ON AN EXTENSION OF SINGULAR INTEGRALS ALONG MANIFOLDS OF FINITE TYPE

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We extend the $L^p$-boundedness of a class of singular integral operators under the $H^1$ kernel condition on a compact manifold from the homogeneous Sobolev space $L^p_0(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$.

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1. Introduction

Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$, $n \geq 2$, with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x')$ be a homogeneous function of degree 0, with $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \quad (1.1)$$

where $x' = x/|x|$ for any $x \neq 0$.

Suppose that $h$ is an $L^\infty(\mathbb{R}^+)$ function; the singular integral operator $SI_{\Omega,h}$ is defined by

$$SI_{\Omega,h}(f)(x) = p.v. \int_{\mathbb{R}^n} h(|y|) \frac{\Omega(y')}{|y|^n} f(x - y)dy \quad (1.2)$$

for all test functions $f$, where $y' = y/|y| \in S^{n-1}$.

We denote $SI_{\Omega,h}(f)$ by $SI_{\Omega}(f)$ if $h = 1$. The operator $SI_{\Omega}$ was first studied by Calderón and Zygmund in their well-known papers (see [1, 2]). They proved that $SI_{\Omega}$ is $L^p(\mathbb{R}^n)$ bounded, $1 < p < \infty$, provided that $\Omega \in L\log^+L(S^{n-1})$ satisfying (1.1). They also showed that the space $L\log^+L(S^{n-1})$ cannot be replaced by any Orlicz space $L^\phi(S^{n-1})$ with a monotonically increasing function $\phi$ satisfying $\phi(t) = o(t\log t)$, $t \to \infty$, that is, $L(\log^+L)^{-1+\varepsilon}(S^{n-1})$, $0 < \varepsilon \leq 1$. The idea of their proof was as follows.

Suppose that $\Omega \in L^1(S^{n-1})$ is an odd function, then one can easily show that

$$SI_{\Omega}(f)(x) = \frac{1}{2} \int_{S^{n-1}} \Omega(y') \left\{ \int_{-\infty}^{\infty} f(x - ty')t^{-1}dt \right\} d\sigma(y'). \quad (1.3)$$
By the method of rotation and the well-known $L^p$-boundedness of the Hilbert transform, one then obtains the $L^p$-boundedness of $SI_\Omega$ under the weak condition $\Omega \in L^1(S^{n-1})$.

For even kernels, the condition $\Omega \in L^1(S^{n-1})$ is insufficient. It turns out that the right condition is $\Omega \in L^{\log^+}L(S^{n-1})$, as far as the size of $\Omega$ is concerned. The idea of Calderón and Zygmund is to compose the operator $SI_\Omega$ with the Riesz transforms $R_j$, $1 \leq j \leq n$, and to show that $R_j(SI_\Omega)$ is a singular integral operator with an appropriate odd kernel. Thus

$$\|R_j(SI_\Omega)(f)\|_p \leq C_p \|f\|_p \quad (1.4)$$

for all test functions $f \in \mathcal{D}$. Furthermore, one can obtain

$$\|SI_\Omega(f)\|_p = \left\| \sum_{j=1}^n R_j^2 SI_\Omega(f) \right\|_p \leq \sum_{j=1}^n \|R_j(R_j SI_\Omega(f))\|_p \leq n \sum_{j=1}^n \|R_j SI_\Omega(f)\|_p \leq n^2 C \|f\|_p \quad (1.5)$$

for all test functions $f \in \mathcal{D}$, since $-\sum_{j=1}^n R_j^2$ is the identity map. Using the above method, Connett [7] and Ricci and Weiss [15] independently obtained the same $L^p$-boundedness of $SI_\Omega$ under the weak condition $\Omega \in H^1(S^{n-1})$, where $H^1(S^{n-1})$ is the Hardy space which contains $L^{\log^+}L(S^{n-1})$ as a proper subspace.

In [12], Fefferman generalized this Calderón-Zygmund singular integral by replacing the kernel $\Omega(x')|x|^{-n}$ by $h(|x|)\Omega(x')|x|^{-n}$, where $h$ is an arbitrary $L^\infty$ function. This allows the kernel to be rough not only on the sphere but also in the radial direction. For the singular integral operator $SI_{\Omega,h}$ with the kernel $K(x) = h(|x|)(\Omega(x')/(|x|^n))$, the formula (1.3) now is

$$SI_{\Omega,h}(f)(x) = \int_{S^{n-1}} \Omega(y') \left\{ \int_0^\infty f(x - ty') h(t) t^{-1} dt \right\} d\sigma(y'). \quad (1.6)$$

Clearly, the method of Calderón and Zygmund can no longer be used to estimate the above integral in (1.6) even if $\Omega$ is odd, since the integral in parentheses cannot be reduced to the Hilbert transform for an arbitrary $h(t)$. Thus, one needs to find a new approach.

Using a method which is different from Calderón and Zygmund, Fefferman showed in [12] that if $\Omega$ satisfies a Lipschitz condition, then $SI_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Later in [8], using Littlewood-Paley theory and Fourier transform methods, Duoandikoetxea and Rubio de Francia improved Fefferman’s results by assuming a roughness condition $\Omega \in L^q(S^{n-1})$ (see also [3, 13, 14]). By modifying the method in [8], recently, Fan and Pan [11] have improved the above results on $SI_{\Omega,h}$ by assuming a roughness condition $\Omega \in H^1(S^{n-1})$. 
Noting that $S^{n-1}$ is an $(n-1)$-dimensional compact manifold in $\mathbb{R}^{n-1}$, Duoandikoetxea and Rubio de Francia [8] introduced the following extension of the operator $SI_{\Omega,h}$.

