We introduce the category $\text{IRel}(H)$ consisting of intuitionistic fuzzy relational spaces on sets and we study structures of the category $\text{IRel}(H)$ in the viewpoint of the topological universe introduced by Nel. Thus we show that $\text{IRel}(H)$ satisfies all the conditions of a topological universe over $\text{Set}$ except the terminal separator property and $\text{IRel}(H)$ is cartesian closed over $\text{Set}$.

1. Introduction

In 1965, Zadeh [30] introduced a concept of a fuzzy set as the generalization of a crisp set. Also, in 1971, he introduced a fuzzy relation naturally, as a generalization of a crisp relation in [31].

Nel [27] introduced the notion of a topological universe which implies concrete quasitopos [1]. Every topological universe satisfies all the properties of a topos except one condition on the subobject classifier. The notion of a topological universe has already been put to effective use in several areas of mathematics in [24, 25, 28]. In 1980, Cerruti [8] introduced the category of $L$-fuzzy relations and investigated some of its properties. After that time, Hur [14] introduced the category $\text{Rel}(H)$ of the fuzzy relational spaces with a complete Heyting algebra $H$ as a codomain and he studied the category $\text{Rel}(H)$ in the sense of a topological universe.

In 1983, Atanassov [2] introduced the concept of an intuitionistic fuzzy set as the generalization of fuzzy sets and he also investigated many properties of intuitionistic fuzzy sets (cf. [3]). After that time, Banerjee and Basnet [4], Biswas [6], and Hur and his colleagues [15, 16, 17, 20] applied the concept of intuitionistic fuzzy sets to algebra. Also, Çoker [9], Hur and his colleagues [21], and S. J. Lee and E. P. Lee [26] applied one to topology. In particular, Hur and his colleagues [18] applied the notion of intuitionistic fuzzy sets to topological group.

In this paper, we introduce the category $\text{IRel}(H)$ of intuitionistic $H$-fuzzy relational spaces and study the category $\text{IRel}(H)$ in a topological universe viewpoint. In particular, we show that $\text{IRel}(H)$ satisfies all the conditions of a topological universe over $\text{Set}$ except...
the terminal separator property. Also $\text{IRel}(H)$ is shown to be cartesian closed over $\text{Set}$. For general categorical background, we refer to Herrlich and Strecker [12].

2. Preliminaries

In this section, we will introduce some basic definitions and well-known results which are needed in the next sections.

Let $X$ be a set, let $(X_i)_{i \in I}$ be a family of sets indexed by a class $I$, and let $f_i$ be a mapping with domain $X$ for each $i \in I$. Then a pair $(X, (f_i)_I)$ (simply, $(f_i)_I$) is called a source of mappings. A sink of mappings is the dual notion of a source of mappings.

**Definition 2.1** [12]. Let $A$ be a concrete category and let $I$ be a class.

1. A source in $A$ is a pair $(X, (f_i)_I)$ (simply, $(X, f_i)$ or $(f_i)_I$), where $X$ is an $A$-object and $(f_i : X \rightarrow X_i)_I$ is a family of $A$-morphisms each with domain $X$. In this case, $X$ is called the domain of the source and the family $(X_i)_I$ is called the codomain of the source.

2. A source $(X, f_i)$ is called a monosource provided that the $f_i$ can be simultaneously canceled from the left; that is, provided that for any pair $Y \xrightarrow{r} X$ of morphisms such that $f_i \circ r = f_i \circ s$ for each $i \in I$, it follows that $r = s$.

Dual notions: sink in $A$ and episink.

**Definition 2.2** [23]. Let $A$ be a concrete category and let $((Y_i, \xi_i))_I$ be a family of objects in $A$ indexed by a class $I$. For any set $X$, let $(f_i : X \rightarrow Y_i)_I$ be a source of mappings indexed by $I$. An $A$-structure $\xi$ on $X$ is said to be initial with respect to $(X, (f_i), ((Y_i, \xi_i)))$ provided that the following conditions hold.

1. For each $i \in I$, $f_i : (X, \xi) \rightarrow (Y_i, \xi_i)$ is an $A$-morphism.

2. If $(Z, \rho)$ is an $A$-object and $g : Z \rightarrow X$ is mapping such that for each $i \in Z$, the mapping $f_i \circ g : (Z, \rho) \rightarrow (Y_i, \xi_i)$ is an $A$-morphism, then $g : (Z, \rho) \rightarrow (X, \xi)$ is an $A$-morphism. In this case, $(f_i : (X, \xi) \rightarrow (Y_i, \xi_i))_I$ is called an initial source in $A$.

Dual notions: final structure and final sink.

**Definition 2.3** [23]. A concrete category $A$ is said to be topological over $\text{Set}$ provided that for each set $X$, for any family $((Y_i, \xi_i))_I$ of $A$-objects, and for any source $(f_i : X \rightarrow Y_i)_I$ of mappings, there exists a unique $A$-structure $\xi$ on $X$ which is initial with respect to $(X, (f_i), ((Y_i, \xi_i)))$.

Dual notions: cotopological category.

**Result 2.4** [23, Theorem 1.5]. A concrete category $A$ is topological if and only if $A$ is cotopological.

**Result 2.5** [23, Theorem 1.6]. Let $A$ be a topological category over $\text{Set}$. Then $A$ is complete and cocomplete.

