a_n z^n. \tag{2.1}
\]

From the definition of \( f(z, q) \), one can show that

\[
f(z, q) = qz^3 f(zq^2, q), \quad f(z, q) = -z^2 f\left(\frac{1}{z}, q\right). \tag{2.2}
\]

The first equality implies that for each \( n \),

\[
a_{3n} = a_0 q^{3n^2 - 2n}, \quad a_{3n+1} = a_1 q^{3n^2}, \quad a_{3n+2} = a_2 q^{3n^2 + 2n}, \tag{2.3}
\]

whereas the second equality implies that \( a_2 = -a_0 \) and \( a_1 = 0 \). By putting all these together, we have

\[
f(z, q) = a_0(q) \sum_{n=\infty}^{\infty} q^{3n^2 + n} (z^{3n} q^{-3n} - z^{-3n-1} q^{3n+1}). \tag{2.4}
\]

Comparing (2.4) to (1.1) shows that all we need to do is to prove that \( a_0(q) = 1 \). Note that \( a_0(0) = 1 \).

We can also write (2.4) in the following forms (which will be useful later):

\[
f(z, q) = a_0(q) \sum_{n=\infty}^{\infty} q^{3n^2 - 2n} \left( z^{3n} - \frac{1}{z^{3n-2}} \right) \tag{2.5a}
\]

\[
= a_0(q) \sum_{n=\infty}^{\infty} q^{3n^2 + n} \left( \left(\frac{z}{q}\right)^{3n} - \left(\frac{q}{z}\right)^{3n+1} \right). \tag{2.5b}
\]

To obtain (2.5a), we let \( n \to n - 1 \) in the second sum on the right-hand side of (2.4). Equation (2.5b) is simply another way of writing (2.4).

By putting \( z = i \) in (2.5a), we have, on the one hand,

\[
f(i, q) = a_0(q) \sum_{n=\infty}^{\infty} q^{3n^2 - 2n} \left( i^{3n} - \frac{1}{i^{3n-2}} \right) = 2a_0(q) \sum_{n=\infty}^{\infty} q^{12n^2 - 4n} (-1)^n. \tag{2.6}
\]

Note that, in the second equality, we have used the fact that

\[
i^{3n} - \frac{1}{i^{3n-2}} = 2 \cos \frac{3n \pi}{2} = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 2(-1)^{n/2}, & \text{if } n \text{ is even}. \end{cases} \tag{2.7}
\]
On the other hand, let us evaluate $f(i, q)$ as an infinite product:

$$f(i, q) = 2 \prod_{n=1}^{\infty} (1 - q^{2n}) \left(1 - iq^{2n-1} \right) \left(1 + iq^{2n-1} \right) \left(1 + q^{4n} \right)^2$$

$$= 2 \prod_{n=1}^{\infty} (1 - q^{2n}) \left(1 + q^{4n-2} \right) \left(1 + q^{4n} \right)^2$$

$$= 2 \prod_{n=1}^{\infty} (1 - q^{2n}) \left(1 + q^{2n} \right) \left(1 + q^{4n} \right)$$

$$= 2 \prod_{n=1}^{\infty} (1 - q^{4n}) \left(1 + q^{4n} \right)$$

$$= 2 \prod_{n=1}^{\infty} (1 - q^{8n}).$$

Note that we have used the fact that $\prod (1 + q^{4n-2})(1 + q^{4n}) = \prod (1 + q^{2n})$ to derive the third equality.

By putting (2.6) and (2.8) together, we arrive at

$$\prod_{n=1}^{\infty} (1 - q^{8n}) = a_0(q) \sum_{n=-\infty}^{\infty} q^{12n^2-4n} (-1)^n. \quad (2.9)$$

Note that, at this stage, if we appeal to Euler’s pentagonal number theorem (with $q$ replaced by $q^8$) [4], we have

$$\prod_{n=1}^{\infty} (1 - q^{8n}) = \sum_{n=-\infty}^{\infty} q^{12n^2-4n} (-1)^n. \quad (2.10)$$

Compared with (2.9), we see that $a_0(q) = 1$. Alternatively, we can find $a_0(q)$ by evaluating $f(z, q)$ in a different way.

Precisely, let us evaluate $f(-q^4, q^4)$. By (2.5b), we have

$$f(-q^4, q^4) = a_0(q^4) \sum_{n=-\infty}^{\infty} q^{12n^2+4n} \left((-1)^{3n} - (-1)^{3n+1} \right)$$

$$= 2a_0(q^4) \sum_{n=-\infty}^{\infty} q^{12n^2+4n} (-1)^n$$

$$= 2a_0(q^4) \sum_{n=-\infty}^{\infty} q^{12n^2-4n} (-1)^n. \quad (2.11)$$

For the second equality, we have used the fact that $(-1)^{3n} - (-1)^{3n+1} = 2(-1)^n$. For the last equality, we let $n \to -n$ in the second line.

Again, evaluating $f(-q^4, q^4)$ as an infinite product gives

$$f(-q^4, q^4) = 2 \prod_{n=1}^{\infty} (1 - q^{8n}) \left( \prod_{k=1}^{\infty} (1 + q^{8k}) (1 - q^{16k-8}) \right)^2 = 2 \prod_{n=1}^{\infty} (1 - q^{8n}). \quad (2.12)$$
Another simple proof of the quintuple product identity

The second equality is obtained by direct computation, similar to the derivation of (2.8). Alternatively, it follows from an identity due to Euler (e.g., see [3, page 60]) that

\[ \prod_{k=1}^{\infty} (1 + q^{8k})(1 - q^{16k-8}) = 1. \]  

(2.13)

By putting together (2.11) and (2.12), we have

\[ \prod_{n=1}^{\infty} (1 - q^{8n}) = a_0(q^4) \sum_{n=-\infty}^{\infty} q^{12n^2-4n}(-1)^n. \]  

(2.14)

Finally, by comparing (2.9) and (2.14), we conclude that \( a_0(q) = a_0(q^4) \). This implies that

\[ a_0(q) = a_0(q^4) = a_0(q^{16}) = \cdots = a_0(q^{4k}) = \cdots = a_0(0) = 1 \]  

(2.15)

and (1.1) is proven.

We remark that the evaluation of \( f(-q^4, q^4) \) above also gives a simple proof of Euler’s pentagonal number theorem.

Acknowledgments

I would like to thank Douglas Woken for his support and encouragement and Shaun Cooper for his valuable comments and encouragement. Also, I would like to thank the referees, whose comments were extremely valuable and encouraging.

References


Hei-Chi Chan: Mathematical Science Program, University of Illinois at Springfield, Springfield, IL 62703-5407, USA

E-mail address: chan.hei-chi@uis.edu
Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>February 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>August 1, 2009</td>
</tr>
</tbody>
</table>

**Guest Editors**

**José Roberto Castilho Piqueira,** Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

**Elbert E. Neher Macau,** Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

**Celso Grebogi,** Department of Physics, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk