SUBORDINATION PROPERTIES OF $p$-VALENT FUNCTIONS DEFINED BY INTEGRAL OPERATORS

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By applying certain integral operators to $p$-valent functions we define a comprehensive family of analytic functions. The subordinations properties of this family is studied, which in certain special cases yield some of the previously obtained results.

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1. Introduction

For the natural numbers $p$ let $A(p)$ denote the class of functions of the form $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$, which are analytic in the open unit disk $U = \{z : |z| < 1\}$. For $f(z) \in A(p)$ we define

$$I^\sigma f(z) = \frac{(p+1)^\sigma}{\Gamma(\sigma)} \int_0^z \left( \log \frac{z}{t} \right)^{\sigma-1} f(t) \, dt$$

$$= z^p + \sum_{n=p+1}^{\infty} \left( \frac{p+1}{n+1} \right)^\sigma a_n z^n, \quad \sigma > 0. \quad (1.1)$$

Also, for $-1 \leq B < A \leq 1$ and $\lambda \geq 0$, let $\Omega^\sigma_p(A,B,\lambda)$ be the class of functions $f \in A(p)$ so that

$$\frac{\lambda}{p} I^{\sigma-1} f(z) + \frac{p - \lambda}{p} I^\sigma f(z) > 1 + Az, \quad 1 + Bz, \quad \lambda \geq 0, \quad (1.2)$$

where “$<$” denotes the usual subordination. See [2].

The family $\Omega^\sigma_p(A,B,\lambda)$ is a comprehensive family containing various well-known as well as new classes of analytic functions. For example, for $\sigma = 0$ and $\lambda = p + 1$ we obtain the class $\Omega^0_p(A,B,p + 1)$ studied by Patel and Mohanty [3] or for nonzero $\sigma$ see Liu [1].

2. Main results

Our first theorem examines the containment properties of the family $\Omega^\sigma_p(A,B,\lambda)$. 
2 $p$-valent functions

**Theorem 2.1.** For $f \in A(p)$ suppose that $f \in \Omega^p_\sigma(A,B,\lambda)$ and $0 \leq \lambda \leq p(p + 1)$. Then $f \in \Omega^p_\sigma(A,B,0)$.

To prove our theorem we will need the following lemma which is due to Miller and Mocanu [2].

**Lemma 2.2.** Let $g(z)$ be analytic and convex univalent in $U$ and $g(0) = 1$. Also let $p(z)$ be analytic in $U$ with $p(0) = 1$. If $p(z) + (zp'(z))/\gamma < g(z)$, where $\gamma \neq 0$ and $\text{Re} \gamma \geq 0$, then

$$p(z) \prec \gamma z - \gamma \int_0^z t^{p-1} g(t) \, dt.$$  

**Proof of Theorem 2.1.** First, we note that

$$z(I^\sigma f(z))' = (p + 1)I^{\sigma-1}f(z) - I^\sigma f(z).$$  \hspace{1cm} (2.1)

Setting $p(z) = (I^\sigma f(z))/z^p$ we also observe that

$$\frac{(I^\sigma f(z))'}{pz^{p-1}} = p(z) + \frac{zp'(z)}{p},$$
$$\frac{I^{\sigma-1}f(z)}{z^p} = p(z) + \frac{zp'(z)}{p + 1}.$$  \hspace{1cm} (2.2)

Therefore, for $f \in \Omega^p_\sigma(A,B,\lambda)$, we conclude that

$$p(z) + \frac{\lambda}{p(p + 1)}zp'(z) < \frac{1 + Az}{1 + Bz}.$$  \hspace{1cm} (2.3)

Now from Lemma 2.2 for $\gamma = p(p + 1)/\lambda$ it follows that

$$\frac{I^\sigma f(z)}{z^p} < \frac{p(p + 1)}{\lambda}z^{-p(p + 1)/\lambda}\int_0^z t^{p(p + 1)/\lambda - 1}\frac{1 + At}{1 + Bt} \, dt = q(z) < \frac{1 + Az}{1 + Bz}.$$  \hspace{1cm} (2.4)

Thus $f \in \Omega^p_\sigma(A,B,0)$.

As a special case to Theorem 2.1, we obtain the following.

**Corollary 2.3.** Let $f \in A(p)$. Then $(1/(p + 1))[(zf'(z) + f(z))/z^p] < (1 + Az)/(1 + Bz)$, implies $f(z)/z^p < (1 + Az)/(1 + Bz)$.

**Theorem 2.4.** For $f \in A(p)$ suppose that $f \in \Omega^p_\sigma(A,B,\lambda)$. If $0 \leq \lambda \leq p(p + 1)$, then

$$\text{Re} \left( \frac{I^\sigma f(z)}{z^p} \right) \geq \frac{p(p + 1)}{\lambda} \int_0^1 u^{p(p + 1)/\lambda - 1}\frac{1 - Au}{1 - Bu} \, du.$$  \hspace{1cm} (2.5)

The result is sharp.

**Proof.** Set $p(z) = I^\sigma f(z)/z^p$. Then, by Theorem 2.1, we have

$$p(z) < \frac{p(p + 1)}{\lambda}z^{-p(p + 1)/\lambda}\int_0^z t^{p(p + 1)/\lambda - 1}\frac{1 + At}{1 + Bt} \, dt < \frac{1 + Az}{1 + Bz}.$$  \hspace{1cm} (2.6)
This is equivalent to

\[
\frac{I^\sigma f(z)}{z^p} = \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \frac{1+uAw(z)}{1+uBw(z)} \, du,
\]

(2.7)

where \(w(z)\) is analytic in \(U\) with \(w(0) = 0\) and \(|w(z)| < 1\) in \(U\). Therefore

\[
\text{Re} \left( \frac{I^\sigma f(z)}{z^p} \right) = \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \text{Re} \left\{ \frac{1+uAw(z)}{1+uBw(z)} \right\} \, du \geq \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \frac{1-Au}{1-Bu} \, du.
\]

(2.8)

Therefore

\[
\frac{I^\sigma f(z)}{z^p} = \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \frac{1+Au}{1+Bu} \, du,
\]

(2.9)

such that for this function we have

\[
\frac{\lambda}{p} \frac{I^\sigma f(z)}{z^p} + \frac{p-\lambda}{p} \frac{I^\sigma f(z)}{z^p} = \frac{1+Az}{1+Bz}.
\]

(2.10)

Letting \(z \to -1\) yields

\[
\frac{I^\sigma f(z)}{z^p} \to \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \frac{1-Au}{1-Bu} \, du.
\]

(2.11)

□

References


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