**Definition 2.6** [11]. A category $A$ is called cartesian closed provided that the following conditions hold.

1. For any $A$-objects $A$ and $B$, there exists a product $A \times B$ in $A$.

2. Exponential exists in $A$, that is, for any $A$-object $A$, the functor $A \times - : A \rightarrow A$ has a right adjoint, that is, for any $A$-object $B$, there exists an $A$-object $B^A$ and an $A$-morphism $e_{A,B} : A \times B^A \rightarrow B$ (called the evaluation) such that for any $A$-object $C$
and any \( A \)-morphism \( f : A \times C \to B \), there exists a unique \( A \)-morphism \( \overline{f} : C \to B^A \) such that the diagram

\[
\begin{array}{ccc}
A \times B^A & \xrightarrow{e_{A,B}} & B \\
\downarrow \exists 1_{A \times f} & & \Downarrow f \\
A \times C & \xrightarrow{\exists 1_{A \times f}} & C
\end{array}
\]

(2.1)

commutes.

**Definition 2.7** [23]. Let \( A \) be a concrete category.

1. The \( A \)-fiber of a set \( X \) is the class of all \( A \)-structures on \( X \).
2. \( A \) is called properly fibered over \( \text{Set} \) provided that the following conditions hold.
   1. **Fiber-smallness.** For each set \( X \), the \( A \)-fiber of \( X \) is a set.
   2. **Terminal separator property.** For each singleton set \( X \), the \( A \)-fiber of \( X \) has precisely one element.
   3. If \( \xi \) and \( \eta \) are \( A \)-structures on a set \( X \) such that \( 1_X : (X, \xi) \to (X, \eta) \) and \( 1_X : (X, \eta) \to (X, \xi) \) are \( A \)-morphisms, then \( \xi = \eta \).

**Definition 2.8** [27]. A category \( A \) is called a topological universe over \( \text{Set} \) provided that the following conditions hold.

1. \( A \) is well structured over \( \text{Set} \), that is, (i) \( A \) is a concrete category; (ii) \( A \) has the fiber-smallness condition; (iii) \( A \) has the terminal separator property.
2. \( A \) is cotopological over \( \text{Set} \).
3. Final episinks in \( A \) are preserved by pullbacks, that is, for any final episink \((g_\lambda : X \to Y)_\Lambda\) and any \( A \)-morphism \( f : W \to Y \), the family \((e_\lambda : U_\lambda \to W)_\Lambda\), obtained by taking the pullback of \( f \) and \( g_\lambda \) for each \( \lambda \), is again a final episink.

**Definition 2.9** [29]. A category \( A \) is called a topos provided that the following conditions hold.

1. There is a terminal object \( U \) in \( A \), that is, for each \( A \)-object \( A \), there exists one and only one \( A \)-morphism from \( A \) to \( U \).
2. \( A \) has equalizers, that is, for any \( A \)-objects \( A \) and \( B \) and \( A \)-morphisms

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow g & & \Downarrow \exists 1_{A \times f} \\
C & \xrightarrow{h} & A
\end{array}
\]

(2.2)

there exist an \( A \)-object \( C \) and an \( A \)-morphism \( h : C \to A \) such that

(a) \( f \circ h = g \circ h \),
(b) for each \( A \)-object \( C' \) and \( A \)-morphism \( h' : C' \to A \) with \( f \circ h' = g \circ h' \), there exists a unique \( A \)-morphism \( \overline{h'} : C' \to C \) such that \( h' = h \circ \overline{h}' \), that is, the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{h} & A & \xrightarrow{f} & B \\
\downarrow \exists 1_{A \times f} & & \Downarrow g & & \Downarrow \exists 1_{A \times f} \\
C' & \xrightarrow{h'} & A & \xrightarrow{f} & B
\end{array}
\]

(2.3)

commutes;
(3) \(A\) is cartesian closed;
(4) there is a subobject classifier in \(A\), that is, there is an \(A\)-object \(\Omega\) and \(A\)-morphism \(\nu : U \to \Omega\) such that for each \(A\)-monomorphism \(m : A' \to A\), there exists a unique \(A\)-morphism \(\phi_m : A \to \Omega\) such that the following diagram is a pullback:

\[
\begin{array}{c}
A' \\
m \downarrow \quad \downarrow \nu \\
A \\
\phi_m \\
\end{array}
\]

\[(2.4)\]

Remark 2.10. Let \(A\) be any category with a subobject classifier. If \(f\) is any bimorphism in \(A\), then \(f\) is an isomorphism in \(A\) (cf. [7]).

3. The category \(\text{IRel}(H)\)

First we will list some concepts and one result which are needed in this section and the next section. Next, we introduce the category \(\text{IRel}(H)\) of intuitionistic \(H\)-fuzzy relational spaces and show that it has similar structures as those of \(\text{ISet}(H)\).

Definition 3.1 [5, 22]. A lattice \(H\) is called a complete Heyting algebra if \(H\) satisfies the following conditions:

(1) \(H\) is a complete lattice;
(2) for any \(a, b \in H\), the set \(\{x \in H : x \land a \leq b\}\) has a greatest element denoted by \(a \to b\) (called pseudocomplement of \(a\) and \(b\)), that is, \(x \land a \leq b\) if and only if \(x \leq (a \to b)\).

In particular, for each \(a \in H\), \(N(a) = a \to 0\) is called the negation or the pseudocomplement of \(a\).

Result 3.2 [5, Example 6, page 46]. Let \(H\) be a complete Heyting algebra and let \(a, b \in H\). Then

(1) if \(a \leq b\), then \(N(b) \leq N(a)\), that is, \(N : H \to H\) is an involutive order-reversing operation in \((H, \leq)\);
(2) \(a \leq NN(a)\);
(3) \(N(a) = NNN(a)\);
(4) \(N(a \lor b) = N(a) \land N(b)\) and \(N(a \land b) = N(a) \land N(b)\).

Throughout this paper, we use \(H\) as a complete Heyting algebra.

