A NOTE INVOLVING $p$-VALENTLY BAZILEVIĆ FUNCTIONS

HÜSEYIN IRMAK, KRZYSZTOF PIEJKO, AND JAN STANKIEWICZ

Received 11 July 2004 and in revised form 6 December 2004

A theorem involving $p$-valently Bazilević functions is considered and then its certain consequences are given.

1. Introduction and definitions

Let \( \mathcal{A}_n(p) \) be the class of normalized functions of the form

\[
f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (n, p \in \mathbb{N} = \{1, 2, 3, \ldots\}),
\]

which are analytic and $p$-valent in the unit disc \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \). A function \( f \in \mathcal{A}_n(p) \) is said to be in the class \( \mathcal{F}_n(p, \alpha) \) if it satisfies the inequality

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p, \ p \in \mathbb{N}, \ z \in \mathbb{D}).
\]

(1.2)

Also a function \( f \in \mathcal{A}_n(p) \) is said to be a $p$-valently Bazilević function of type \( \beta \ (\beta \geq 0) \) and order \( \gamma \ (0 \leq \gamma < p; \ p \in \mathbb{N}) \) if there exists a function \( g \) belonging to the class \( \mathcal{F}_n(p) := \mathcal{F}_n(p, 0) \) such that

\[
\Re \left\{ \frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^\beta} \right\} > \gamma \quad (z \in \mathbb{D}).
\]

(1.3)

We denote the class of all such functions by \( \mathcal{B}_n(p, \beta, \gamma) \). In particular, when \( \beta = 1 \), a function \( f \in \mathcal{B}_n(p, 1, \gamma) := \mathcal{B}_n(p, 1) \) is said to be $p$-valently close-to-convex of order \( \gamma \) in \( \mathbb{D} \). Moreover, \( \mathcal{B}_n(p, 0, \gamma) =: \mathcal{F}_n(p, \gamma) \) when \( \beta = 0 \).

2. Main results and their consequences

We begin with the following lemma due to Jack [2].
Lemma 2.1. Let \( \omega(z) \) be nonconstant and regular in \( \mathbb{U} \) with \( \omega(0) = 0 \). If \( |\omega(z)| \) attains its maximum value on the circle \( |z| = r \) (0 < r < 1) at the point \( z_0 \), then \( z_0 \omega'(z_0) = c\omega(z_0) \), where \( c \geq 1 \).

With the aid of the above lemma, we prove the following result.

Theorem 2.2. Let \( f \in \mathcal{A}_n(p) \), \( w \in \mathbb{C} \setminus \{0\} \), \( \beta \geq 0 \), \( 0 \leq \alpha < p \), \( p \in \mathbb{N} \), \( z \in \mathbb{U} \), and also let the function \( \mathcal{H} \) be defined by

\[
\mathcal{H}(z) = \left( \frac{zf'(z)}{zf'(z) - p[f(z)]^{1-\beta}[g(z)]^\beta} \right) \cdot \left( 1 + \frac{zf''(z)}{f'(z)} - (1 - \beta) \frac{zf'(z)}{f(z)} - \beta \frac{g'(z)}{g(z)} \right),
\]

where \( g \in \mathcal{F}_n(p) \). If \( \mathcal{H}(z) \) satisfies one of the following conditions:

\[
\Re\{\mathcal{H}(z)\} \begin{cases} < |w|^{-2}\Re\{w\} & \text{when } \Re\{w\} > 0, \\ \neq 0 & \text{when } \Re\{w\} = 0, \\ > |w|^{-2}\Re\{w\} & \text{when } \Re\{w\} < 0, \end{cases}
\]

or

\[
\Im\{\mathcal{H}(z)\} \begin{cases} < |w|^{-2}\Im\{\bar{w}\} & \text{when } \Im\{\bar{w}\} > 0, \\ \neq 0 & \text{when } \Im\{\bar{w}\} = 0, \\ > |w|^{-2}\Im\{\bar{w}\} & \text{when } \Im\{\bar{w}\} < 0, \end{cases}
\]

then

\[
\left| \left( \frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^\beta} - p \right)^w \right| < p - \alpha,
\]

where the value of complex power in (2.4) is taken to be as its principal value.

Proof. We define the function \( \Omega \) by

\[
\left( \frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^\beta} - p \right)^w = (p - \alpha)\Omega(z),
\]

where \( \beta \geq 0 \), \( w \in \mathbb{C} \setminus \{0\} \), \( 0 \leq \alpha < p \), \( p \in \mathbb{N} \), \( z \in \mathbb{U} \), \( f \in \mathcal{A}_n(p) \), and \( g \in \mathcal{F}_n(p) \).

