We introduce the concepts of lifting modules and (quasi-)discrete modules relative to a given left module. We also introduce the notion of SSRS-modules. It is shown that (1) if $M$ is an amply supplemented module and $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ an exact sequence, then $M$ is $N$-lifting if and only if it is $N'$-lifting and $N''$-lifting; (2) if $M$ is a Noetherian module, then $M$ is lifting if and only if $M$ is $R$-lifting if and only if $M$ is an amply supplemented SSRS-module; and (3) let $M$ be an amply supplemented SSRS-module such that $\text{Rad}(M)$ is finitely generated, then $M = K \oplus K'$, where $K$ is a radical module and $K'$ is a lifting module.

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1. Introduction and preliminaries

Extending modules and their generalizations have been studied by many authors (see [2, 3, 8, 7]). The motivation of the present discussion is from [2, 8], where the concepts of extending modules and (quasi-)continuous modules with respect to a given module and CESS-modules were studied, respectively. In this paper, we introduce the concepts of lifting modules and (quasi-)discrete modules relative to a given module and SSRS-modules. It is shown that (1) if $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ is an exact sequence and $M$ an amply supplemented module, then $M$ is $N$-lifting if and only if it is both $N'$-lifting and $N''$-lifting; (2) if $M$ is a Noetherian module, then $M$ is lifting if and only if $M$ is $R$-lifting if and only if $M$ is an amply supplemented SSRS-module; and (3) let $M$ be an amply supplemented SSRS-module such that $\text{Rad}(M)$ is finitely generated, then $M = K \oplus K'$, where $K$ is a radical module and $K'$ is a lifting module.

Throughout this paper, $R$ is an associative ring with identity and all modules are unital left $R$-modules. We use $N \leq M$ to indicate that $N$ is a submodule of $M$. As usual, $\text{Rad}(M)$ and $\text{Soc}(M)$ stand for the Jacobson radical and the socle of a module $M$, respectively.

Let $M$ be a module and $S \leq M$. $S$ is called small in $M$ (notation $S \ll M$) if $M \neq S + T$ for any proper submodule $T$ of $M$. Let $N$ and $L$ be submodules of $M$, $N$ is called a supplement of $L$ in $M$ if $N + L = M$, and $N$ is minimal with respect to this property. Equivalently,
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\[ M = N + L \text{ and } N \cap L \ll N. \]  
\( N \) is called a \textit{supplement submodule} if \( N \) is a supplement of some submodule of \( M \). \( M \) is called an \textit{amply supplemented} module if for any two submodules \( A \) and \( B \) of \( M \) with \( A + B = M \), \( B \) contains a supplement of \( A \). \( M \) is called a \textit{weakly supplemented module} (see [5]) if for each submodule \( A \) of \( M \) there exists a submodule \( B \) of \( M \) such that \( M = A + B \) and \( A \cap B \ll M \). Let \( B \leq A \leq M \). If \( A/B \ll M/B \), then \( B \) is called a \textit{coessential submodule} of \( A \) and \( A \) is called a \textit{coessential extension} of \( B \) in \( M \). A submodule \( A \) of \( M \) is called \textit{coclosed} if \( A \) has no proper coessential submodules in \( M \). Let \( B \leq A \leq M \). If \( A/B \ll M/B \), then \( B \) is called a \textit{coessential submodule} of \( A \) and \( A \) is called a \textit{coessential extension} of \( B \) in \( M \).

Let \( B \leq A \leq M \). If \( A/B \ll M/B \), then \( B \) is called a \textit{coessential submodule} of \( A \) and \( A \) is called a \textit{coessential extension} of \( B \) in \( M \).

\( A \) is called \textit{coclosed} if \( A \) has no proper coessential submodules in \( M \). Following [5], \( B \) is called an \textit{s-closure} of \( A \) in \( M \) if \( B \) is a coessential submodule of \( A \) and \( B \) is coclosed in \( M \).

Lemma 1.1 (see [12, 19.3]). Let \( M \) be a module and \( K \leq L \leq M \).