Let $m, n \in \mathbb{N}$, $m \leq n - 1$, and let $\mathcal{M}$ be a compact, smooth, $m$-dimensional manifold in $\mathbb{R}^n$. Suppose that $\mathcal{M} \cap \{rv : r > 0\}$ contains at most one point for any $v \in S^{n-1}$. Let $\mathcal{C}(\mathcal{M})$ denote the cone $\{r\theta : r > 0, \theta \in \mathcal{M}\}$ equipped with the measure $ds(\theta) = r^m dr d\sigma(\theta)$, where $d\sigma$ represents the induced Lebesgue measure on $\mathcal{M}$. For a locally integrable function in $\mathcal{C}(\mathcal{M})$ of the form

$$K(r\theta) = r^{-m-1} h(r) \Omega(\theta),$$

where $\Omega$ satisfies

$$\int_{\mathcal{M}} \Omega(\theta) d\sigma(\theta) = 0,$$  \hspace{1cm} (1.8)

they defined the corresponding singular integral operator $SI_{\mathcal{M},\Omega,h}$ on $\mathbb{R}^n$ by

$$(SI_{\mathcal{M},\Omega,h} f)(x) = \text{p.v.} \int_{\mathcal{C}(\mathcal{M})} f(x-y)K(y)ds(y)$$

$$= \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} \int_{\mathcal{M}} f(x-r\theta)\Omega(\theta)h(r)r^{-1} d\sigma(\theta)dr$$

initially for $f \in \mathcal{S}(\mathbb{R}^n)$.

In [8], Duoandikoetxea and Rubio de Francia obtained the following results regarding $SI_{\mathcal{M},\Omega,h}$.

**Theorem 1.1.** Let $SI_{\mathcal{M},\Omega,h}$ be given as in (1.7)–(1.9). Suppose that

(i) $\Omega \in L^q(\mathcal{M})$,  
(ii) $\sup_{R>0} ((1/R) \int_0^R |h(r)|^2 dr) < \infty$,  
(iii) $\mathcal{M}$ has a contact of finite order with every hyperplane.

Then $SI_{\mathcal{M},\Omega,h}$ extends to a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Inspired by the earlier result of Fan and Pan regarding $\Omega \in H^1(S^{n-1})$, Cheng and Pan [5] established the following.

**Theorem 1.2.** Let $SI_{\mathcal{M},\Omega,h}$ be given as in Theorem 1.1, and let $h$ and $\mathcal{M}$ satisfy (ii) and (iii), respectively. If $\Omega \in H^1(\mathcal{M})$, then $SI_{\mathcal{M},\Omega,h}$ extends to a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

The main purpose of this paper is to extend Theorem 1.2 to the case $\Omega \in H^r(\mathcal{M})$ with $0 < r < 1$. The space $H^r(\mathcal{M})$ is a distribution space when $0 < r < 1$. The definition of $H^r(\mathcal{M})$ can be found in Section 2, but here we must define the operator in the sense of distribution.
Let \( \langle \Omega, \phi \rangle \) be the pairing between \( \Omega \in H^r(\mathcal{M}) \) and a \( C^\infty \) function \( \phi \) on \( \mathcal{M} \). For \( 0 \leq \alpha \), we define the singular integral operator \( SI_{\mathcal{M},\Omega,h,\alpha} f(x) \) by

\[
SI_{\mathcal{M},\Omega,h,\alpha} f(x) = \lim_{\epsilon \to 0^+} \int_{\Omega} \langle f(x - r \cdot), \Omega(\cdot) \rangle h(r) r^{1-\alpha} dr,
\]

where \( f \in \mathcal{S}(\mathbb{R}^n) \), \( h, \Omega \) satisfy (ii) and (iii) in Theorem 1.1, respectively, and \( \Omega \in H^r(\mathcal{M}) \) satisfies

\[
\langle \Omega, P_m |_{\mathcal{M}} \rangle = 0
\]

for all polynomials on \( \mathbb{R}^n \) with degree \( m \leq [\alpha] \) and \( r = m/m + \alpha \).

When \( \mathcal{M} = S^{n-1} \), the operator \( SI_{S^{n-1},\Omega,h,\alpha} \) was studied in [4]. It is not difficult to check that (1.10) is well defined and it is finite for all \( x \in \mathbb{R}^n \).

When \( \alpha = 0 \), the operator \( SI_{S^{n-1},\Omega,h,0} \) is exactly the operator \( SI_{\mathcal{M},\Omega,h} \).

The main result of this paper is as follows.

**Theorem 1.3.** Let \( SI_{\mathcal{M},\Omega,h,\alpha} \) be given as in (1.10), and let \( h, \mathcal{M} \) satisfy (ii) and (iii) as in Theorem 1.1, respectively. If \( \Omega \in H^r(\mathcal{M}) \) satisfies (1.11), then \( SI_{\mathcal{M},\Omega,h,\alpha} \) extends to a bounded operator from the homogeneous Sobolev space \( \dot{L}^p_\alpha(\mathbb{R}^n) \) to the Lebesgue space \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \).

2. Definitions and lemmas

Let \( \mathcal{M} \) be a compact, smooth, \( m \)-dimensional manifold in \( \mathbb{R}^n \), \( m \leq n - 1 \). The Hardy spaces \( H^p(\mathcal{M}) \) can be defined by using the maximal operator

\[
\mathcal{A} : f \mapsto (\mathcal{A} f)(x) = \sup_{t>0} |u(t,x)|,
\]

where \( u(t,x) \) is the solution of the boundary value problem

\[
\left( \frac{\partial}{\partial t} - \Delta_x \right) u = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathcal{M},
\]

\[
u(0,x) = f(x), \quad x \in \mathcal{M}.
\]

Here \( \Delta_x \) denotes the Laplace-Beltrami operator of \( \mathcal{M} \).

**Definition 2.1.** Define

\[
H^p(\mathcal{M}) = \{ f \in \mathcal{S}'(\mathcal{M}) : \| \mathcal{A} f \|_{L^p(\mathcal{M})} < \infty \}.
\]

For \( f \in H^p(\mathcal{M}) \), we set \( \| f \|_{H^p(\mathcal{M})} = \| \mathcal{A} f \|_{L^p(\mathcal{M})} \).