We see clearly that the function \( \Omega \) is regular in \( \mathbb{U} \) and \( \Omega(0) = 0 \). Making use of the logarithmic differentiation of both sides of (2.5) with respect to the known complex variable \( z \), and if we make use of equality (2.5) once again, then we find that

\[
wz \left( \frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^\beta} - p \right)^{-1} \left( \frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^\beta} - p \right)' = \frac{z\Omega'(z)}{\Omega(z)},
\]
which yields

\[ \mathcal{H}(z) := \frac{w z \Omega'(z)}{|w|^2 \Omega(z)} \quad (w \in \mathbb{C} \setminus \{0\}; \ z \in \mathcal{U}). \] (2.7)

Assume that there exists a point \( z_0 \in \mathcal{U} \) such that

\[ \max_{|z| \leq |z_0|} |\Omega(z)| = |\Omega(z_0)| = 1 \quad (z \in \mathcal{U}). \] (2.8)

Applying Lemma 2.1, we can then write

\[ z_0 \Omega'(z_0) = c \Omega(z_0) \quad (c \geq 1). \] (2.9)

Then (2.7) yields

\[ \Re\{\mathcal{H}(z_0)\} = \Re\left\{ \frac{w z_0 \Omega'(z_0)}{|w|^2 \Omega(z_0)} \right\} = \Re\{c |w|^{-2}\}, \] (2.10)

so that

\[ \Re\{\mathcal{H}(z_0)\} = \frac{c}{|w|^2} \Re\{\bar{w}\} \begin{cases} \geq |w|^{-2} \Re\{w\} & \text{if } \Re\{w\} > 0, \\ = 0 & \text{if } \Re\{w\} = 0, \\ \leq |w|^{-2} \Re\{w\} & \text{if } \Re\{w\} < 0, \end{cases} \] (2.11)

\[ \Im\{\mathcal{H}(z_0)\} = \frac{c}{|w|^2} \Im\{\bar{w}\} \begin{cases} \geq |w|^{-2} \Im\{\bar{w}\} & \text{if } \Im\{\bar{w}\} > 0, \\ = 0 & \text{if } \Im\{\bar{w}\} = 0, \\ \leq |w|^{-2} \Im\{\bar{w}\} & \text{if } \Im\{\bar{w}\} < 0. \end{cases} \] (2.12)

But the inequalities in (2.11) and (2.12) contradict, respectively, the inequalities in (2.2) and (2.3). Hence, we conclude that \( |\Omega(z)| < 1 \) for all \( z \in \mathcal{U} \). Consequently, it follows from (2.5) that

\[ \left| \left( \frac{z f'(z)}{[f(z)]^{1-\beta}[g(z)]^\beta} - p \right)^w \right| = (p - \alpha) |\Omega(z)| < p - \alpha. \] (2.13)

Therefore, the desired proof is completed. \( \square \)

This theorem has many interesting and important consequences in analytic function theory and geometric function theory. We give some of these with their corresponding geometric properties.

First, if we choose the value of the parameter \( w \) as a real number with \( w := \delta \in \mathbb{R} \setminus \{0\} \) in the theorem, then we obtain the following corollary.

**Corollary 2.3.** Let \( f \in A_n(p) \), \( \delta \in \mathbb{R} \setminus \{0\} \), \( \beta \geq 0 \), \( 0 \leq \alpha < p \), \( p \in \mathbb{N} \), \( z \in \mathcal{U} \), and let the function \( \mathcal{H} \) be defined by (2.1). Also, if \( \mathcal{H} \) satisfies the following conditions:

\[ \Re\{\mathcal{H}(z)\} \begin{cases} < |\delta|^{-2} & \text{when } \delta > 0, \\ > |\delta|^{-2} & \text{when } \delta < 0, \end{cases} \] (2.14)
then
\[
\text{Re}\left\{ \frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^\beta} \right\} > p - (p - \alpha)^{1/\delta}. \tag{2.15}
\]

Putting \( w = 1 \) in the theorem, we get the following corollary.

**Corollary 2.4.** Let \( f \in \mathcal{A}_n(p) \), \( g \in \mathcal{F}_n(p) \), \( \beta \geq 0 \), \( 0 \leq \alpha < p \), \( p \in \mathbb{N} \), \( z \in \mathbb{U} \), and let the function \( \mathcal{H} \) be defined by (2.1). If \( \mathcal{H}(z) \) satisfies one of the following conditions:
\[
\text{Re}\{\mathcal{H}(z)\} < 1 \quad \text{or} \quad \text{Im}\{\mathcal{H}(z)\} \neq 0, \tag{2.16}
\]
then \( f \in \mathcal{B}_n(p, \beta, \alpha) \), that is, \( f \) is a \( p \)-valently Bazilevi\'c function of type \( \beta \) and order \( \alpha \) in \( \mathbb{U} \).