Definition 3.3 [19]. Let \(X\) be a set. A triple \((X, \mu, \nu)\) is called an intuitionistic \(H\)-fuzzy set (in short, \(\text{IHFS}\)) on \(X\) if the following conditions holds:

(i) \(\mu, \nu \in H^X\), that is, \(\mu\) and \(\nu\) are \(H\)-fuzzy sets;
(ii) \(\mu \leq N(\nu)\), that is, \(\mu(x) \leq N(\nu(x))\) for each \(x \in X\), where \(N : H \to H\) is an involutive order-reversing operation in \((H, \leq)\).

Definition 3.4 [19]. Let \((X, \mu_X, \nu_X)\) and \((Y, \mu_Y, \nu_Y)\) be \(\text{IHFSs}\). A mapping \(f : X \to Y\) is called a morphism if \(\mu_X \leq \mu_Y \circ f\) and \(\nu_X \geq \nu_Y \circ f\).

From Definitions 3.3 and 3.4, we can form a concrete category \(\text{ISet}(H)\) consisting of all \(\text{IHFSs}\) and morphisms between them. In this case, each \(\text{ISet}(H)\)-morphism will be called an \(\text{ISet}(H)\)-mapping.
It is clear that if \( f : (X, \mu_X, \nu_X) \to (Y, \mu_Y, \nu_Y) \) is an ISet\((H)\)-mapping, then \( f : (X, \mu_X) \to (Y, \mu_Y) \) is a Set \((H)\)-mapping (cf. [13]).

**Definition 3.5** [14]. (1) Let \( X \) be a set. \( R \) is called an \( H\)-fuzzy relation (or simply, a fuzzy relation) on \( X \) if \( \mu_R : X \times X \to H \) is a mapping. In this case, \( (X, R) \) is called an \( H\)-fuzzy relational space (or simply, a fuzzy relational space).

(2) Let \( (X, R_X) \) and \( (Y, R_Y) \) be any fuzzy relational spaces. A map \( f : X \to Y \) is called a relation-preserving map provided that \( \mu_R \leq \mu_R \circ f^2 \), where \( f^2 = f \times f \).

From Definition 3.5, we can form a concrete category \( \text{Rel}(H) \) consisting of all relational spaces and relation preserving mappings between them. Every \( \text{Rel}(H) \)-morphism will be called a \( \text{Rel} \)(\(H\))-mapping.

**Definition 3.6.** Let \( X \) be a set. A pair \( R = (\mu_R, \nu_R) \) is called an intuitionistic \( H\)-fuzzy relation (in short, IFRS) on \( X \) if it satisfies the following conditions:

(i) \( \mu_R : X \times X \to H \) and \( \nu_R : X \times X \to H \) are mappings, where \( \mu_R \) and \( \nu_R \) denote the degree of membership (namely, \( \mu_R(x, y) \)) and the degree of nonmembership (namely, \( \nu_R(x, y) \)) of each \( (x, y) \in X \times X \) to \( R \);

(ii) \( \mu_R \leq N(\nu_R) \), that is, \( \mu_R(x, y) \leq N(\nu_R(x, y)) \) for each \( (x, y) \in X \times X \).

In this case, \( (X, R) \) or \( (X, \mu_R, \nu_R) \) is called an intuitionistic \( H\)-fuzzy relational space (in short, IFRS).

**Definition 3.7.** Let \( (X, R_X) \) and \( (Y, R_Y) \) be an IFRSs. A mapping \( f : X \to Y \) is called a relation-preserving mapping if \( \mu_{R_X} \leq \mu_{R_Y} \circ f^2 \) and \( \nu_{R_X} \geq \nu_{R_Y} \circ f^2 \), where \( f^2 = f \times f \).

The following is the immediate result of Definition 3.7.

**Proposition 3.8.** Let \( (X, R_X), (Y, R_Y), \) and \( (Z, R_Z) \) be IFRSs.

(1) \( 1_X : (X, R_X) \to (X, R_X) \) is a relation-preserving mapping.

(2) If \( f : (X, R_X) \to (Y, R_Y) \) and \( g : (Y, R_Y) \to (Z, R_Z) \) are relation-preserving mappings, then \( g \circ f : (X, R_X) \to (Z, R_Z) \) is a relation-preserving mapping.

From Definitions 3.6 and 3.7, and Proposition 3.8, we can form a concrete category \( \text{IRel}(H) \) consisting of all IFRSs and relation-preserving mappings between them. Every \( \text{IRel}(H) \)-morphism will be called an \( \text{IRel}(H) \)-mapping. Moreover, it is clear that if \( f : (X, R_X) \to (Y, R_Y) \) is an \( \text{IRel}(H) \)-mapping, then \( f : (X, \mu_{R_X}) \to (Y, \mu_{R_Y}) \) is a \( \text{Rel}(H) \)-mapping.

**Theorem 3.9.** \( \text{IRel}(H) \) is topological over Set.

**Proof.** Let \( X \) be any set and let \( ((X_a, R_a)) \) be any family of IFRSs indexed by a class \( \Gamma \). Let \( (f_a : X \to X_a) \) be any source of mappings. We define two mappings \( \mu_R : X \times X \to H \) and \( \nu_R : X \times X \to H \) by \( \mu_R = \bigwedge_{\Gamma} \mu_{R_a} \circ f_a^2 \) and \( \nu_R = \bigvee_{\Gamma} \nu_{R_a} \circ f_a^2 \). Then, by the definition of \( R = (\mu_R, \nu_R) \), \( \mu_R \leq N(\nu_R) \). Thus \( (X, R) \in \text{IRel}(H) \). Moreover, \( f_a : (X, R) \to (X_a, R_a) \) is an \( \text{IRel}(H) \)-mapping for each \( \alpha \in \Gamma \).