1. \( L \ll M \) if and only if \( K \ll M \) and \( L/K \ll M/K \).
2. If \( M' \) is a module and \( \phi : M \to M' \) a homomorphism, then \( \phi(L) \ll M' \) whenever \( L \ll M \).

Lemma 1.2 (see Lemma 1.1 in [5]). Let \( M \) be a weakly supplemented module and \( N \leq M \). Then the following statements are equivalent.

1. \( N \) is a supplement submodule of \( M \).
2. \( N \) is coclosed in \( M \).
3. For all \( X \leq N \), \( X \ll M \) implies \( X \ll N \).

Lemma 1.3 (see Proposition 1.5 in [5]). Let \( M \) be an amply supplemented module. Then every submodule of \( M \) has an \textit{s-closure}.

Lemma 1.4 (see [12, 41.7]). Let \( M \) be an amply supplemented module. Then every coclosed submodule of \( M \) is amply supplemented.

2. Relative lifting modules

To define the concepts of relative lifting and (quasi-)discrete modules, we dualize the concepts of relative extending and (quasi-)continuous modules introduced in [8] in this section. We start with the following.

Let \( N \) and \( M \) be modules. We define the family

\[ $(N,M) = \left\{ A \leq M \mid \exists X \leq N, \exists f \in \text{Hom}(X,M), \exists A f(X) \ll M f(X) \right\}. \]  
(2.1)
Proposition 2.1. $(N,M)$ is closed under small submodules, isomorphic images, and coessential extensions.

Proof. We only show that $(N,M)$ is closed under coessential extensions. Let $A \in (N,M)$, $A \preceq A' \preceq M$ and $A'/A \ll M/A$. There exist $X \subseteq N$ and $f \in \text{Hom}(X,M)$ such that $f(X) \preceq A$ and $A/f(X) \ll M/f(X)$ since $A \in (N,M)$. Note that $A'/A \ll M/A$, so $A'/f(X) \ll M/f(X)$ by Lemma 1.1(1). Thus $A' \in (N,M)$. \hfill $\square$

Lemma 2.2. Let $A \in (N,M)$ and $A$ be coclosed in $M$. Then $B \in (N,M)$ for any submodule $B$ of $A$.

Proof. There exist $X \subseteq N$ and $f \in \text{Hom}(X,M)$ such that $f(X) \preceq A$ and $A/f(X) \ll M/f(X)$ by hypothesis. Since $A$ is coclosed in $M$, $f(X) = A$. Let $B$ be any submodule of $A$ and $Y = f^{-1}(B) \subseteq X \subseteq N$. Then $f|_Y : Y \rightarrow M$ is a homomorphism such that $f|_Y(Y) = B$ for $f(X) = A$. Clearly $B/f|_Y(Y) \ll M/f|_Y(Y)$. Therefore $B \in (N,M)$. \hfill $\square$

Lemma 2.3. Let $C \preceq A \preceq B \preceq M$ and $A$ be a coessential submodule of $B$. If $C$ is an $s$-closure of $A$, then it is also an $s$-closure of $B$.

Proof. It is clear by Lemma 1.1(1). \hfill $\square$

Proposition 2.4. Let $M$ be an amply supplemented module. Then every $A$ in $(N,M)$ has an $s$-closure $\overline{A}$ in $(N,M)$.

Proof. Since $A \in (N,M)$, there exist $X \subseteq N$ and $f \in \text{Hom}(X,M)$ such that $A/f(X) \ll M/f(X)$. Note that $M$ is amply supplemented, and so $f(X)$ has an $s$-closure $\overline{A}$ in $M$ by Lemma 1.3. Thus $\overline{A}$ is also an $s$-closure of $A$ by Lemma 2.3. The verification for $\overline{A} \in (N,M)$ is analogous to that for $B \in (N,M)$ in Lemma 2.2. \hfill $\square$

Let $N$ be a module. Consider the following conditions for a module $M$.

\begin{itemize}
  \item (\$N,M\$)-D$_1$ For every submodule $A \in (N,M)$, there exists a direct summand $K$ of $M$ such that $K \preceq A$ and $A/K \ll M/K$.
  \item (\$N,M\$)-D$_2$ If $A \in (N,M)$ such that $M/A$ is isomorphic to a direct summand of $M$, then $A$ is a direct summand of $M$.
  \item (\$N,M\$)-D$_3$ If $A$ and $L$ are direct summands of $M$ with $A \in (N,M)$ and $A + L = M$, then $A \cap L$ is a direct summand of $M$.
\end{itemize}

Definition 2.5. Let $N$ be a module. A module $M$ is said to be $N$-lifting, $N$-discrete, or $N$-quasidiscrete if $M$ satisfies $(N,M)$-D$_1$, $(N,M)$-D$_1$ and $(N,M)$-D$_2$ or $(N,M)$-D$_1$ and $(N,M)$-D$_3$, respectively.

One easily obtains the hierarchy: $M$ is $N$-discrete $\Rightarrow$ $M$ is $N$-quasidiscrete $\Rightarrow$ $M$ is $N$-lifting. Clearly, the notion of relative discreteness generalizes the concept of discreteness. For any module $N$, lifting modules are $N$-lifting. But the converse is not true as shown in the following examples.

Example 2.6. Since, for any module $M$, $(0,M) = \{A \mid A \ll M\}$ and $0$ is a direct summand of $M$ such that $A/0 \ll M/0$ for any $A \in (0,M)$, all modules are $0$-lifting. However, the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ is not lifting since the supplement submodule $\langle(1,2)\rangle$
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$\langle (1,2) \rangle$ is a supplement of $\langle (1,1) \rangle$ and is not a direct summand of it though it is amply supplemented.

Example 2.7. Let $M$ be a module with zero socle and $S$ a simple module. Then $M$ is $S$-lifting since $\$(S,M)$ is a family only containing all small submodules of $M$. So all torsion-free $\mathbb{Z}$-modules are $S$-lifting for any simple $\mathbb{Z}$-module $S$ (see [12, Exercise 21.17]). In particular, $\mathbb{Z}$ and $\mathbb{Q}$ are $S$-lifting for any simple $\mathbb{Z}$-module, but each one is not a lifting module.

Lemma 2.8. Let $M$ be a module. Then $\$(M,M) = \{A \mid A \leq M\}$ $\bigcup_{N \in \text{R-Mod}} \$(N,M)$, where $\text{R-Mod}$ denotes the category of left $\text{R}$-module.

Proof. It is straight forward. □

Proposition 2.9. Let $M$ be a module. Then $M$ is lifting or (quasi-)discrete if and only if $M$ is $M$-lifting or $M$-(quasi-)discrete if and only if $M$ is $N$-lifting or $N$-(quasi-)discrete for any module $N$.

Proof. It is clear by Lemma 2.8. □

Proposition 2.10. Let $M$ be an amply supplemented module. Then the condition $\$(N,M)$-D_i$ is inherited by coclosed submodules of $M$.

Proof. Let $M$ satisfy $\$(N,M)$-D_i$ and $H$ be a coclosed submodule of $M$. $H$ is amply supplemented by Lemma 1.4. For any $A \in \$(N,H)$, $A$ has an s-closure $\overline{A} \in \$(N,H)$ in $H$ by Proposition 2.4. Since $\overline{A} \in \$(N,H)$ $\subseteq \$(N,M)$ and $M$ satisfies $\$(N,M)$-D_i$, there is a direct summand $K$ of $M$ such that $K \leq \overline{A}$ and $\overline{A}/K \ll M/K$. By Lemma 1.2, $\overline{A}/K \ll H/K$. Now $\overline{A} = K$ since $\overline{A}$ is coclosed in $H$. Thus $H$ satisfies $\$(N,H)$-D_i$. □

Corollary 2.11. Let $M$ be an amply supplemented module. Then the condition $\$(N,M)$-D_i$ is inherited by direct summands of $M$.

Proposition 2.12. Let $M$ be an amply supplemented module. Then $\$(N,M)$-D_i$ ($i = 2, 3$) is inherited by direct summands of $M$.

Proof. (1) Let $M$ satisfy $\$(N,M)$-D_2$ and $H$ be a direct summand of $M$. We will show that $H$ satisfies $\$(N,H)$-D_2$.

