ON A SUBCLASS OF \( n \)-STARLIKE FUNCTIONS

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In 1999, Kanas and Rønning introduced the classes of starlike and convex functions, which are normalized with \( f(w) = f'(w) - 1 = 0 \) and \( w \) a fixed point in \( U \). In 2005, the authors introduced the classes of functions close to convex and \( \alpha \)-convex, which are normalized in the same way. All these definitions are somewhat similar to the ones for the uniform-type functions and it is easy to see that for \( w = 0 \), the well-known classes of starlike, convex, close-to-convex, and \( \alpha \)-convex functions are obtained. In this paper, we continue the investigation of the univalent functions normalized with \( f(w) = f'(w) - 1 = 0 \), where \( w \) is a fixed point in \( U \).

1. Introduction

Let \( \mathcal{H}(U) \) be the set of functions which are regular in the unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \), \( A = \{ f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0 \} \), and \( S = \{ f \in A : f \) is univalent in \( U \} \).

We recall here the definitions of the well-known classes of starlike and convex functions:

\[
S^* = \left\{ f \in A : \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > 0, \ z \in U \right\},
\]

\[
S^c = \left\{ f \in A : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \ z \in U \right\}.
\]

Let \( w \) be a fixed point in \( U \) and \( A(w) = \{ f \in \mathcal{H}(U) : f(w) = f'(w) - 1 = 0 \} \).

In [3], Kanas and Rønning introduced the following classes:

\[
S(w) = \{ f \in A(w) : f \) is univalent in \( U \} ,
\]

\[
\text{ST}(w) = S^*(w) = \left\{ f \in S(w) : \text{Re} \left( \frac{(z-w)f'(z)}{f(z)} \right) > 0, \ z \in U \right\} ,
\]

\[
\text{CV}(w) = S^c(w) = \left\{ f \in S(w) : 1 + \text{Re} \left( \frac{(z-w)f''(z)}{f'(z)} \right) > 0, \ z \in U \right\} ,
\]

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It is obvious that a natural “Alexander relation” exists between the classes $S^*(w)$ and $S^c(w)$:

$$g \in S^c(w) \quad \text{iff} \quad f(z) = (z - w)g'(z) \in S^*(w). \quad (1.3)$$

Denote with $\mathcal{P}(w)$ the class of all functions $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z - w)^n$ that are regular in $U$ and satisfy $p(w) = 1$ and $\Re p(z) > 0$ for $z \in U$.

2. Preliminary results

If is easy to see that a function $f(z) \in A(w)$ has the series of expansions:

$$f(z) = (z - w) + a_2(z - w)^2 + \ldots \quad (2.1)$$

In [8], Wald gives the sharp bounds for the coefficients $B_n$ of the function $p \in \mathcal{P}(w)$.

**Theorem 2.1.** If $p(z) \in \mathcal{P}(w)$, $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z - w)^n$, then

$$|B_n| \leq \frac{2}{(1 + d)(1 - d)^n}, \quad \text{where} \quad d = |w|, \quad n \geq 1. \quad (2.2)$$

Using the above result, Kanas and Rønning obtain the following theorem in [3].

**Theorem 2.2.** Let $f \in S^*(w)$ and $f(z) = (z - w) + b_2(z - w)^2 + \ldots$. Then

$$|b_2| \leq \frac{2}{1 - d^2}, \quad |b_3| \leq \frac{3 + d}{(1 - d^2)^2},$$

$$|b_4| \leq \frac{2}{3} \cdot \frac{(2 + d)(3 + d)}{(1 - d^2)^3}, \quad |b_5| \leq \frac{1}{6} \cdot \frac{(2 + d)(3 + d)(3d + 5)}{(1 - d^2)^4}, \quad (2.3)$$

where $d = |w|$.

**Remark 2.3.** It is clear that the above theorem also provides bounds for the coefficients of functions in $S^c(w)$, due to the relation between $S^c(w)$ and $S^*(w)$.

In [1], are also defined the following sets:

$$D(w) = \left\{ z \in U : \Re \left[ \frac{w}{z} \right] < 1, \Re \left[ \frac{z(1 + z)}{(z - w)(1 - z)} \right] > 0 \right\} \quad \text{for} \ w \neq 0, \ D(0) = U;$$

$$s(w) = \{ f : D(w) \rightarrow \mathbb{C} \} \cap S(w); \quad s^*(w) = S^*(w) \cap s(w), \quad (2.4)$$

where $w$ is a fixed point in $U$. 
The authors consider the integral operator \( L_a : A(w) \to A(w) \) defined by
\[
f(z) = L_a F(z) = \frac{1 + a}{(z - w)^a} \int_w^z F(t) \cdot (t - w)^{a-1} dt, \quad a \in \mathbb{R}, \ a \geq 0.
\]

The next theorem is a result of the so-called “admissible functions method” introduced by Mocanu and Miller (see [3, 4, 6]).

