

# MATRIX TRANSFORMATIONS AND WALSH'S EQUICONVERGENCE THEOREM

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In 1977, Jacob defines  $G_\alpha$ , for any  $0 \leq \alpha < \infty$ , as the set of all complex sequences  $x$  such that  $\limsup |x_k|^{1/k} \leq \alpha$ . In this paper, we apply  $G_u - G_v$  matrix transformation on the sequences of operators given in the famous Walsh's equiconvergence theorem, where we have that the difference of two sequences of operators converges to zero in a disk. We show that the  $G_u - G_v$  matrix transformation of the difference converges to zero in an arbitrarily large disk. Also, we give examples of such matrices.

## 1. Introduction

If  $x = (x_k)$  is a complex number sequence and  $A = [a_{nk}]$  is an infinite matrix, then  $Ax$  is the sequence whose  $n$ th term is given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k. \quad (1.1)$$

The matrix  $A$  is called  $X - Y$  matrix if  $Ax$  is in the set  $Y$  whenever  $x$  is in  $X$ . For  $0 \leq \alpha < \infty$ , let  $G_\alpha = \{x : \limsup |x_k|^{1/k} \leq \alpha\}$ . For various values of  $\alpha$ , this sequence space has been studied extensively by many authors (see [3, 8, 9]). In particular, Jacob [5, page 186] proves the following result.

**THEOREM 1.1.** *An infinite matrix  $A$  is a  $G_u - G_v$  matrix if and only if for each number  $w$  such that  $0 < w < 1/v$ , there exist numbers  $B$  and  $s$  such that  $0 < s < 1/u$  and*

$$|a_{nk}| w^n \leq B s^k \quad (1.2)$$

for all  $n$  and  $k$ .

## 2. Preliminaries

Let  $f$  be an analytic function in the disk  $\mathbf{D}_R = \{z \in \mathbf{C} : |z| < R\}$  for some  $R > 1$ . If  $f(z)$  has the Taylor series expansion  $f(z) = \sum_{k=0}^{\infty} a_k z^k$ , then for each positive integer  $n$ , let

$$S_n(z; f) = \sum_{k=0}^n a_k z^k \quad (2.1)$$

be the  $n$ th partial sum of  $f(z)$ . Also, let  $L_n(z; f)$  denote the unique Lagrange interpolation polynomial of degree at most  $n$  which interpolates  $f(z)$  in the  $(n + 1)$ st roots of unity, that is,

$$L_n(\omega^k; f) = f(\omega^k) \quad \text{for } k = 0, 1, \dots, n, \tag{2.2}$$

where  $\omega = e^{2\pi i/(n+1)}$ . Then the well-known Walsh's equiconvergence theorem [10] states that

$$\lim_{n \rightarrow \infty} [L_n(z; f) - S_n(z; f)] = 0 \quad \text{for } z \in D_{R^2}, \tag{2.3}$$

the convergence being uniform and geometric on any closed subdisk of  $D_{R^2}$ .

This theorem has been extended in various ways by several authors. In [7], Price used certain arithmetical means and in [6], Lou used commutators of interpolation operators to enlarge the disk  $D_{R^2}$  of equiconvergence. In [1], Brück applied certain summability methods to the difference  $L_n - S_n$  in order to enlarge the disk  $D_{R^2}$ . Also, in [2], the authors extended the disk of convergence by substituting the  $n$ th partial sum  $S_n(z; f)$  by polynomials

$$Q_{l,n}(z; f) = \sum_{k=0}^n \sum_{j=0}^{l-1} a_{k+j(n+1)} z^k, \tag{2.4}$$

where  $l$  is a fixed positive integer.

Our aim is to apply a certain class of matrices to  $L_n$  and  $S_n$  and enlarge the disk  $D_{R^2}$  of Walsh's equiconvergence to  $D_\rho$  for any  $\rho > R^2$ .