Let $A \in \$(N,H)$ $\subseteq \$(N,M)$ and $H/A$ is isomorphic to a direct summand of $H$. Since $H$ is a direct summand of $M$, there exists $H' \leq M$ such that $M = H \oplus H'$. Thus $M/A = (H \oplus H')/A \cong (H/A) \oplus H'$, and so $M/A$ is isomorphic to a direct summand of $M$. $A$ is a direct summand of $M$ since $M$ satisfies $\$(N,M)$-D_2$, and hence $A$ is a direct summand of $H$.

(2) Let $A \in \$(N,H)$ $\subseteq \$(N,M)$ and $A$, $L$ be direct summands of $H$ with $A + L = H$. We will show that $A \cap L$ is a direct summand of $H$. Since $H$ is a direct summand of $M$, there exists $H' \leq M$ such that $M = H \oplus H'$. Thus $M = (A + L) \oplus H' = A + (L \oplus H')$. Now $A \cap (L \oplus H')$ is a direct summand of $M$ since $M$ satisfies $\$(N,M)$-D_3$. Note that $A \cap (L \oplus H') = A \cap L$, so $A \cap L$ is a direct summand of $H$. □

Theorem 2.13. Let $M$ be an amply supplemented module and $A \in \$(N,M)$ a direct summand of $M$. If $M$ is $N$-(quasi-)discrete, then $A$ is (quasi-)discrete.
**Proof.** The proof follows from Lemma 2.2, Corollary 2.11, and Proposition 2.12. \(\square\)

**Proposition 2.14.** Let \(0 \to N' \to N \to N'' \to 0\) be an exact sequence. Then \(\$N',M\) \(\cup\) \(\$N'',M\) \(\subseteq\) \(\$N,M\). Therefore, if \(M\) is \(N'\)-lifting (resp., (quasi-)discrete), then \(M\) is \(N''\)-lifting and \(N''\)-lifting (resp., (quasi-)discrete).

**Proof.** Without loss of generality we can assume that \(N' \leq N\) and \(N'' = N/N'\). By definition, \(N' \leq N\) implies \(\$N',M\) \(\subseteq\) \(\$N,M\). Next, let \(A_2 \in \$N'',M\). Then there exist \(X \leq N'' = N/N'\) and \(f \in \text{Hom}(X,M)\) such that \(A_2/f(X) \ll M/f(X)\). Write \(X = Y/N'\), \(Y \leq N\) and let \(\delta : Y \to Y/N'\) be the canonical homomorphism. It is clear that \(g = f\delta \in \text{Hom}(Y,M)\) and \(g(Y) = f(X)\), hence \(A_2/g(Y) \ll M/g(Y)\). Thus \(A_2 \in \$N,M\). Therefore \(\$N',M\) \(\cup\) \(\$N'',M\) \(\subseteq\) \(\$N,M\). The rest is obvious. \(\square\)

Dual to [8, Proposition 2.7], we have the following.

**Theorem 2.15.** Let \(0 \to N' \to N \to N'' \to 0\) be an exact sequence and \(M\) an amply supplemented module. Then \(M\) is \(N'\)-lifting if and only if it is both \(N'\)-lifting and \(N''\)-lifting.