For any \( (Y, R_Y) \in \text{IRel}(H) \), let \( g : Y \to X \) be any mapping for which \( f_a \circ g : (Y, R_Y) \to (X_a, R_a) \) is an \( \text{IRel}(H) \)-mapping for each \( \alpha \in \Gamma \). Then we can easily check that \( g : (Y, R_Y) \to (X, R) \) is an \( \text{IRel}(H) \)-mapping. Hence \( R = (\mu_R, \nu_R) \) is the initial structure on \( X \) with respect to \( (X, (f_a), ((X_a, R_a))) \). This completes the proof. \( \square \)
Example 3.10. (1) Inverse image of an IHFR. Let $X$ be a set, let $(Y, R_Y)$ be an IHFRS, and let $f : X \rightarrow Y$ be any mapping. Then there exists the initial IHFR $R$ on $X$ for which $f : (X, R) \rightarrow (Y, R_Y)$ is an $\text{IRel}(H)$-mapping. In this case, $R$ is called the inverse image of $R_Y$ under $f$. In particular, if $X \subset Y$ and $f : X \rightarrow Y$ is the canonical mapping, then $(X, R)$ is called an intuitionistic $H$-fuzzy relational subspace of $(Y, R_Y)$, where $R = (\mu_R, \nu_R)$ is the inverse image of $R_Y$ under $f$. In fact, $\mu_R = \mu_{R_Y} \mid_X \times_X$ and $\nu_R = \nu_{R_Y} \mid_X \times_X$.

(2) Intuitionistic fuzzy product structure. Let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IHFRSs and let $X = \prod X_\alpha$ be the product set of $(X_\alpha)_\Gamma$. Then there exists the initial IHFR $R$ on $X$ for which each projection $\pi_\alpha : (X, R) \rightarrow (X_\alpha, R_\alpha)$ is an $\text{IRel}(H)$-mapping. In this case, $R$ is called the product of $(R_\alpha)_\Gamma$ and is denoted by $R = \prod R_\alpha$ and $(\prod X_\alpha, \prod R_\alpha)$ is called the intuitionistic $H$-fuzzy product relational space of $((X_\alpha, R_\alpha))_\Gamma$. In fact, $\mu_{\prod R} = \bigwedge \mu_{R_\alpha} \circ \pi_\alpha^2$ and $\nu_{\prod R} = \bigvee \nu_{R_\alpha} \circ \pi_\alpha^2$.

In particular, if $H = \{1, 2\}$, then $\mu_{R_1 \times R_2}((x_1, y_1), (x_2, y_2)) = \mu_{R_1}(x_1, x_2) \land \mu_{R_2}(y_1, y_2)$ and $\nu_{R_1 \times R_2}((x_1, y_1), (x_2, y_2)) = \nu_{R_1}(x_1, x_2) \lor \nu_{R_2}(y_1, y_2)$ for any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$.

Corollary 3.11. $\text{IRel}(H)$ is complete and cocomplete. Moreover, by definition, it is easy to show that $\text{IRel}(H)$ is well powered and co-well-powered.

From Result 2.4 and Theorem 3.9, it is clear that $\text{IRel}(H)$ is cotopological. However, we show directly that $\text{IRel}(H)$ is cotopological.

Theorem 3.12. $\text{IRel}(H)$ is cotopological over $\text{Set}$.

Proof. Let $X$ be any set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IHFRSs indexed by a class $\Gamma$. Let $(f_\alpha : X_\alpha \rightarrow X)_\Gamma$ be any sink of mappings. We define two mappings $\mu_R : X \times X \rightarrow H$ and $\nu_R : X \times X \rightarrow H$ by, for each $(x, y) \in X \times X$,

\[ \mu_R(x, y) = \bigvee_{\Gamma} \bigwedge \{ \mu_{R_\alpha}(x, y_\alpha) \mid (x_\alpha, y_\alpha) \in f_\alpha^{-1}(x, y) \} \]

\[ \nu_R(x, y) = \bigwedge_{\Gamma} \bigvee \{ \nu_{R_\alpha}(x, y_\alpha) \mid (x_\alpha, y_\alpha) \in f_\alpha^{-1}(x, y) \} \]  \hspace{1cm} (3.1)

where $f_\alpha^{-1} = f_\alpha^{-1} \times f_\alpha^{-1}$. Then clearly $(X, R) \in \text{IRel}(H)$. Moreover, $f_\alpha : (X_\alpha, R_\alpha) \rightarrow (X, R)$ is an $\text{IRel}(H)$-mapping for each $\alpha \in \Gamma$.

For any $(Y, R_Y) \in \text{IRel}(H)$, let $g : X \rightarrow Y$ be any mapping for which $g \circ f_\alpha : (X_\alpha, R_\alpha) \rightarrow (Y, R_Y)$ is an $\text{IRel}(H)$-mapping for each $\alpha \in \Gamma$. Then we can easily check that $g : (X, R) \rightarrow (Y, R_Y)$ is an $\text{IRel}(H)$-mapping. Hence $R = (\mu_R, \nu_R)$ is the final structure on $X$ with respect to $(((X_\alpha, R_\alpha)), (f_\alpha), X)$. This completes the proof. \square

Example 3.13. (1) Intuitionistic $H$-fuzzy quotient relation. Let $(X, R) \in \text{IRel}(H)$, let $\sim$ be an equivalence relation on $X$, and let $\varphi : X \rightarrow X/R$ the canonical mapping. Then there exists the final intuitionistic $H$-fuzzy relation $(\mu_{X/\sim}, \nu_{X/\sim})$ on $X/\sim$ for which $\varphi : (X, R) \rightarrow (X/\sim, \mu_{X/\sim}, \nu_{X/\sim})$ is an $\text{IRel}(H)$-mapping. In this case, $(\mu_{X/\sim}, \nu_{X/\sim})$ is called the intuitionistic $H$-fuzzy quotient relation of $X$ by $R$.