Throughout this paper, we let  $\Gamma$  be any circle  $|t| = r$  with  $1 < r < R$ . For any function  $f$  analytic in  $D_R$ , we have by Cauchy integral formula

$$\begin{aligned} L_n(z; f) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{t^{n+1} - z^{n+1}}{t^{n+1} - 1} \frac{f(t)}{t - z} dt \\ &= \frac{1}{2\pi i} \int_{\Gamma} \left[ 1 - \left(\frac{z}{t}\right)^{n+1} \right] \frac{t^{n+1}}{t^{n+1} - 1} \frac{f(t)}{t - z} dt. \end{aligned} \tag{2.5}$$

Since  $|t| = r > 1$ , we get that

$$L_n(z; f) = \frac{1}{2\pi i} \int_{\Gamma} \left[ 1 - \left(\frac{z}{t}\right)^{n+1} \right] \sum_{j=0}^{\infty} \left(\frac{1}{t^{n+1}}\right)^j \frac{f(t)}{t - z} dt. \tag{2.6}$$

Interchanging the summation and the integral, we see that

$$\begin{aligned} L_n(z; f) &= \frac{1}{2\pi i} \int_{\Gamma} \left[ 1 - \left(\frac{z}{t}\right)^{n+1} \right] \frac{f(t)}{t - z} dt \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma} \left[ 1 - \left(\frac{z}{t}\right)^{n+1} \right] \sum_{j=1}^{\infty} \frac{1}{t^{j(n+1)}} \frac{f(t)}{t - z} dt. \end{aligned} \tag{2.7}$$

Similarly, we can express  $S_n(z; f)$  as follows:

$$S_n(z; f) = \frac{1}{2\pi i} \int_{\Gamma} \left[ 1 - \left( \frac{z}{t} \right)^{n+1} \right] \frac{f(t)}{t-z} dt. \tag{2.8}$$

Therefore,

$$L_n(z; f) = S_n(z; f) + \frac{1}{2\pi i} \int_{\Gamma} \left[ 1 - \left( \frac{z}{t} \right)^{n+1} \right] \sum_{j=1}^{\infty} \frac{1}{t^{j(n+1)}} \frac{f(t)}{t-z} dt. \tag{2.9}$$

For simplicity, we will denote  $L_n(z; f)$  by  $L_n(z)$  and  $S_n(z; f)$  by  $S_n(z)$ .

**3. Main result**

For  $1 < r < R$ , choose  $\rho > R^2$ ,  $u > \rho/r$ , and  $0 < v < 1$ . Let  $A$  be a  $G_u - G_v$  matrix. Therefore, by Theorem 1.1, for any  $w$  such that  $1 < w < 1/v$ , there exist numbers  $B$  and  $s$  such that  $0 < s < 1/u$  and

$$|a_{nk}| w^n \leq B s^k \quad \forall n, k. \tag{3.1}$$

Consequently, the matrix  $A$  is a summability matrix which transforms null sequences into null sequences. This is because

$$\sum_{k=0}^{\infty} |a_{nk}| \leq \frac{B}{(1-s)w^n} \leq \frac{B}{(1-s)}, \tag{3.2}$$

$$\sum_{k=0}^{\infty} a_{nk} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad a_{nk} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We define  $\lambda_n(z) = \sum_{k=0}^{\infty} a_{nk} L_k(z)$  and  $\sigma_n(z) = \sum_{k=0}^{\infty} a_{nk} S_k(z)$ . Then, for  $|z| < \rho$ , we obtain that

$$\begin{aligned} \sigma_n(z) &= \sum_{k=0}^{\infty} a_{nk} \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z} \left[ 1 - \left( \frac{z}{t} \right)^{k+1} \right] dt \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z} \left[ \sum_{k=0}^{\infty} a_{nk} - \left( \frac{z}{t} \right) \sum_{k=0}^{\infty} a_{nk} \left( \frac{z}{t} \right)^k \right] dt. \end{aligned} \tag{3.3}$$