**Proof.** Let \(M\) be \(N'\)-lifting. Then it is both \(N'\)-lifting and \(N''\)-lifting by Proposition 2.14. Conversely suppose that \(M\) is both \(N'\)-lifting and \(N''\)-lifting. For any submodule \(A \in \$N,M\), \(A\) has an \(s\)-closure \(\tilde{A} \in \$N,M\) by Proposition 2.4. Since \(\tilde{A} \in \$N,M\), there exist \(X \leq N\) and \(f \in \text{Hom}(X,M)\) such that \(\tilde{A}/f(X) \ll M/f(X)\). Since \(\tilde{A}\) is coclosed in \(M\), \(f(X) = \tilde{A}\). Write \(Y = X \cap N' \leq N'\) and \(f|_Y : Y \to M\) is a homomorphism, then \(f(Y) \leq f(X) = \tilde{A}\). Let \(\overline{f(Y)}\) be an \(s\)-closure of \(f(Y)\) in \(\tilde{A}\) (for \(\tilde{A}\) is amply supplemented). Thus we conclude that \(f(Y)/\overline{f(Y)} \ll M/\tilde{A}\) and \(\overline{f(Y)} \in \$N',M\). Since \(M\) is \(N'\)-lifting, there exists a direct summand \(K\) of \(M\) such that \(\overline{f(Y)}/K \ll M/K\). It is easy to see \(\overline{f(Y)}\) is coclosed in \(M\), hence \(\overline{f(Y)} = K\) is a direct summand of \(M\). Write \(M = \overline{f(Y)} \oplus K'\), \(K' \leq M\) and \(\tilde{A} = \tilde{A} \cap M = \overline{f(Y)} \oplus (\tilde{A} \cap K')\). Define \(h : W = (X + N')/N' \to M\) by \(h(x + N') = \pi f(x)\), where \(\pi : \tilde{A} \to \tilde{A} \cap K'\) denotes the canonical projection. It is clear that \(h(W) = \tilde{A} \cap K'\), thus \((\tilde{A} \cap K')/h(W) \ll M/h(W)\), and hence \((\tilde{A} \cap K') \in \$N'',M\). Since \(M\) is \(N''\)-lifting, there exists a direct summand \(K''\) of \(M\) such that \((\tilde{A} \cap K')/K'' \ll M/K''\). Since \(\tilde{A} \cap K'\) is coclosed in \(M\), \(\tilde{A} \cap K' = K''\). Now \(\tilde{A} \cap K'\) is a direct summand of \(K'\). Thus \(\tilde{A}\) is a direct summand of \(M\). It follows that \(M\) is \(N'\)-lifting. \(\square\)

**Corollary 2.16.** Let \(M\) be an amply supplemented module. If \(M\) is \(N_i\)-lifting for \(i = 1,2,\ldots,n\) and \(N = \bigoplus^n_i N_i\), then \(M\) is \(N\)-lifting.

**Corollary 2.17.** Let \(M\) be an amply supplemented module. Then \(M\) is lifting if and only if \(M\) is \(N\)-lifting and \(M/N\)-lifting for every submodule \(N\) of \(M\) if and only if \(M\) is \(N\)-lifting and \(M/N\)-lifting for some submodule \(N\) of \(M\).

Recall that a module \(M\) is said to be **distributive** if \(N \cap (K + L) = (N \cap K) + (N \cap L)\) for all submodules \(N, K, L\) of \(M\). A module \(M\) has SSP (see [4]) if the sum of any pair of direct summands of \(M\) is a direct summand of \(M\).

**Corollary 2.18.** Let \(0 \to N' \to N \to N'' \to 0\) be an exact sequence and let \(M\) be a distributive and amply supplemented module with SSP. If \(M\) is both \(N'\)-quasidiscrete and \(N''\)-quasidiscrete, then \(M\) is \(N\)-quasidiscrete.
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Proof. We only need to show that $M$ satisfies $(N,M)\text{-}D_3$ when $M$ satisfies $(N',M)\text{-}D_3$ and $(N'',M)\text{-}D_3$ by Theorem 2.15. Let $A \in (N,M)$ and $A,H$ be direct summands of $M$ with $A + H = M$. We know that $A = A_1 \oplus A_2$, where $A_1 \in (N',M)$, $A_2 \in (N'',M)$ from the proof of Theorem 2.15. Since $M$ is a distributive module with SSP, $A_1 \cap H$ and $A_2 \cap H$ are direct summands of $M$. This implies that $A \cap H$ is a direct summand of $M$. Thus $M$ satisfies $(N,M)\text{-}D_3$. □

3. SSRS-modules

In [2], a module is called a CESS-module if every complement with essential socle is a direct summand. As a dual of CESS-modules, the concept of SSRS-modules is given in this section. It is proven that: (1) let $M$ be an amply supplemented SSRS-module such that $\text{Rad}(M)$ is finitely generated, then $M = K \oplus K'$, where $K$ is a radical module and $K'$ is a lifting module; (2) let $M$ be a finitely generated amply supplemented module, then $M$ is an SSRS-module if and only if $M/K$ is a lifting module for every coclosed submodule $K$ of $M$.