(2) Sum of intuitionistic $H$-fuzzy relations. Let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IHFRSs, let $X$ be the sum of $(X_\alpha)_\Gamma$ and let $j_\alpha : X_\alpha \rightarrow X$ be the canonical (injection) mapping for
each \( \alpha \in \Gamma \). Then there exists the final IHFR \( R \) on \( X \). In fact, for each \((x_\alpha, \alpha), (y_\beta, \beta)\) \( X \times X \), \( \mu_R((x_\alpha, \alpha), (y_\beta, \beta)) = \sqrt{\Gamma} \mu_R(x, y) \) and \( \nu_R((x_\alpha, \alpha), (y_\beta, \beta)) = \bigwedge \nu_R(x, y) \). In this case, \( R \) is called the sum of \((R_\alpha) \Gamma \) and \((X, R)\) is called the sum of \((X_\alpha, R_\alpha) \Gamma \).

**Theorem 3.14.** Final episinks in \( \text{IRel}(H) \) are preserved by pullbacks.

**Proof.** Let \((g_\alpha : (U_\alpha, R_\alpha) \rightarrow (Y, R_Y)) \Gamma \) be any final episink in \( \text{IRel}(H) \) and let \( f : (W, R_W) \rightarrow (Y, R_Y) \) be any \( \text{IRel}(H) \)-mapping. For each \( \alpha \in \Gamma \), let \( U_\alpha = \{(w, x_\alpha) \in W \times X_\alpha : f(w) = g_\alpha(x_\alpha)\} \) and let us define two mappings \( \mu_{R_{U_\alpha}} : U_\alpha \times U_\alpha \rightarrow H \) and \( \nu_{R_{U_\alpha}} : U_\alpha \times U_\alpha \rightarrow H \) by for each \(( (w, x_\alpha), (w', x'_\alpha)) \) \( U_\alpha \times U_\alpha \),

\[
\begin{align*}
\mu_{R_{U_\alpha}}((w, x_\alpha), (w', x'_\alpha)) &= \mu_{R_W}(w, w') \land \mu_{R_\alpha}(x_\alpha, x'_\alpha), \\
\nu_{R_{U_\alpha}}((w, x_\alpha), (w', x'_\alpha)) &= \nu_{R_W}(w, w') \lor \nu_{R_\alpha}(x_\alpha, x'_\alpha).
\end{align*}
\]  

(3.2)

Let \( e_\alpha : U_\alpha \rightarrow W \) and \( p_\alpha : U_\alpha \rightarrow X_\alpha \) denote the usual projections of \( U_\alpha \). Then clearly \((U_\alpha, R_{U_\alpha}) \in \text{IRel}(H)\) for each \( \alpha \in \Gamma \). Moreover, \( e_\alpha : (U_\alpha, R_{U_\alpha}) \rightarrow (W, R_W) \) and \( p_\alpha : (U_\alpha, R_{U_\alpha}) \rightarrow (X_\alpha, R_\alpha) \) are \( \text{IRel}(H) \)-mappings for each \( \alpha \in \Gamma \). And the following diagram is a pullback square in \( \text{IRel}(H) \):

\[
\begin{array}{ccc}
(U_\alpha, R_{U_\alpha}) & \xrightarrow{p_\alpha} & (X_\alpha, R_\alpha) \\
e_\alpha \downarrow & & \downarrow g_\alpha \\
(W, R_W) & \xrightarrow{f} & (Y, R_Y)
\end{array}
\]  

(3.3)

We will show that \((e_\alpha : (U_\alpha, R_{U_\alpha}) \rightarrow (W, R_W)) \Gamma \) is a final episink in \( \text{IRel}(H) \). By the process of the proof of \([14, \text{Theorem 2.5}] \), \((e_\alpha)_\Gamma \) is an episink in \( \text{IRel}(H) \). Suppose \( R = (\mu_R, \nu_R) \) is another final IHFR on \( W \) with respect to \((e_\alpha)_\Gamma \). By the process of the proof of \([14, \text{Theorem 2.5}] \), \( \mu_R = \mu_{R_W} \). Thus it is sufficient to show that \( \nu_R = \nu_{R_W} \). Let \((w, w') \in W \times W \). Then

\[
\begin{align*}
\nu_{R_W}(w, w') &= \nu_{R_W}(w, w') \lor \nu_{R_W}(w, w') \\
&\geq \nu_{R_W}(w, w') \lor \left[ \nu_{R_{Y}} \circ f^2(w, w') \right] \\
&\quad \text{(since } f : (W, R_W) \rightarrow (Y, R_Y) \text{is an } \text{IRel}(H) \text{-mapping)} \\
&= \nu_{R_W}(w, w') \lor \nu_{R_Y}(f(w), f(w')) \\
&= \nu_{R_W}(w, w') \lor \left[ \bigwedge_{(x_\alpha, x'_\alpha)} \gamma_{R_{U_\alpha}}(x_\alpha, x'_\alpha) \right] \text{ (3.4)}
\end{align*}
\]