It is well known that since \( \mathcal{M} \) is compact,

\[
H^p(\mathcal{M}) = L^p(\mathcal{M}) \subset L\log^+ L(\mathcal{M}) \subset H^1(\mathcal{M}) \subset H^r(\mathcal{M}), \quad 0 < r < 1 < p,
\]

and all the inclusions are proper.
Let $B_n(x, r) = \{ y \in \mathbb{R}^n : |y - x| < r \}$. To give the atomic characterization of $H^r$, we need to define atoms on $\mathcal{M}$.

**Definition 2.2.** A function $a(\cdot)$ on $\mathcal{M}$ is called an $H^r$ atom if there are $\rho > 0$ and $\theta_0 \in \mathcal{M}$ such that

1. $\text{supp}(a) \subseteq B_n(\theta_0, \rho) \cap \mathcal{M}$,
2. $\|a\|_{\infty} \leq \rho^{-m/r}$,
3. $\int_{\mathcal{M}} a(\theta) P_k(\theta) d\sigma(\theta) = 0$,

for all polynomials $P_k$ on $\mathbb{R}^n$, with degrees $k \leq \lfloor m(1/r - 1) \rfloor$.

If $\Omega \in H^r(\mathcal{M})$, then there exist $H^r$ atoms $\{a_j\}$ and complex numbers $\{c_j\}$ such that

$$\Omega = \sum c_j a_j, \quad \sum |c_j|^r \equiv \|\Omega\|_{H^r(\mathcal{M})}^r \quad (\text{see [6]}).$$

**Definition 2.3.** A smooth mapping $\phi$ from an open set $U$ in $\mathbb{R}^m$ into $\mathbb{R}^n$ is said to be of finite type at $u_0 \in U$ if, for every $\eta \in S^{n-1}$, there exists a nonzero multi-index $\omega = \omega(\eta)$ such that

$$\frac{\partial^\omega [\eta \cdot \phi(u)]}{\partial u^\omega}\bigg|_{u = u_0} \neq 0.$$

By the smoothness and compactness of $\mathcal{M}$, we may assume that there is a smooth mapping $\phi$ from a neighborhood of $\overline{B_m(0, 1)}$ into $\mathbb{R}^n$ such that

(i) $\theta_0 \in \phi(B_m(0, 1/2))$ and $\mathcal{M} \cap B_n(\theta_0, \rho) \subseteq \phi(B_m(0, 1)) \subseteq \mathcal{M}$;

(ii) the vectors $\partial \phi/\partial u_1, \ldots, \partial \phi/\partial u_m$ are linearly independent for each $u \in \overline{B_m(0, 1)}$;

(iii) $\phi$ is of finite type at every point in $\overline{B_m(0, 1)}$ (see [16, page 350]).

Thus there is a smooth function $J(u)$ such that

$$\int_{\phi(B_m(0, 1))} Fd\sigma = \int_{B_m(0, 1)} F(\phi(u)) J(u) du,$$

for any integrable function $F$ on $\mathcal{M}$. Since $\mathcal{M}$ is compact, we may assume that all $\phi$ raised from atoms $a$ satisfy $|\phi(u) - \phi(u_0)| \leq |u - u_0|$.

Now given $\Omega \in H^r(\mathcal{M})$, then for each $H^r$ atom, $a(\theta)$ supported in $\mathcal{M} \cap B_m(\theta_0, \rho)$, write $b(u) = a(\phi(u)) J(u) \chi_{B_m(0, 1)}$. Let $u_0 = \phi^{-1}(\theta_0)$. It follows from (i)–(iii) that

$$\text{supp}(b) \subseteq B_m(u_0, \rho),$$

$$\|b\|_{\infty} \leq C \rho^{-m/r}, \quad \text{we may assume that } C = 1,$$

$$\int_{\mathbb{R}^m} b(u)(\phi(u) - \phi(u_0))^k du = 0,$$

for all $|k| \leq \lfloor \alpha \rfloor$, where $k = (k_1, k_2, \ldots, k_m)$ is a multi-index and $k = \sum_{i=1}^m k_i$.

We will need the following result (see [8]).
6  Singular integrals along manifolds

Lemma 2.4. Let \( \{a_k\} \) be a lacunary sequence of positive numbers such that \( a_k > 0 \) and \( \inf_{k \in \mathbb{Z}} |a_{k+1}/a_k| = \tau > 1 \). Let \( \tau_k \) be a sequence of Borel measures in \( \mathbb{R}^n \). Suppose that \( \|\tau_k\| \leq 1 \) and

(i) \( |\hat{\tau_k}| \leq C|a_{k+1}\xi|^\gamma \),

(ii) \( |\hat{\tau_k}| \leq C|a_k\xi|^{-\gamma} \),

for all \( k \in \mathbb{Z} \), and suppose also that for some \( q > 1 \),

(iii) \( |\tau_k| \leq C \|f\|_q \),

where \( \tau^* \) is the maximal operator: \( \tau^*(f) = \sup_k \|\tau_k * f\| \). Then

\[
Tf(x) = \sum_{k=-\infty}^{\infty} \tau_k * f(x) \tag{2.9}
\]

is a bounded operator on \( L^p(\mathbb{R}^n) \) for \( |1/p - 1/2| < 1/2q \).

We will also need the following result (see [8, 9, 11]).