Definition 3.1. A module is called an SSRS-module if every supplement with small radical is a direct summand.

Lifting modules are SSRS-modules, but the converse is not true. For example, $\mathbb{Z}\mathbb{Z}$ is an SSRS-module which is not a lifting module.

Proposition 3.2. Let $M$ be an SSRS-module. Then any direct summand of $M$ is an SSRS-module.

Proof. Let $K$ be a direct summand of $M$ and $N$ a supplement submodule of $K$ such that $\text{Rad}(N) \ll N$. Let $N$ be a supplement of $L$ in $K$, that is, $N + L = K$ and $N \cap L \ll N$. Since $K$ is a direct summand of $M$, there exists $K' \leq M$ such that $M = K \oplus K'$. Note that $M = N + (L \oplus K')$ and $N \cap (L \oplus K') = N \cap L \ll N$. Therefore $N$ is a supplement of $L \oplus K'$ in $M$. Thus $N$ is a direct summand of $M$ since $M$ is an SSRS-module. So $N$ is a direct summand of $K$. The proof is complete. □

Proposition 3.3. Let $M$ be a weakly supplemented SSRS-module and $K$ a coclosed submodule of $M$. Then $K$ is an SSRS-module.

Proof. It follows from the assumption and [4, Lemma 2.6(3)]. □

Proposition 3.4. Let $M$ be an amply supplemented module. Then $M$ is an SSRS-module if and only if for every submodule $N$ with small radical, there exists a direct summand $K$ of $M$ such that $K \leq N$ and $N/K \ll M/K$.

Proof. “$\Leftarrow$” Let $N$ be a supplement submodule with small radical. By assumption, there exists a direct summand $K$ of $M$ such that $K \leq N$ and $N/K \ll M/K$. Since $N$ is coclosed in $M$, $N = K$. Thus $N$ is a direct summand of $M$.

“$\Rightarrow$” Let $N \leq M$ with $\text{Rad}(N) \ll N$. There exists an $s$-closure $\overline{N}$ of $N$ since $M$ is amply supplemented. Since $\text{Rad}(N) \ll M$ (for $\text{Rad}(N) \ll N$) and $\text{Rad}(\overline{N}) \leq \text{Rad}(N)$,
Rad(N) ≪ N and N is a supplement submodule by Lemma 1.2. Therefore N is a direct summand of M by assumption. This completes the proof. □

Corollary 3.5. Let M be an amply supplemented SSRS-module. Then every simple submodule of M is either a direct summand or a small submodule of M.

Proposition 3.6. Let M be an amply supplemented module. Then M is an SSRS-module if and only if for every submodule N of M, every s-closure of N with small radical is a lifting module and a direct summand of M.

Proof. It is straight forward. □

Proposition 3.7. Let M be an amply supplemented SSRS-module. Then M = K ⊕ K', where K is semisimple and K' has small socle.

Proof. For Soc(M), there exists a direct summand K of M such that Soc(M)/K ≪ M/K by Proposition 3.4. It is easy to see that K is semisimple. Since K is a direct summand of M, there exists K' ≤ M such that M = K ⊕ K'. Note that Soc(M) = Soc(K) ⊕ Soc(K'). So Soc(M)/K = (K ⊕ Soc(K'))/K ≪ M/K = (K ⊕ K')/K. Thus Soc(K') ≪ K'. □

Recall that a module M is called a radical module if Rad(M) = M. Dual to [2, Theorem 2.6], we have the following.

Theorem 3.8. Let M be an amply supplemented SSRS-module such that Rad(M) is finitely generated. Then M = K ⊕ K', where K is a radical module and K' is a lifting module.

Proof. Rad(Rad(M)) ≪ Rad(M) since Rad(M) is finitely generated. There exists a direct summand K of M such that Rad(M)/K ≪ M/K by Proposition 3.4. Since K is a direct summand of M, there exists K' ≤ M such that M = K ⊕ K'. Note that Rad(M) = Rad(K) ⊕ Rad(K'). Therefore M = K ∩ Rad(M) = Rad(K) and Rad(M)/K = (Rad(K) ⊕ Rad(K'))/K ≪ M/K = (K ⊕ K')/K. Thus Rad(K) = K and Rad(K') ≪ K'.