(since \((g_\alpha)_\Gamma \) is final)

\[
\begin{align*}
&= \bigwedge_{(x_\alpha, x'_\alpha)} \left[ \nu_{R_{U_\alpha}}(w, w') \lor \nu_{R_{U_\alpha}}(x_\alpha, x'_\alpha) \right] \\
&= \bigwedge_{((w, x_\alpha), (w', x'_\alpha))} \nu_{R_{U_\alpha}}((w, x_\alpha), (w', x'_\alpha)).
\end{align*}
\]
Thus $\nu_{R_W}(w, w') \geq \nu_R(w, w')$ for each $(w, w') \in W \times W$. So $\nu_{R_W} \geq \nu_R$. On the other hand, since $(e_\alpha : (U_\alpha, R_{U_\alpha}) \rightarrow (W, R))_\alpha$ is final, $1_W : (W, R) \rightarrow (W, R_W)$ is an $\text{IRel}(H)$-mapping. Thus $\nu_R \geq \nu_{R_W}$. So $\nu_R = \nu_{R_W}$. Hence $R = R_W$. This completes the proof. \hfill \square

For any singleton set $\{a\}$, since the IHFR $R$ on $\{a\}$ is not unique, the category $\text{IRel}(H)$ is not properly fibered over $\text{Set}$. Hence, by Theorems 3.12 and 3.14, we obtain the following result.

**Theorem 3.15.** $\text{IRel}(H)$ satisfies all the conditions of a topological universe over $\text{Set}$ except the terminal separator property.

**Theorem 3.16.** $\text{IRel}(H)$ is cartesian closed over $\text{Set}$.

**Proof.** It is clear that $\text{IRel}(H)$ has products by Corollary 3.11. We will show that $\text{IRel}(H)$ has exponential objects.

For any IHFRs $X = (X, R_X)$ and $Y = (Y, R_Y)$, let $Y^X$ be the set of all mappings from $X$ into $Y$. We define two mappings $\mu_R : Y^X \times Y^X \rightarrow H$ and $\nu_R : Y^X \times Y^X \rightarrow H$ as follows: for each $(f, g) \in Y^X \times Y^X$,

\[
\mu_R(f, g) = \bigwedge \{ h \in H : \mu_{R_X}(x, y) \wedge h \leq \mu_{R_Y}(f(x), g(y)) \text{ for each } (x, y) \in X \times X \},
\]

\[
\nu_R(f, g) = \bigvee \{ h \in H : \nu_{R_X}(x, y) \vee h \geq \nu_{R_Y}(f(x), g(y)) \text{ for each } (x, y) \in X \times X \}. \tag{3.5}
\]

Then clearly $(Y^X, R) \in \text{IRel}(H)$. Let $Y^X = (Y^X, R)$. Then, by the definition of $R$,

\[
\mu_{R_X}(x, y) \wedge \mu_R(f, g) \leq \mu_{R_Y}(f(x), g(y)),
\]

\[
\nu_{R_X}(x, y) \vee \nu_R(f, g) \geq \nu_{R_Y}(f(x), g(y)) \tag{3.6}
\]

for each $(f, g) \in Y^X$ and $(x, y) \in X \times X$.

Define $e_{X,Y} : X \times Y^X \rightarrow Y$ by $e_{X,Y}(x, f) = f(x)$ for each $(x, f) \in X \times Y^X$. Let $((x, f), (y, g)) \in (X \times Y^X) \times (X \times Y^X)$. Then, by the process of the proof of [14, Theorem 2.7], $\mu_{R_X \times R}((x, f), (y, g)) \leq \mu_{R_Y} \circ e_{X,Y}^2((x, f), (y, g))$. So $\mu_{R_X \times R} \leq \mu_{R_Y} \circ e_{X,Y}^2$. On the other hand,

\[
\nu_{R_X \times R}((x, f), (y, g)) = \nu_{R_X}(x, y) \vee \nu_R(f, g)
\]

\[
\geq \nu_{R_Y}(f(x), g(y))
\]

\[
= \nu_{R_Y}(e_{X,Y}(x, f), e_{X,Y}(y, g))
\]

\[
= \nu_{R_Y} \circ e_{X,Y}^2((x, f), (y, g)). \tag{3.7}
\]

Thus $\nu_{R_X \times R} \geq \nu_{R_Y} \circ e_{X,Y}^2$. Hence $e_{X,Y} : X \times Y^X \rightarrow Y$ is an $\text{IRel}(H)$-mapping.

For any $Z = (Z, R_Z) \in \text{IRel}(H)$, let $h : X \times Z \rightarrow Y$ be an $\text{IRel}(H)$-mapping. We define $\tilde{h} : Z \rightarrow Y^X$ by $[\tilde{h}(z)](x) = h(x, z)$ for each $z \in Z$ and each $x \in X$. Let $z, z' \in Z$ and
let \( x, x' \in X \). Then, by the process of the proof of [14, Theorem 2.7], \( \mu_{R_2}(z, z') \leq \mu_R \circ \overline{h}^2(z, z') \). So \( \mu_{R_2} \leq \mu_R \circ \overline{h}^2 \). On the other hand,

\[
\nu_{R_2 \times R_2}((x, z), (x', z')) = \nu_{R_2}(x, x') \lor \nu_{R_2}(z, z') \\
\geq \nu_{R_2} \circ \overline{h}^2((x, z), (x', z')) \\
(\text{since } h : X \times Z \rightarrow Y \text{ is an } IRel(H)-\text{mapping}) \tag{3.8}
\]

\[
= \nu_{R_2}(h(x, z), h(x', z')) \\
= \nu_{R_2}([\overline{h}(z)](x), [\overline{h}(z')](x')).
\]

Thus, by the definition of \( R \), \( \nu_{R_2}(z, z') \geq \nu_R(\overline{h}(z), \overline{h}(z')) = \nu_{R_2} \circ \overline{h}^2(z, z') \). So \( \nu_{R_2} \geq \nu_R \circ \overline{h}^2 \).