Setting \( w = 1 \) and \( \beta = 0 \) in the theorem, we have the following corollary.

**Corollary 2.5.** Let \( f \in \mathcal{A}_n(p) \), \( 0 \leq \alpha < p \), \( p \in \mathbb{N} \), \( z \in \mathbb{U} \), and let the function \( \mathcal{G} \) be defined by
\[
\mathcal{G}(z) = \left( \frac{zf'(z)}{zf'(z) - pf(z)} \right) \left( 1 + \frac{zf''(z)}{zf'(z)} - \frac{zf'(z)}{f(z)} \right). \tag{2.17}
\]
If \( \mathcal{G}(z) \) satisfies one of the following conditions:
\[
\text{Re}\{\mathcal{G}(z)\} < 1 \quad \text{or} \quad \text{Im}\{\mathcal{G}(z)\} \neq 0, \tag{2.18}
\]
then \( f \in \mathcal{F}_n(p, \alpha) \), that is, \( f \) is \( p \)-valently starlike of order \( \alpha \) in \( \mathbb{U} \).

By taking \( w = 1 \) and \( \beta = 1 \) in the theorem, we obtain the following corollary.

**Corollary 2.6.** Let \( f \in \mathcal{A}_n(p) \), \( g \in \mathcal{F}_n(p) \), \( 0 \leq \alpha < p \), \( p \in \mathbb{N} \), \( z \in \mathbb{U} \), and let the function \( \mathcal{F} \) be defined by
\[
\mathcal{F}(z) = \left( \frac{zf'(z)}{zf'(z) - pg(z)} \right) \left( 1 + \frac{zf''(z)}{zf'(z)} - \frac{zf'(z)}{g(z)} \right). \tag{2.19}
\]
If \( \mathcal{F}(z) \) satisfies one of the following conditions:
\[
\text{Re}\{\mathcal{F}(z)\} < 1 \quad \text{or} \quad \text{Im}\{\mathcal{F}(z)\} \neq 0, \tag{2.20}
\]
then \( f \in \mathcal{K}_n(p, \alpha) \), that is, \( f \) is \( p \)-valently close-to-convex of order \( \alpha \) in \( \mathbb{U} \).

Lastly, if we take \( p = 1 \) in Corollaries 2.4, 2.5, and 2.6, then we easily obtain the three important results involving Bazilevi\'c functions of type \( \beta \) (\( \beta \geq 0 \)) and order \( \alpha \) (\( 0 \leq \alpha < 1 \)) in \( \mathbb{U} \), starlike functions of order \( \alpha \) (\( 0 \leq \alpha < 1 \)) in \( \mathbb{U} \), and close-to-convex functions of order \( \alpha \) (\( 0 \leq \alpha < 1 \)) in \( \mathbb{U} \), respectively, (see, e.g., [1, 3, 4, 5]).
Acknowledgments

This work has been carried out by the help of four-month financial support (June–September, 2004) from the TÜBİTAK (The Scientific and Technical Research Council of Turkey) which is given to the first author during the scientific research at the University of Rzeszów and Rzeszów University of Technology in Poland. This present investigation was also supported by NATO and Baškent University (Ankara, Turkey). The first author would also like to acknowledge Professor Mehmet Haberal, Rector of Baškent University, who generously supports scientific researches in all aspects. I would like to extend my thanks to Professor J. Stankiewicz and Professor J. Dziok for their kind invitation to Poland and their invaluable support for this research.

References


Hüseyin Irmak: Department of Mathematics Education, Faculty of Education, Baškent University, Bağlıca Campus, 06530 Etimesgut, Ankara, Turkey
E-mail address: hisimya@baskent.edu.tr

Krzysztof Piejko: Department of Mathematics, Faculty of Management and Marketing, Rzeszów University of Technology, 2 Wincentego Pola Street, 35-959 Rzeszów, Poland
E-mail address: piejko@prz.rzeszow.pl

Jan Stankiewicz: Department of Mathematics, Faculty of Management and Marketing, Rzeszów University of Technology, 2 Wincentego Pola Street, 35-959 Rzeszów, Poland; Institute of Mathematics, University of Rzeszów, 16A Rejtana Street, 35-310 Rzeszów, Poland
E-mail address: jstan@prz.rzeszow.pl
Special Issue on
Decision Support for Intermodal Transport

Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

Before submission authors should carefully read over the journal’s Author Guidelines, which are located at http://www.hindawi.com/journals/jamds/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>June 1, 2009</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>September 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>December 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Gerrit K. Janssens, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; Gerrit.Janssens@uhasselt.be

Guest Editor

Cathy Macharis, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; Cathy.Macharis@vub.ac.be