**Theorem 2.4.** Let \( h \) be convex in \( U \) and \( \text{Re}[\beta h(z) + \gamma] > 0, \ z \in U \). If \( p \in \mathcal{H}(U) \) with \( p(0) = h(0) \) and \( p \) satisfied the Briot-Bouquet differential subordination
\[
p(z) + \frac{z p'(z)}{\beta p(z) + \gamma} < h(z),
\]
then \( p(z) < h(z) \).

### 3. Main results

**Definition 3.1.** Let \( w \) be a fixed point in \( U \), \( n \in \mathbb{N} \). \( D_w^n \) denotes the differential operator:
\[
D_w^0 : A(w) \longrightarrow A(w) \quad \text{with,}
\]
\[
D_w^0 f(z) = f(z),
\]
\[
D_w^1 f(z) = D_w f(z) = (z - w) \cdot f'(z),
\]
\[
D_w^n f(z) = D_w(D_w^{n-1} f(z)).
\]

**Remark 3.2.** For \( f \in A(w), \ f(w) = (z - w) + \sum_{j=2}^{\infty} a_j (z - w)^j \), we have
\[
D_w^n f(z) = (z - w) + \sum_{j=2}^{\infty} j^n \cdot a_j \cdot (z - w)^j.
\]

It easy to see that if we take \( w = 0 \), we obtain the Sălăgean differential operator (see [7]).

**Definition 3.3.** Let \( w \) be a fixed point in \( U \), \( n \in \mathbb{N} \) and \( f \in S(w) \). \( f \) is said to be an \( n-w \)-starlike function if
\[
\text{Re} \left( \frac{D_w^{n+1} f(z)}{D_w^n f(z)} \right) > 0, \quad z \in U.
\]

The class of all these functions is denoted by \( S^*_n(w) \).

**Remark 3.4.** (1) \( S^*_0(w) = S^*(w) \) and \( S^*_n(0) = S^*_n \), where \( S^*_n \) is the class of \( n \)-starlike functions introduced by Sălăgean in [7].

(2) If \( f(z) \in S^*_n(w) \) and we denote \( D_w^n f(z) = g(z) \), we obtain \( g(z) \in S^*(w) \).

(3) Using the class \( s(w) \), we obtain \( s_n^*(w) = S^*_n(w) \cap s(w) \).

**Theorem 3.5.** Let \( w \) be a fixed point in \( U \) and \( n \in \mathbb{N} \). If \( f(z) \in s_{n+1}^*(w) \) then \( f(z) \in s_n^*(w) \). This means
\[
s_{n+1}^*(w) \subset s_n^*(w).
\]
By hypothesis, we have $\Re(D_w^p f(z)/D_w^{p+1} f(z)) > 0$, $z \in U$. We denote $p(z) = (D_w^{p+1} f(z)/D_w^p f(z))$, where $p(0) = 1$ and $p(z) \in \mathcal{H}(U)$. We obtain

$$
\frac{D_w^{p+2} f(z)}{D_w^{p+1} f(z)} = \frac{D_w(D_w^{p+1} f(z))}{D_w^p f(z)} = \frac{(z-w)(D_w^{p+1} f(z))'}{(z-w)(D_w^p f(z))} = \frac{(D_w^{p+1} f(z))'}{(D_w^p f(z))},
$$

$$
p'(z) = \frac{(D_w^{p+1} f(z))' \cdot (D_w^p f(z))'}{(D_w^p f(z))} - p(z) \cdot \frac{(D_w^p f(z))'}{(D_w^p f(z))},
$$

(3.5)

Thus we have

$$
(z - w) \cdot p'(z) = \frac{(D_w^{p+1} f(z))'}{(D_w^p f(z))} \cdot \frac{(z-w) \cdot (D_w^p f(z))'}{D_w^p f(z)} - p(z) \cdot \frac{(z-w) \cdot (D_w^p f(z))'}{D_w^p f(z)},
$$

$$
(z - w) \cdot p'(z) = \frac{(D_w^{p+1} f(z))'}{(D_w^p f(z))} \cdot p(z) - [p(z)]^2,
$$

$$
\frac{(D_w^{p+1} f(z))'}{(D_w^p f(z))} = p(z) + \frac{1}{p(z)} \cdot (z-w) \cdot p'(z).
$$

(3.6)

From $\Re(D_w^{p+2} f(z)/D_w^{p+1} f(z)) > 0$ we obtain $p(z) + (1/p(z)) \cdot (z-w) \cdot p'(z) < ((1 + z)/(1 - z))$ or

$$
p(z) + \frac{zp'(z)}{1/(1 - (w/z)) \cdot p(z)} < \frac{1+z}{1-z} \equiv h(z), \quad \text{with } h(0) = 1.
$$

(3.7)

By hypothesis, we have $\Re[1/(1 - (w/z)) \cdot h(z)] > 0$, and thus from Theorem 2.4 we obtain $p(z) < h(z)$ or $\Re p(z) > 0$. This means $f \in s_n^*(w)$. 

\[ \square \]

**Remark 3.6.** From Theorem 3.5, we obtain $s_n^*(w) \subset s_0^*(w) \subset S^*(w)$, $n \in \mathbb{N}$.

**Theorem 3.7.** If $F(z) \in s_n^*(w)$ then $f(z) = L_a F(z) \in S_n^*(w)$, where $L_a$ is the integral operator defined by (2.5).