Hence \( \overline{h} : Z \rightarrow Y^X \) is an \( IRel(H)-\text{mapping} \). Moreover, \( \overline{h} \) is the unique \( IRel(H)-\text{mapping} \) such that \( \epsilon_{X,Y} \circ (1_X \times \overline{h}) = h \). This completes the proof. \( \square \)

**Remark 3.17.** \( IRel(H) \) has no subobject classifier. Hence \( IRel(H) \) is not topos.

**Example 3.18.** Let \( H = \{0, 1\} \) be the two points chain and let \( X = \{a\} \). Let \( R_1 \) and \( R_2 \) be the IHFRs on \( X \) given by \( \mu_{R_1}(a, a) = 0, \nu_{R_1}(a, a) = 1 \) and \( \mu_{R_2}(a, a) = 1, \nu_{R_2}(a, a) = 0 \). Let \( 1_X : (X, R_1) \rightarrow (X, R_2) \) be the identity mapping. Then clearly, \( 1_X \) is both a monomorphism and an epimorphism in \( IRel(H) \). But, \( 1_X \) is not an isomorphism in \( IRel(H) \). Hence \( IRel(H) \) has no subobject classifier (see [7]).

4. The relations between \( IRel(H) \) and \( Rel(H) \)

**Lemma 4.1.** Define \( G_1, G_2 : IRel(H) \rightarrow Rel(H) \) by

\[
G_1(X, \mu_R, \nu_R) = (X, \mu_R), \\
G_2(X, \mu_R, \nu_R) = (X, N(\nu_R)), \tag{4.1}
\]

Then \( G_1 \) and \( G_2 \) are functors.

**Proof.** Clearly \( G_1(X, \mu_{R_1}, \nu_{R_1}) = (X, \mu_{R_1}) \in Rel(H) \) for each \( (X, \mu_R, \nu_R) \in IRel(H) \). Let \( (X, \mu_{R_1}, \nu_{R_1}), (Y, \mu_{R_1}, \nu_{R_1}) \in IRel(H) \) and let \( f : (X, \mu_{R_1}, \nu_{R_1}) \rightarrow (Y, \mu_{R_1}, \nu_{R_1}) \) be an \( IRel(H)-\text{mapping} \). Then \( \mu_{R_1} \leq \mu_{R_2} \circ f^2 \). Thus \( G_1(f) = f : (X, \mu_{R_1}) \rightarrow (Y, \mu_{R_1}) \) is a \( Rel(H)-\text{mapping} \). Hence \( G_1 : IRel(H) \rightarrow Rel(H) \) is a functor. Also \( G_2(X, \mu_{R_1}, \nu_{R_1}) = (X, N(\nu_{R_1})) \in Rel(H) \) for each \( (X, \mu_{R_1}, \nu_{R_1}) \in IRel(H) \). Now let \( (X, \mu_{R_1}, \nu_{R_1}), (Y, \mu_{R_1}, \nu_{R_1}) \in IRel(H) \) and let \( f : (X, \mu_{R_1}, \nu_{R_1}) \rightarrow (Y, \mu_{R_1}, \nu_{R_1}) \) be an \( IRel(H)-\text{mapping} \). Then \( \nu_{R_1} \leq \nu_{R_2} \circ f^2 \). Thus \( N(\nu_{R_1}) \leq N(\nu_{R_1} \circ f^2) \). So \( G_2(f) = \overline{f} : (X, N(\nu_{R_1})) \rightarrow (Y, N(\nu_{R_1})) \) is a \( Rel(H)-\text{mapping} \). Hence \( G_2 : IRel(H) \rightarrow Rel(H) \) is a functor. \( \square \)

**Lemma 4.2.** Define \( F_1 : Rel(H) \rightarrow IRel(H) \) by \( F_1(X, \mu_R) = (X, \mu_R, N(\mu_R)) \) and \( F_1(f) = f \).

Then \( F_1 \) is a functor.
Theorem 4.5. $f$ is a $\text{Rel}$ a mapping. Let $(X, \mu_{RX}) \in \text{Rel}(H)$ and let $f : (X, \mu_{RX}) \rightarrow (Y, \mu_{RY})$ be an $\text{Rel}(H)$-mapping. Then $\mu_{RY} \leq \mu_{RY} \circ f$. Consider the mapping $F_1(f) = f : (X, \mu_{RX}) \rightarrow (Y, \mu_{RY})$. Since $\mu_{RY} \leq \mu_{RY} \circ f$, $N(\mu_{RX}) \geq N(\mu_{RY}) \circ f$. So $f : (X, \mu_{RX}) \rightarrow (Y, \mu_{RY})$ is an $\text{IRel}(H)$-mapping. Hence $F_1$ is a functor.

Lemma 4.3. Define $F_2 : \text{Rel}(H) \rightarrow \text{IRel}(H)$ by $F_2(X, \mu_R) = (X, N(\mu_R), N(\mu_R))$ and $F_2(f) = f$. Then $F_2$ is a functor.

Proof. It is clear that $F_2(X, \mu_{RX}) \in \text{IRel}(H)$ for each $(X, \mu_{RX}) \in \text{Rel}(H)$. Let $(X, \mu_{RX})$, $(Y, \mu_{RY}) \in \text{Rel}(H)$ and let $f : (X, \mu_{RX}) \rightarrow (Y, \mu_{RY})$ be an $\text{Rel}(H)$-mapping. Consider the mapping $F_2(f) = f : F_2(X, \mu_{RX}) \rightarrow (Y, N(\mu_R), N(\mu_R))$, where $F_2(X, \mu_{RX}) = (X, N(\mu_{RX}), N(\mu_{RX}))$ and $F_2(Y, \mu_{RY}) = (Y, N(\mu_{RY}), N(\mu_{RY}))$. Since $f : (X, \mu_{RX}) \rightarrow (Y, \mu_{RY})$ is a $\text{Rel}(H)$-mapping, $\mu_{RX} \leq \mu_{RY} \circ f^2$. Thus $N(\mu_{RX}) \geq N(\mu_{RY}) \circ f^2$. Moreover $N(\mu_{RX}) \geq N(\mu_{RY}) \circ f^2$. So $F_2(f) = f : F_2(X, \mu_{RX}) \rightarrow F_2(Y, \mu_{RY})$ is an $\text{IRel}(H)$-mapping. Hence $F_2$ is a functor.