The interchange of the integral and the summation is justified by showing that the series  $\sum_k a_{nk}$  and  $\sum_k a_{nk}(z/t)^k$  converge absolutely as follows. Using (3.1), we get that the series

$$\sum_{k=0}^{\infty} |a_{nk}| \leq \frac{B}{w^n} \sum_{k=0}^{\infty} s^k, \tag{3.4}$$

which converges for each  $n$  since  $s < 1/u < 1$  and that the series

$$\begin{aligned} \sum_{k=0}^{\infty} |a_{nk}| \left| \frac{z}{t} \right|^k &\leq \frac{B}{w^n} \sum_{k=0}^{\infty} \left( \frac{|z|s}{|t|} \right)^k, \quad t \in \Gamma, \\ &= \frac{B}{w^n} \sum_{k=0}^{\infty} \left( \frac{|z|s}{r} \right)^k, \end{aligned} \tag{3.5}$$

which also converges for each  $n$ , since  $|z|s/r < |z|/ru < |z|/\rho < 1$ . Also,

$$\begin{aligned} \lambda_n(z) &= \sum_{k=0}^{\infty} a_{nk} \left[ S_k(z) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z} \left( 1 - \left( \frac{z}{t} \right)^{k+1} \right) \sum_{j=1}^{\infty} \frac{1}{t^{j(k+1)}} dt \right] \\ &= \sigma_n(z) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z} \sum_{j=1}^{\infty} \left[ \sum_{k=0}^{\infty} a_{nk} \frac{1}{t^{j(k+1)}} - \sum_{k=0}^{\infty} a_{nk} \left( \frac{z}{t} \right)^{k+1} \frac{1}{t^{j(k+1)}} \right] dt. \end{aligned} \tag{3.6}$$

The interchange of the integral and the summation is justified as follows. Using (3.1), we see that for each  $n$  and each  $j$ ,

$$\begin{aligned} \sum_{k=0}^{\infty} |a_{nk}| \frac{1}{|t|^{j(k+1)}} &\leq \frac{B}{w^n r^j} \sum_{k=0}^{\infty} \left( \frac{s}{r^j} \right)^k \\ &\leq \frac{B}{w^n r^j} \frac{r^j}{(r^j - s)} = \frac{B}{w^n (r^j - s)} \end{aligned} \tag{3.7}$$

because  $s/r^j < 1/ur^j < 1/\rho r^{j-1} < 1$ , and similarly

$$\begin{aligned} \sum_{k=0}^{\infty} |a_{nk}| \left| \frac{z}{t} \right|^{k+1} \frac{1}{|t|^{j(k+1)}} &\leq \frac{B|z|}{w^n r^{j+1}} \sum_{k=0}^{\infty} \left( \frac{|z|s}{r^{j+1}} \right)^k \\ &\leq \frac{B|z|}{w^n r^{j+1}} \frac{r^{j+1}}{(r^{j+1} - |z|s)} \\ &= \frac{B|z|}{w^n (r^{j+1} - |z|s)} \end{aligned} \tag{3.8}$$

because  $|z|s/r^{j+1} < |z|s/r < 1$ .

**THEOREM 3.1.** *Let  $\rho > R^2$ . Choose  $u > \rho/r$ , where  $1 < r < R$  and  $0 < v < 1$  and let  $A$  be a  $G_u - G_v$  matrix. Then*

$$\lim_{n \rightarrow \infty} [\lambda_n(z) - \sigma_n(z)] = 0 \quad \forall z \in D_{\rho}. \tag{3.9}$$

*Proof.* Using the expressions obtained for  $\lambda_n(z)$  and  $\sigma_n(z)$ , we get that

$$\lambda_n(z) - \sigma_n(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z} \sum_{j=1}^{\infty} \left[ \sum_{k=0}^{\infty} a_{nk} \frac{1}{t^{j(k+1)}} - \sum_{k=0}^{\infty} a_{nk} \left( \frac{z}{t} \right)^{k+1} \frac{1}{t^{j(k+1)}} \right] dt. \tag{3.10}$$

Therefore using (3.7) and (3.8), for each  $n$ , we have that

$$|\lambda_n(z) - \sigma_n(z)| \leq \frac{B}{2\pi w^n} \int_{\Gamma} \frac{|f(t)|}{|t-z|} \left[ \sum_{j=1}^{\infty} \frac{1}{r^j - s} + \sum_{j=1}^{\infty} \frac{|z|}{(r^{j+1} - |z|s)} \right] dt. \tag{3.11}$$