Lemma 2.5. Let \( l,n \in \mathbb{N} \), and \( \{\tau_{s,k} : 0 \leq s \leq l, \text{and } k \in \mathbb{Z}\} \) be a family of measures on \( \mathbb{R}^n \) with \( \tau_{0,k} = 0 \) for every \( k \in \mathbb{Z} \). Let \( \{\alpha_{ij} : 1 \leq s \leq l, \text{and } j = 1,2\} \subset \mathbb{R}^n, \{\eta_s : 1 \leq s \leq l\} \subset \mathbb{R}^{-\{1\}}, \{M_s : 1 \leq s \leq l\} \subset \mathbb{N}, \text{and } L_s : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be linear transformations for \( 1 \leq s \leq l \). Suppose that

(i) \( \|\tau_{s,k}\| \leq 1 \) for \( k \in \mathbb{Z} \) and \( 0 \leq s \leq l \);

(ii) \( \|\hat{\tau}_{s,k}(\xi)\| \leq C(\eta_s^k|L_s\xi|)^{-\alpha_s} \) for \( \xi \in \mathbb{R}^m, k \in \mathbb{Z}, \text{and } 0 \leq s \leq l \);

(iii) \( \|\hat{\tau}_{s,k}(\xi) - \hat{\tau}_{s-1,k}(\xi)\| \leq C(\eta_s^k|L_s\xi|)^{\alpha_s} \) for \( \xi \in \mathbb{R}^m, k \in \mathbb{Z}, \text{and } 0 \leq s \leq l \);

(iv) for some \( \rho_0 > 2 \), there exists a \( C > 0 \) such that

\[
\left\| \sum_{k \in \mathbb{Z}} \left( |\tau_{s,k} \ast g_k|^2 \right)^{1/2} \right\|_{L^n(\mathbb{R}^n)} \leq C \left\| \sum_{k \in \mathbb{Z}} \left( |g_k|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \tag{2.10}
\]

for all \( \{g_k\} \in L_0^p(\mathbb{R}^n, L^2) \) and \( 1 \leq s \leq l \).

Then for every \( p \in (p_0, p_0) \), there exists a positive constant \( C_p \) such that

\[
\left\| \sum_{k \in \mathbb{Z}} \tau_{l,k} \ast f \right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \tag{2.11}
\]

\[
\left( \sum_{k \in \mathbb{Z}} \left| \tau_{l,k} \ast f \right|^2 \right)^{1/2} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}
\]

hold for all \( f \in L^p(\mathbb{R}^n) \). The constant \( C_p \) is independent of the linear transformations \( \{L_s\}_{s=1}^l \).

3. Proof of theorem

We will prove the theorem in three different cases: \( 0 < \alpha < 1, \alpha = 1,2,3,\ldots, \) and \( \alpha > 1, \alpha \notin \mathbb{Z} \). Without loss of generality, we may assume that \( \Omega(\theta) = a(\theta) \) is an \( H' \) atom as defined in Definition 2.2, the details can be found in [4].
Case 1 \((0 < \alpha < 1)\). Using the “lift” property of the Riesz potential and the definition of the space \(L^p_a(\mathbb{R}^n)\), it is known that for any \(\alpha > 0\) and \(f \in L^p_a(\mathbb{R}^n)\), one can write \(f = G_\alpha \ast f_a\) with \(|\hat{G}_\alpha(\xi)| \approx |\xi|^{-\alpha}, |G_\alpha(y)| \approx |y|^{-\alpha},\) and \(\|f_a\|_p \approx \|f\|_{L^p_a}\).

We write
\[
(\text{SI}_{\mu,\Omega,a}f)(x) = \sum_k \mu_{k,a} \ast f_a(x),
\]
where
\[
\mu_{k,a}(x) = \int_{2^k}^{2^{k+1}} \int_{\Omega} G_{\alpha}(x - r\theta)\Omega(\theta) h(r)r^{-\alpha} d\sigma(\theta) dr.
\]

In light of Lemma 2.4, in order to show that \(\|\text{SI}_{\mu,\Omega,a}f\|_{L^p} \leq C\|f\|_{L^p_a}\), it suffices to show that

(i) \(\|\mu_{k,a}\|_{L^1(\mathbb{R}^n)} \leq C\),

(ii) \(|\hat{\mu}_{k,a}(\xi)| \leq C|2^k \xi|^1\alpha\),

(iii) \(|\hat{\mu}_{k,a}(\xi)| \leq C|2^k \xi|^\alpha\),

(iv) \(\sup_{k \in \mathbb{Z}} |\mu_{k,a} \ast f|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}\), for all \(q \in (1, \infty)\).

Now, by the cancellation condition of \(b(u) = \Omega(\phi(u))J(u)\chi_{B_m(0,1)}(u)\), we have
\[
\|\mu_{k,a}\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left| \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} (G_{\alpha}(x - r\phi(u)) - G_{\alpha}(x - r\phi(u_0)))b(u) du \right| |h(r)| r^{-\alpha} dr dx
\]
\[
\leq \int_{2^k}^{2^{k+1}} r^{-\alpha} \int_{B_m(0,1)} |b(u)| dx
\]
\[
\times \int_{\mathbb{R}^n} |G_{\alpha}(x - r\phi(u)) - G_{\alpha}(x - r\phi(u_0))| dx |h(r)| du dr.
\]
\[
(3.3)
\]
Letting \(y = x - r\phi(u_0)\), we have
\[
\int_{\mathbb{R}^n} |G_{\alpha}(x - r\phi(u)) - G_{\alpha}(x - r\phi(u_0))| dx = \int_{\mathbb{R}^n} |G_{\alpha}(y + r(\phi(u) - \phi(u_0))) - G_{\alpha}(y)| dy.
\]
\[
(3.4)
\]
As we mentioned before, \(|\phi(u) - \phi(u_0)| \leq |u - u_0| \leq \rho, \) for \(u \in \text{supp}(b)\).