Next, we show that K' is a lifting module. K' is amply supplemented since it is a direct summand of M. So we only prove that every supplement submodule of K' is a direct summand of K'. Let N be a supplement submodule of K'. By Lemma 1.2 and Rad(K') ≪ K', we know that Rad(N) ≪ N. N is a direct summand of K' since K' is an SSRS-module by Proposition 3.2. The proof is complete. □

Corollary 3.9. Let M be an amply supplemented module with small radical. Then M is an SSRS-module if and only if M is a lifting module.

Theorem 3.10. Let M be a finitely generated amply supplemented module. Then the following statements are equivalent.

(1) M is an SSRS-module.
(2) M is a lifting module.
(3) M/K is a lifting module for every coclosed submodule K of M.

Proof. (1) ⇒ (2) follows from Corollary 3.9.
(3) ⇒ (1) is clear.
(1) ⇒ (3) we only prove that any supplement submodule of M/K is a direct summand. Let A/K be a supplement submodule of M/K. A is coclosed in M since A/K is coclosed in
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\( M/K \) and \( K \) is coclosed in \( M \). \( \text{Rad}(A) \ll A \) since \( M \) is finitely generated and \( A \) is coclosed in \( M \). \( A \) is a direct summand of \( M \) by assumption. Thus \( A/K \) is a direct summand of \( M/K \).

Lemma 3.11. Let \( M \) be a module. Then the following statements are equivalent.

1. For every cyclic submodule \( N \) of \( M \), there exists a direct summand \( K \) of \( M \) such that \( K \leq N \) and \( N/K \ll M/K \).
2. For every finitely generated submodule \( N \) of \( M \), there exists a direct summand \( K \) of \( M \) such that \( K \leq N \) and \( N/K \ll M/K \).

Proof. See [12, 41.13].

Corollary 3.12. Let \( M \) be a Noetherian module. Then the following statements are equivalent.

1. \( M \) is \( R \)-lifting.
2. \( M \) is \( F \)-lifting, for any free module \( F \).
3. \( M \) is lifting.
4. \( M \) is an amply supplemented SSRS-module.

Proof. It is easy to see that \( (R, M) \) and \( (F, M) \) are closed under cyclic submodules. The rest follows immediately from Theorem 3.10 and Lemma 3.11.

Corollary 3.13. Let \( R \) be a left perfect (semiperfect) ring. Then every SSRS-module (finitely generated SSRS-module) is a lifting module.

Proof. It follows from the fact that every module over a left perfect ring has small radical, [11, Theorems 1.6 and 1.7] and Corollary 3.9.

A module \( M \) is uniserial (see [6]) if its submodules are linearly ordered by inclusion and it is serial if it is a direct sum of uniserial submodules. A ring \( R \) is right (left) serial if the right (left) \( R \)-module \( R_R(R) \) is serial and it is serial if it is both right and left serial.

Corollary 3.14. The following statements are equivalent for a ring \( R \) with radical \( J \).

1. \( R \) is an artinian serial ring and \( J^2 = 0 \).
2. \( R \) is a left semiperfect ring and every finitely generated module is an SSRS-module.
3. \( R \) is a left perfect ring and every module is an SSRS-module.

Proof. It holds by [6, Theorem 3.15], [10, Theorem 1 and Proposition 2.13], and Corollary 3.13.

References


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Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics. It means topics where a satellite, a rocket, or any kind of man-made object is travelling in space governed by the gravitational forces of celestial bodies and/or forces generated by propulsion systems that are available in those objects. Many topics are related to orbit determination, propagation, and orbital maneuvers related to those spacecrafts. Several other topics that are related to this subject are numerical methods, nonlinear dynamics, chaos, and control.

The main objective of this Special Issue is to publish topics that are under study in one of those lines. The idea is to get the most recent researches and published them in a very short time, so we can give a step in order to help scientists and engineers that work in this field to be aware of actual research. All the published papers have to be peer reviewed, but in a fast and accurate way so that the topics are not outdated by the large speed that the information flows nowadays.

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