**Proof.** From (2.5) we obtain

$$
(1 + a) \cdot F(z) = a \cdot f(z) + (z-w) \cdot f'(z).
$$

(3.8)

By means of the application of the operator $D_w^{p+1}$ we obtain

$$
(1 + a) \cdot D_w^{p+1} F(z) = a \cdot D_w^{p+1} f(z) + D_w^{p+1}[(z-w) \cdot f'(z)]
$$

(3.9)
or
\[(1 + a) \cdot D^{n+1}_w F(z) = a \cdot D^{n+1}_w f(z) + D^{n+2}_w f(z). \quad (3.10)\]

Similarly, by means of the application of the operator $D^n_w$, we obtain
\[(1 + a) \cdot D^n_w F(z) = a \cdot D^n_w f(z) + D^{n+1}_w f(z). \quad (3.11)\]

Thus
\[
\frac{D^{n+1}_w F(z)}{D^n_w F(z)} = \frac{(D^{n+2}_w f(z) / D^{n+1}_w f(z)) \cdot (D^{n+1}_w f(z) / D^n_w f(z)) + a \cdot (D^{n+1}_w f(z) / D^n_w f(z))}{(D^{n+1}_w f(z) / D^n_w f(z)) + a}. \quad (3.12)
\]

Using the notation $D^{n+1}_w f(z) / D^n_w f(z) = p(z)$, with $p(0) = 1$, we have
\[
\frac{(z - w) \cdot p'(z)}{p(z)} = \frac{D^{n+2}_w f(z)}{D^{n+1}_w f(z)} - p(z) \quad (3.13)
\]
or
\[
\frac{D^{n+2}_w f(z)}{D^{n+1}_w f(z)} = p(z) + \frac{(z - w) \cdot p'(z)}{p(z)}. \quad (3.14)
\]

Thus
\[
\frac{D^{n+1}_w F(z)}{D^n_w F(z)} = p(z) \left[ p(z) + \frac{(z - w) \cdot p'(z)}{p(z)} + a \right] \frac{p(z) + a}{p(z) + a} = p(z) + \frac{zp'(z)}{(1/(1 - (w/z))) \cdot p(z) + (a/(1 - (w/z)))}. \quad (3.15)
\]

From $F(z) \in s^*_n(w)$ we obtain $(D^{n+1}_w F(z) / D^n_w F(z)) < ((1 + z)/(1 - z)) \equiv h(z)$ or
\[
p(z) + \frac{zp'(z)}{(1/(1 - (w/z))) \cdot p(z) + (a/(1 - (w/z)))} < h(z). \quad (3.16)
\]

By hypothesis, we have $\text{Re}[(1/(1 - (w/z))) \cdot h(z) + (a/(1 - (w/z)))] > 0$ and from Theorem 2.4 we obtain $p(z) < h(z)$ or $\text{Re} \{D^{n+1}_w f(z) / D^n_w f(z)\} > 0$, $z \in U$. This means $f(z) = L_a F(z) \in S^*_n(w)$.

**Remark 3.8.** If we consider $w = 0$ in Theorem 3.7 we obtain that the integral operator defined by (2.5) preserves the class of $n$-starlike functions, and if we consider $w = 0$ and $n = 0$ in the above theorem we obtain that the integral operator defined by (2.5) preserves the well-known class of starlike functions.
Theorem 3.9. Let \( w \) be a fixed point in \( U \) and \( f \in S^*_n(w) \) with \( f(z) = (z - w) + \sum_{j=2}^\infty a_j \cdot (z - w)^j \). Then

\[
\begin{align*}
|a_2| & \leq \frac{1}{2^{n-1} \cdot (1 - d^2)}, \\
|a_3| & \leq \frac{3 + d}{3^n \cdot (1 - d^2)^2}, \\
|a_4| & \leq \frac{(2 + d)(3 + d)}{2^{2n-1} \cdot 3 \cdot (1 - d^2)^3}, \\
|a_5| & \leq \frac{(2 + d)(3 + d)(3d + 5)}{5^n \cdot 6 \cdot (1 - d^2)^4}, \tag{3.17}
\end{align*}
\]

where \( d = |w| \).

Proof. From Remark 3.4 for \( f \in S^*_n(w) \) we obtain

\[ D^n_w f(z) = g(z) \in S^*_n(w). \tag{3.18} \]

If we consider \( g(z) = (z - w) + \sum_{j=2}^\infty b_j \cdot (z - w)^j \), using Remark 3.2, from (3.18) we obtain \( j^n \cdot a_j = b_j, \quad j = 2, 3, \ldots \).

Thus we have \( a_j = 1/j^n \cdot b_j, \quad j = 2, 3, \ldots \), and from the estimates (2.3) we get the result. \( \square \)

References


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<thead>
<tr>
<th>Manuscript Due</th>
<th>May 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>August 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
</tr>
</tbody>
</table>

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