We write
\[
\int_{\mathbb{R}^n} |G_{\alpha}(y + r(\phi(u) - \phi(u_0))) - G_{\alpha}(y)| dy
\]
\[
= \int_{|y| \geq 3\rho} |G_{\alpha}(y + r(\phi(u) - \phi(u_0))) - G_{\alpha}(y)| dy
\]
\[
+ \int_{|y| < 3\rho} |G_{\alpha}(y + r(\phi(u) - \phi(u_0))) - G_{\alpha}(y)| dy
\]
\[
= I_1 + I_2, \quad \text{where } u \text{ is in the support of } b(u).
\]
\[
(3.5)
\]
By the definition of $G_\alpha(x)$, we have, if $y \geq 3r\rho \geq 3r|\phi(u) - \phi(u_0)|$,

$$|G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)| \leq C \frac{rp}{|y|^{n-\alpha+1}}.$$  \hfill (3.6)

Thus,

$$I_1 \leq C \int_{|y| \geq 3r\rho} \frac{rp}{|y|^{n-\alpha+1}} dy \approx (rp)^\alpha. \hfill (3.7)$$

It is easy to see that

$$I_2 \leq 2 \int_{|y| \leq 5r\rho} |G_\alpha(y)| \, dy \leq C \int_{|y| \leq 5r\rho} \frac{dy}{|y|^{n-\alpha}} \leq C(rp)^\alpha. \hfill (3.8)$$

Thus,

$$\|\mu_{k,\alpha}\|_{L^1(\mathbb{R}^n)} \leq \int_{2^{k-1}}^{2^k} r^{-1-\alpha} \int_{B_m(0,1)} |b(u)| \times \int_{\mathbb{R}^n} |G_\alpha(x - r\phi(u)) - G_\alpha(x - r\phi(u_0))| \, dx \, h(r) \, du \, dr \leq C \int_{2^{k-1}}^{2^k} r^{-1-\alpha} \int_{B_m(0,1)} |b(u)| \, (rp)^\alpha \, h(r) \, du \, dr \leq C. \hfill (3.9)$$

To prove (ii), we write

$$|\hat{\mu}_{k,\alpha}(\xi)| = |(\sigma_{k,\alpha} * G_\alpha)(\xi)| = |\hat{\sigma}_{k,\alpha}(\xi)| |\hat{G}_\alpha(\xi)| \leq C|\xi|^{-\alpha} |\hat{\sigma}_{k,\alpha}(\xi)|. \hfill (3.10)$$

Thus,

$$|\hat{\mu}_{k,\alpha}(\xi)| \leq C|\xi|^{-\alpha} \int_{2^{k-1}}^{2^k} \left( \int_{B_m(0,1)} e^{-ir\xi \cdot \phi(u)} b(u) \, du \right) r^{-1-\alpha} h(r) \, dr$$

$$\leq C|\xi|^{-\alpha} 2^{-ka} \int_{2^{k-1}}^{2^k} \left( \int_{B_m(0,1)} (e^{-ir\xi \cdot \phi(u)} - e^{ir\xi \cdot \phi(u_0)}) b(u) \, du \right) r^{-1} h(r) \, dr$$

$$\leq C|\xi|^{-\alpha} 2^{-ka} 2^k |\phi(u) - \phi(u_0)| |b(u) - b(u_0)| du \leq C |2^k \xi \rho|^{-\alpha}, \hfill (3.11)$$

which proves (ii).

On the other hand,

$$|\hat{\mu}_{k,\alpha}(\xi)| \leq C|\xi|^{-\alpha} 2^{-ka} \int_{2^{k-1}}^{2^k} |b(u)| \, du \, r^{-1} \, h(r) \, dr = C |2^k \xi \rho|^{-\alpha}, \hfill (3.12)$$

which proves (iii).
It remains to show that
\[
\left\| \sup_{k \in \mathbb{Z}} |\mu_{k,\alpha} \ast f| \right\|_p \leq C \|f\|_p. \tag{3.13}
\]

Without loss of generality, assume that \(h(r) \geq 0\). Then
\[
\left\| \sup_{k \in \mathbb{Z}} |\mu_{k,\alpha} \ast f| \right\|_{L^q(\mathbb{R}^n)} \leq C \sup_{k \in \mathbb{Z}} 2^{-k - \alpha} \int_{2^k}^{2^{k+1}} h(r) \int_{B_m(0,1)} |b(u)| \int_{\mathbb{R}^n} |f(x - z)| |G_\alpha(z - r\phi(u)) - G_\alpha(z - r\phi(u_0))| \, dz \, dudr.
\tag{3.14}
\]

In the above integral, we write
\[
\int_{\mathbb{R}^n} |f(x - z)| |G_\alpha(z - r\phi(u)) - G_\alpha(z - r\phi(u_0))| \, dz
= \int_{|z - r\phi(u_0)| > 3\rho} |f(x - z)| |G_\alpha(z - r\phi(u)) - G_\alpha(z - r\phi(u_0))| \, dz
+ \int_{|z - r\phi(u_0)| \leq 3\rho} |f(x - z)| |G_\alpha(z - r\phi(u)) - G_\alpha(z - r\phi(u_0))| \, dz \tag{3.15}
\]

where \(u \in B_n(u_0, \rho) \cap \mathcal{M}\).

In the integral \(I_1(f)\), we change variables \(z - r\phi(u_0) \to y\) and again write \(y\) as \(z\), then
\[
I_1(f)(x) = C \int_{|z| > 3\rho} |f(x - z + r\phi(u_0))| |G_\alpha(z + r\phi(u_0) - r\phi(u)) - G_\alpha(z)| \, dz.
\tag{3.16}
\]

Note that \(|r\phi(u_0) - r\phi(u)| \leq r\rho < |z|/2\). By the mean value theorem,
\[
I_1(f)(x) \leq C \int_{|z| > 3\rho} r\rho |f(x - z + r\phi(u_0))| |z|^{\alpha - 1 - n} \, dz
\approx \int_{S^{n-1}} \int_{3\rho}^{\infty} r\rho s^{\alpha - 2} |f(x - sz + r\phi(u_0))| \, dsd\sigma(z').
\tag{3.17}
\]