It can be easily proved that the two series on the right-hand side of the above inequality converge by using the ratio test. Therefore,  $w > 1$  implies that

$$\lim_{n \rightarrow \infty} [\lambda_n(z) - \sigma_n(z)] = 0 \tag{3.12}$$

for each  $|z| < \rho$ . □

### 4. Examples

First, we give below an obvious example for such a matrix  $A$ . Choose  $u > \rho/r$  and  $v$  such that  $0 < v < 1$ . Define the matrix  $A$  by

$$a_{nk} = \frac{v^n}{t^k}, \quad t > u. \tag{4.1}$$

For each  $w$  so that  $0 < w < 1/v$ , we have

$$|a_{nk}| w^n = \frac{(vw)^n}{t^k} < \frac{1}{t^k}, \tag{4.2}$$

where  $1/t < 1/u$ . Hence by Theorem 1.1,  $A$  is a  $G_u - G_v$  matrix.

Our next example is the Sonnenschein matrix  $A(g) = [a_{nk}]$  which is defined by [4, page 257]

$$[g(z)]^n = \sum_{k=0}^{\infty} a_{nk} z^k \quad \text{for } n \geq 1, \tag{4.3}$$

where  $g$  is analytic at  $z = 0$  and  $a_{00} = 1$ , and  $a_{0k} = 0$  for  $k \geq 1$ . Clearly, for each  $n \geq 1$ ,

$$a_{nk} = \frac{1}{k!} \frac{d^k}{dz^k} [g(z)]^n \Big|_{z=0}. \tag{4.4}$$

As we easily see that the first  $(n - 1)$  derivatives of  $[g(z)]^n$  contains  $g(z)$  as its factor. So, if  $g(0) = 0$ , then the first  $(n - 1)$  terms of the series  $\sum_{k=0}^{\infty} a_{nk} z^k$  vanish and the matrix  $A(g) = [a_{nk}]$  reduces to an upper triangular matrix.

Now, for  $u > \rho/r$  and  $0 < v < 1$ , choose

$$l > \max \left\{ u \left( 1 + \frac{1}{v} \right), \frac{3}{2v} \right\}. \tag{4.5}$$

Let  $g(z) = 1/(z - 2l) + 1/2l$  so that  $g(0) = 0$ . Therefore, the Sonnenschein matrix  $A(g) = [a_{nk}]$  is an upper triangular matrix. Since  $g(z)$  is analytic at  $z = 0$  and on  $D_{2l}$ ,  $[g(z)]^n$  is analytic on  $D_{2l}$ . Let  $C = \{z : |z| = l\}$ . Then on  $C$ ,

$$|g(z)| \leq \frac{1}{|z - 2l|} + \frac{1}{2l} \leq \frac{3}{2l}. \tag{4.6}$$

Therefore by Cauchy integral formula,

$$\begin{aligned} |a_{nk}| &= \left| \frac{1}{2\pi i} \int_C \frac{[g(z)]^n}{t^{k+1}} dt \right| \\ &\leq \left(\frac{3}{2l}\right)^n \frac{1}{l^k} \quad \text{for } k \geq n > 0. \end{aligned} \quad (4.7)$$

Then for any  $w$  such that  $0 < w < 1/\nu$ , we have

$$\begin{aligned} |a_{nk}| w^n &\leq \left(\frac{3}{2l}\right)^n \frac{w^n}{l^k} \\ &\leq \left(\frac{3}{2l}\right)^n \left(\frac{1}{\nu l}\right)^k \quad \text{for } k \geq n \quad (0 < \nu < 1) \\ &< \nu^n \left(\frac{1}{\nu l}\right)^k \quad \text{since } l > \frac{3}{2\nu}, \\ &< (1 + \nu)^n \left(\frac{1}{\nu l}\right)^k \\ &= \left(\frac{1 + \nu}{\nu l}\right)^k \quad \text{for } k \geq n, \end{aligned} \quad (4.8)$$

where  $(1 + \nu)/\nu l = (1/l)(1 + 1/\nu) < 1/u$ . Therefore by Theorem 1.1,  $A(g)$  is a  $G_u - G_\nu$  matrix.

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