Theorem 4.4. The functor $F_1 : \text{Rel}(H) \rightarrow \text{IRel}(H)$ is a left adjoint of the functor $G_1 : \text{IRel}(H) \rightarrow \text{Rel}(H)$.

Proof. For each $(X, \mu_R) \in \text{Rel}(H)$, $1_X : (X, \mu_R) \rightarrow G_1F_1(X, \mu_R) = (X, \mu_R)$ is a $\text{Rel}(H)$-mapping. Let $(Y, \mu_{RY}, \nu_{RY}) \in \text{IRel}(H)$ and let $f : (X, \mu_R) \rightarrow G_1(Y, \mu_{RY}, \nu_{RY})$ be an $\text{IRel}(H)$-mapping. We will show that $f : F_1(X, \mu_R) = (X, \mu_R, \nu_{RY}) \rightarrow (Y, \mu_{RY}, \nu_{RY})$ is an $\text{IRel}(H)$-mapping. Since $f : (X, \mu_R) = G_1(Y, \mu_{RY}, \nu_{RY}) \rightarrow (Y, \mu_{RY})$ is a $\text{Rel}(H)$-mapping, $\mu_R \leq \mu_{RY} \circ f^2$. Then $N(\mu_R) \geq N(\mu_{RY}) \circ f^2$. Since $\mu_{RY} \leq N(\nu_{RY})$, $\nu_{RY} \leq N(\nu_{RY}) \leq N(\mu_{RY})$. Thus $N(\mu_R) \geq \nu_{RY} \circ f^2$. So $f : F_1(X, \mu_R) = (X, \mu_R, \nu_{RY})$ is an $\text{IRel}(H)$-mapping. Hence $1_X$ is a $G_1$-universal map for $(X, \mu_R)$ in $\text{Rel}(H)$. This completes the proof.

For each $(X, \mu_R) \in \text{Rel}(H)$, $F_1(X, \mu_R) = (X, \mu_R, N(\mu_R))$ is called an intuitionistic $H$-fuzzy set in $X$ induced by $(X, \mu_R)$. Let us denote the category of all induced intuitionistic $H$-fuzzy sets and $\text{IRel}(H)$-mappings as $\text{IRel}^*(H)$. Then it is clear $\text{IRel}^*(H)$ is a full subcategory of $\text{IRel}(H)$.

Theorem 4.5. Two categories $\text{Rel}(H)$ and $\text{IRel}^*(H)$ are isomorphic.

Proof. It is clear that $F_1 : \text{Rel}(H) \rightarrow \text{IRel}^*(H)$ is a functor by Lemma 4.2. Consider the restriction $G_1 : \text{IRel}^*(H) \rightarrow \text{Rel}(H)$ of the functor $G_1$ in Lemma 4.1. Let $(X, \mu_R) \in \text{Rel}(H)$. Then, by Lemma 4.2, $F_1(X, \mu_R) = (X, \mu_R, N(\mu_R))$. Thus $G_1F_1(X, \mu_R) = G_1(X, \mu_R, N(\mu_R)) = (X, \mu_R)$. So $G_1 \circ F = 1_{\text{Rel}(H)}$. Now let $(X, \mu_R, N(\mu_R)) \in \text{IRel}^*(H)$. Then, by Lemma 4.1, $G_1(X, \mu_R, N(\mu_R)) = (X, \mu_R)$. Thus $F G_1(X, \mu_R, N(\mu_R)) = (X, \mu_R, N(\mu_R))$. So $F \circ G_1 = 1_{\text{IRel}^*(H)}$. Hence $F : \text{Rel}(H) \rightarrow \text{IRel}^*(H)$ is an isomorphism. This completes the proof.

Remark 4.6. We are going to investigate “intuitionistic $H$-fuzzy reflexive relations,” “some subcategories of the category $\text{IRelk}(H)$,” and “intuitionistic $H$-fuzzy relations on intuitionistic $H$-fuzzy sets” in the viewpoint of topological universe.

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References


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Kul Hur: Division of Mathematics and Informational Statistics and Institute of Basic Natural Science, Wonkwang University, Iksan, Chonbuk 579-792, Korea
E-mail address: kulhur@wonkwang.ac.kr

Su Youn Jang: Division of Mathematics and Informational Statistics and Institute of Basic Natural Science, Wonkwang University, Iksan, Chonbuk 579-792, Korea
E-mail address: suyoun123@yahoo.co.kr

Hee Won Kang: Department of Mathematics Education, Woosuk University, Hujong-Ri, Samrae-Eup, Wanju-kun, Chonbuk 565-701, Korea
E-mail address: khwon@woosuk.ac.kr
Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

It means topics where a satellite, a rocket, or any kind of man-made object is travelling in space governed by the gravitational forces of celestial bodies and/or forces generated by propulsion systems that are available in those objects. Many topics are related to orbit determination, propagation, and orbital maneuvers related to those spacecrafts. Several other topics that are related to this subject are numerical methods, nonlinear dynamics, chaos, and control.

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