Using integration by parts, it is easy to see that
\[
I_1(f)(x) \leq C \int_{S^{n-1}} (r\rho)^{\alpha} \int_{0}^{3\rho} |f(x - tz' + r\phi(u_0))| \, dt \, d\sigma(z')
+ C \int_{S^{n-1}} \int_{3\rho}^{\infty} r\rho s^{\alpha - 3} \int_{0}^{s} |f(x - tz' + r\phi(u_0))| \, ds \, dt \, d\sigma(z').
\tag{3.18}
\]
Let $M_z f(x)$ be the maximal function

$$M_z f(x) = \sup_{t > 0} t^{-1} \int_0^t |f(x - rz)| dr. \quad (3.19)$$

It is known in [16, page 477] that there is a constant $C$ independent of $z$ such that

$$\|M_z(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \quad (3.20)$$

Thus we have

$$I_1(f)(x) \leq C(r\rho)^\alpha \int_{S^{n-1}} M_z f(x + r\phi(u)) d\sigma(z'). \quad (3.21)$$

For the second integral $I_2(f)(x)$, we have $I_2(f)(x) \leq J_1(f)(x) + J_2(f)(x)$, where

$$J_1(f)(x) = \int_{|z - r\phi(u)| < 3r\rho} |f(x - z)G_\alpha(z - r\phi(u))| dz,$$

$$J_2(f)(x) = \int_{|z| < 3r\rho} |f(x - z + r\phi(u))G_\alpha(z)| dz. \quad (3.22)$$

Let $w = z - r\phi(u)$. Then, in $J_1(f)(x)$, we have

$$|w| \leq |z - r\phi(u)| + |r\phi(u) - r\phi(u_0)| \leq 4r\rho. \quad (3.23)$$

This gives (again write $z$ instead of $w$)

$$J_1(f)(x) \leq C \int_{|z| < 4r\rho} |f(x - z)G_\alpha(z - r\phi(u))| |z|^{n-\eta} dz$$

$$= C \int_{S^{n-1}} \int_0^{4r\rho} t^{\alpha-1} |f(x - tz' - r\phi(u))| dt d\sigma(z'). \quad (3.24)$$

Using integration by parts, we obtain

$$J_1(f)(x) \leq C \int_{S^{n-1}} (r\rho)^\eta M_z' \{f(x - r\phi(u))\} d\sigma(z'). \quad (3.25)$$

Similarly, we can have the same estimate on $J_2(f)(x)$ so that

$$J_2(f)(x) \leq C \int_{S^{n-1}} (r\rho)^\eta \{M_z' f(x + r\phi(u_0)) + M_z f(x - r\phi(u))\} d\sigma(z'). \quad (3.26)$$

Thus

$$\int_{\mathbb{R}^n} |f(x - z)| \ |G_\alpha(z - r\phi(u)) - G_\alpha(z - r\phi(u_0))| dz$$

$$\leq C(r\rho)^\eta \int_{S^{n-1}} \{M_z f(x + r\phi(u_0)) + M_z f(x - r\phi(u))\} d\sigma(z'). \quad (3.27)$$
Therefore, we have

\[
\left\| \sup_{k \in \mathbb{Z}} |\mu_{k,\alpha}| \ast f \right\|_{L^q(\mathbb{R}^n)} \leq C \int_{B_m(0,1) \times S^{n-1}} |b(u)| \rho^\alpha \left\{ |M_{\phi(u)}M_{\mathcal{A}}(f)|_{L^q(\mathbb{R}^n)} + |M_{\phi(u)}M_{\mathcal{A}}f|_{L^q(\mathbb{R}^n)} \right\} d\sigma(z') du.
\]

(3.28)

Since \(b\) is an \((r, \infty)\) atom supported in \(B_m(u_0, \rho) \cap M\) with \(r = m/(m + \alpha)\), it is easy to see that

\[
\int_{B_m(0,1)} |b(u)| \rho^\alpha du \leq C
\]

uniformly for \(b\) and \(\rho\). Thus

\[
\left\| \sup_{k \in \mathbb{Z}} |\mu_{k,\alpha}| \ast f \right\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^q(\mathbb{R}^n)}.
\]

(3.30)

By Lemma 2.4, Case 1 is established.

Case 2 \((\alpha = 1, 2, 3, \ldots)\). Using Taylor's expansion about \(\theta_0\), we have, for \(j = (j_1, \ldots, j_m)\),

\[
(SL_{\mu, \Omega, h, \alpha} f)(x) = \sum_{|j| = \alpha} C_j \int_0^1 (1-t)^{\alpha-1} \int_0^\infty \int_{B_m(0,1)} \mathcal{B}(u)r^{-1}h(r) \times D^j f(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) du dr dt,
\]

(3.31)

where \(C_j\)'s are constants and \(\mathcal{B}(u) = b(u)(\phi(u) - \phi(u_0))^j\). Clearly, \(\mathcal{B}(u)\) is an \(H^1\) atom with the same support as \(b\).

For each \(j, |j| = \alpha\), define the measures \(\{\sigma_{\phi, \mathcal{B}, h, k, \alpha} |k \in \mathbb{Z}\}\) on \(\mathbb{R}^n\) by

\[
\int_{\mathbb{R}^n} F(x) d\sigma_{\phi, \mathcal{B}, h, k, \alpha} = \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} F(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) \mathcal{B}(u)r^{-1}h(r) du dr dt.
\]

(3.32)

**Lemma 3.1.** Suppose that \(h\) satisfies (ii) in Theorem 1.1. Then for \(1 < p < \infty\), there exists a constant \(C_p > 0\) such that

\[
\left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{\phi, \mathcal{B}, h, k, \alpha} \ast g_k|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left( \sum_{k \in \mathbb{Z}} |g_k|^2 \right)^{1/2} \right\|_p.
\]

(3.33)

holds for all continuous mappings \(\phi\) and measurable functions \(\{g_k\}\) on \(\mathbb{R}^n\).
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Proof. For $\xi \in \mathbb{R}^n$, we define the maximal operator $M_\xi$ on $\mathbb{R}^n$ by

$$(M_\xi f)(x) = \sup_{k \in \mathbb{Z}} \left[ 2^{-k} \int_{2^k}^{2^{k+1}} |f(x + r\xi)| dr \right]. \quad (3.34)$$

It follows from the $L^p$-boundedness of the one-dimensional Hardy-Littlewood maximal operator that

$$||M_\xi f||_p \leq A_p ||f||_p, \quad (3.35)$$

for $1 < p \leq \infty$, where $A_p$ is independent of $\xi$.

By duality, we may assume that $p > 2$, then for $\{g_k\} \in L^p(\mathbb{R}^n, l^2)$, there exists a function $w \in L^{(p/2)'}(\mathbb{R}^n)$ such that $\|w\|_{(p/2)'} = 1$ and

$$\left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{\phi, h, k, \alpha} * g_k|^2 \right)^{1/2} \right\|_p^2 = \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z}} |\sigma_{\phi, h, k, \alpha} * g_k|^2 \right) w(x) dx. \quad (3.36)$$

By Hölder’s inequality and (3.35),

$$\left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{\phi, h, k, \alpha} * g_k|^2 \right)^{1/2} \right\|_p^2 \leq \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \left( \int_0^1 (1 - t)^{\alpha - 1} \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} g_k(xr\phi(u_0) + rt(\phi(u_0) - \phi(u))) \right) \times B(u)r^{-1} h(r) du dr dt \left\| w(x) dx \right\|^2$$

$$\leq C \|B\|_1 \sum_{k \in \mathbb{Z}} 2^{-k} \int_{\mathbb{R}^n} \int_0^1 \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} |g_k(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) |^2 \times B(u) w(x) | dx dr dt dx$$

$$= C \|B\|_1 \int_0^1 \int_{B_m(0,1)} |B(u)|$$

$$\times \left[ \sum_{k \in \mathbb{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} \int_{\mathbb{R}^n} g_k(x)^2 |w(x + r\phi(u_0) + rt(\phi(u_0) - \phi(u))) | dx dr \right] du dt$$

$$\leq C \|B\|_1 \int_0^1 \int_{B_m(0,1)} \left[ \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z}} |g_k(x)|^2 \right) (M_{\phi(u_0) + rt(\phi(u_0) - \phi(u))) w)(x) dx \right] |B(u)| | du dt$$

$$\leq C \|B\|_1 \left\| \left( \sum_{k \in \mathbb{Z}} |g_k|^2 \right)^{1/2} \right\|_p^2. \quad (3.37)$$

We also have the following estimates for $\sigma_{\phi, h, k, \alpha}$. □
Lemma 3.2. Suppose that $\phi$ is smooth and of finite type at every point in $B_m(0,1)$ and $h$ satisfies (ii) in Theorem 1.1. Then there exists a $\delta > 0$ such that

$$|\hat{\sigma}_{\phi, h, k, \alpha}(\xi)| \leq C\|B\|_2 (2^k|\xi|)^{-\delta}. \tag{3.38}$$

Proof.

$$|\hat{\sigma}_{\phi, h, k, \alpha}(\xi)| = \left| \int_0^1 (1-t)^{a-1} \int_{2^k}^{2^{k+1}} h(r) r^{-1} e^{i\xi r \phi(u)} e^{-i\xi r t \phi(u)} \int_{B_m(0,1)} B(u) e^{i\xi r t \phi(u)} du dr dt \right|. \tag{3.39}$$

Changing variables ($s = rt$), we have

$$|\hat{\sigma}_{\phi, h, k, \alpha}(\xi)| = \left| \int_0^1 (1-t)^{a-1} \int_{2^k}^{2^{k+1}} h\left(\frac{s}{t}\right) s^{-1} e^{i\xi s \phi(u)} e^{-i\xi s t \phi(u)} \int_{B_m(0,1)} B(u) e^{i\xi s \phi(u)} du ds dt \right| \tag{3.40}$$

The remainder of the proof is similar to the proof of Lemma 3.3 in [5].

The following result is similar to those in [10], see also [5].

Lemma 3.3. Let $B(\cdot)$ be a function satisfying $\text{supp}(B) \subset B_m(0,\rho)$ and $\|B\|_\infty \leq \rho^{-m}$ for some $\rho < 1$. Suppose that $h$ satisfies (ii) in Theorem 1.1. Then there exists a constant $C > 0$ such that

$$\left| \int_0^1 (1-t)^{a-1} \int_{2^k}^{2^{k+1}} h(r) r^{-1} \left( \int_{B_m(0,1)} B(u) e^{-i\xi r t [Q(u) + \sum_{|\beta|=s} d_{\beta} u^\beta]} du \right) dr dt \right| \leq C \left( 2^k \rho^s \sum_{|\beta|=s} |d_{\beta}| \right)^{-1/(4s)} \tag{3.41}$$

holds for all polynomials $Q: \mathbb{R}^m \to \mathbb{R}$ with $\text{deg}(Q) < s$ and $\{d_{\beta}\} \subset \mathbb{R}$. The constant $C$ is independent of $\rho$.

Now, by Lemma 3.2, there exists a $\delta > 0$ such that

$$|\hat{\sigma}_{\phi, h, k, \alpha}(\xi)| \leq C(2^k|\xi|)^{-\delta} \rho^{-m/2}. \tag{3.42}$$
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Let \( l = \left\lfloor m/(2\delta) \right\rfloor + 1 \). Following the proof of Theorem 3.7 in [5], we define a sequence of mappings \( \{\Phi^s\}_{s=0}^l \) by

\[
\Phi^s(u) = \left( \sum_{|\beta| \leq s} \frac{1}{\beta!} \partial^\beta \phi_j(u_0) (u - u_0)^\beta, \ldots, \sum_{|\beta| \leq s} \frac{1}{\beta!} \partial^\beta \phi_n(u_0) (u - u_0)^\beta \right)
\]

for \( s = 0, 1, \ldots, l - 1 \).

Let

\[
\sigma_{s,k,\alpha} = \sigma_{\Phi^s, B, h, k, \alpha}
\]

for \( 0 \leq s \leq l \) and \( k \in \mathbb{Z} \).

In order to show that \( \| S_{l,0,\Omega, h, \alpha} f \|_{L^p} \leq C \| f \|_{L^q} \), it suffices to show that the family of measures \( \{\sigma_{s,k,\alpha}\} \) satisfies the conditions of Lemma 2.5.

By its definition and Lemma 3.2, the family of measures \( \{\sigma_{s,k,\alpha}\} \) satisfies conditions (i) and (iv) in Lemma 2.5, for any \( p_0 > 2 \).

It is easy to see that

\[
\| \sigma_{s,k,\alpha} \| \leq \| B \|_1 \int_0^1 |(1-t)^{a-1}| \int_{2^k}^{2^{k+1}} |h(r)| dr dt \leq C.
\]

Also we have

\[
\sigma_{0,k,\alpha}(x) = 0, \quad \text{by the cancellation condition of } B(u).
\]

For \( j = 1, \ldots, n \), let

\[
d_{j,\beta} = \frac{1}{\beta!} \partial^\beta \phi_j(u_0).
\]

By (3.42) and Lemma 3.3, we have

\[
|\hat{\sigma}_{l,k,\alpha}(\xi)| \leq C(2^k \rho^l |\xi|)^{-\delta},
\]

\[
|\hat{\sigma}_{s,k,\alpha}(\xi)| \leq C \left( 2^k \rho^s \sum_{|\beta| = s} \left| \sum_{j=1}^n d_{j,\beta} \xi_j \right| \right)^{-1/(4s)}
\]

for \( 1 \leq s \leq l - 1, k \in \mathbb{Z} \) and \( \xi \in \mathbb{R}^n \). We also have,

\[
|\hat{\sigma}_{l,k,\alpha}(\xi) - \hat{\sigma}_{l-1,k,\alpha}(\xi)|
\]

\[
\leq \int_0^1 |(1-t)^{a-1}| \int_{2^k}^{2^{k+1}} |h(r)| r^{-1} \int_{B_m(0,1)} |B(u)| \left| e^{i\xi r_\phi(u)} - e^{i\xi r_\phi(u-1)}(u) \right| du dr dt
\]

\[
\leq C|\xi|^2 \int_{B_m(0,1)} |B(u)| \left| (\phi(u) - \phi^{l-1}(u)) \right| du \leq C(2^k |\xi| \rho^l).
\]
Similarly,
\[
\left| \hat{\sigma}_{s,k,a}(\xi) - \hat{\sigma}_{s-1,k,a}(\xi) \right| \leq C2^k \int_{B_m(0,1)} |\mathcal{B}(u)| \left| \xi \cdot (\phi^s(u) - \phi^{s-1}(u)) \right| du
\]
\[
\leq C2^k \rho^s \sum_{|\beta|=s} \left| \sum_{j=1}^n d_{ij} \xi_j \right|
\]
(3.50)
for \(1 \leq s \leq l - 1, k \in \mathbb{Z} \) and \(\xi \in \mathbb{R}^n\).

Invoking Lemma 2.5, Case 2 is established.

**Case 3** (\(\alpha > 1, \alpha \notin \mathbb{Z}\)). Write \(\alpha = [\alpha] + \gamma, \gamma \in (0,1)\).

Similar to the case \(\alpha = 1, 2, 3, \ldots\), by Taylor’s expansion, we have
\[
\left( SI_{l,l,\Omega,h,a} f \right)(x) = \sum_{|j|=\alpha} C_j \int_0^1 (1 - t)^{\alpha-1} \int_0^\infty r^{1-\gamma} h(r) \int_{B_m(0,1)} \mathcal{B}(u) \times D^j f(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) du dt dr,
\]
(3.51)
where \(\mathcal{B}(u) = b(u)(\phi(u) - \phi(u_0))^t\). Clearly, \(\mathcal{B}(u)\) is an \(H^r\) atom, where \(r = m/(m + \gamma)\).

Similar to Case 1, again using the “lift” property of the Riesz potential and the definition of the space \(L^p_{\alpha}(\mathbb{R}^n)\), it is known that for any \(\gamma > 0\) and \(f \in L^p_{\alpha}(\mathbb{R}^n)\), one can write \(f = G_\gamma * f_\gamma\) with \(|\hat{G}_\gamma(\xi)| \approx |\xi|^{-\gamma}, |G_\gamma(y)| \approx |y|^{-n+\gamma}\), and \(\|f_\gamma\|_p \approx \|f\|_{L^p_{\alpha}}\).

We write
\[
\left( SI_{l,l,\Omega,h,a,k} f \right)(x) = \sum_k \sigma_{k,y} * f_\gamma,
\]
(3.52)
where
\[
\sigma_{k,y} = \int_0^1 (1 - t)^{\alpha-1} \int_{2^k}^{2^{k+1}} r^{1-\gamma} h(r) \int_{B_m(0,1)} \mathcal{B}(u) G_\gamma(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) du dt dr
t
\]
\[
= \int_0^1 (1 - t)^{\alpha-1} \int_{2^k}^{2^{k+1}} r^{1-\gamma} h(r) \int_{B_m(0,1)} \mathcal{B}(u) \times [G_\gamma(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) - G_\gamma(x - r\phi(u_0))] du dt dr.
\]
(3.53)
Again, by Lemma 2.4, in order to show that \(\| SI_{l,l,\Omega,h,a,k} f \|_{L^p} \leq C \| f \|_{L^p_{\alpha}}\), it suffices to show that
(i) \(\|\sigma_{k,y}\|_{L^1(\mathbb{R}^n)} \leq C\),
(ii) \(|\hat{\sigma}_{k,y}(\xi)| \leq C|2^k \xi \rho|^{1-\gamma}\),
(iii) \(|\hat{\sigma}_{k,y}(\xi)| \leq C|2^k \xi \rho|^{-\gamma}\),
(iv) \(\sup_{k \in \mathbb{Z}} |\sigma_{k,y}||f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}\).

The proof is similar to the proof for Case 1. We leave the details to the reader.

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