THE UNIVERSAL SEMILATTICE COMPACTIFICATION OF A SEMIGROUP

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ABSTRACT. The universal abelian, band, and semilattice compactifications of a semitopological semigroup are characterized in terms of three function algebras. Some relationships among these function algebras and some well-known ones, from the universal compactification point of view, are also discussed.

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1. Introduction. The notion of semigroup compactification has been produced in several principal ways, in whose main approach the Gelfand-Naimark theory of commutative $C^*$-algebras is employed. In fact, the spectrum of every $m$-admissible algebra of functions is a semigroup compactification. Moreover, some of these compactifications enjoy a universal property $P$. For instance, De Leeuw and Glicksberg in their influential paper [2], characterized the universal property of (weakly) almost periodic compactification. The existence of the universal $P$-compactification (using the subdirect product methods) for a broad variety of properties $P$, is guaranteed by Junghenn and Pandian [7]. The construction of some of the better known universal $P$-compactifications in terms of $m$-admissible algebras of functions are collected in Berglund et al. [1], which is our ground reference. The universal right simple, left simple, and group compactifications are characterized in terms of some types of distal functions [6]. In two recent papers [9, 10], Pandian has examined the universal mapping property of generalized distal, and quasiminimal distal functions. Also, in an earlier paper [3], we have characterized the universal nilpotent group compactification. The present paper deals with the construction of three $m$-admissible algebras $AB, BD$, and $SL$, which characterize the universal abelian, band, and semilattice compactifications of a semitopological semigroup.

2. Preliminaries. For background and notations we follow Berglund et al. [1] as much as possible. In what follows, $S$ is a semitopological semigroup unless otherwise stipulated. A (semigroup) compactification of $S$ is a pair $(\psi, X)$, where $X$ is compact, Hausdorff, right topological semigroup and $\psi: S \rightarrow X$ is a continuous homomorphism with dense image such that, for all $s \in S$, the mapping $x \mapsto \psi(s)x: X \rightarrow X$ is continuous.

The $C^*$-algebra of all continuous bounded complex-valued functions on a topological space $Y$ is denoted by $C(Y)$. For $C(S)$ left and right translations, $L_s$ and $R_t$, are
defined for all $s, t \in S$ by $(L_s f)(t) = f(st) = (R_t f)(s)$, $f \in C(S)$. A left translation invariant $C^*$-subalgebra $F$ of $C(S)$ (i.e., $L_s f \in F$ for all $s \in S$ and $f \in F$), containing the constant functions, is called $m$-admissible if the function $s \mapsto (T_\mu f)(s) = \mu(L_s f)$ is in $F$ for all $f \in F$ and $\mu \in SF$ (i.e., $S^F$ under the multiplication $\mu \nu = \mu \circ T_\nu$ ($\mu, \nu \in SF$), furnished with the Gelfand topology, makes $(\varepsilon, S^F)$ a compactification (called the $F$-compactification) of $S$, where $\varepsilon : S \rightarrow SF$ is the evaluation mapping. Conversely, if $(\psi, X)$ is a compactification of $S$, then $\psi^*(C(X))$ is an $m$-admissible subalgebra of $C(S)$, where $\psi^*$ is the dual mapping of $\psi$, and this correspondence between compactifications of $S$ and $m$-admissible subalgebras of $C(S)$ is one-to-one (see [1, Thm. 3.1.7]).

A compactification $(\psi, X)$ of $S$, possessing a certain property $P$, is called the universal $P$-compactification if for any other compactification $(\varphi, Z)$, having the property $P$, there exists a homomorphism $\pi : (\psi, X) \rightarrow (\varphi, Z)$, where $\pi$ is a continuous mapping from $X$ into $Y$ with $\pi \circ \psi = \varphi$, or equivalently, $\varphi^*(C(Z)) \subseteq \psi^*(C(X))$ (see [1, Thm. 3.1.9]).

Some of the usual $m$-admissible subalgebras of $C(S)$, that are needed in the sequel, are the left multiplicatively continuous, weakly almost periodic, almost periodic, strongly almost periodic, distal, minimal distal, and strongly distal functions on $S$. These are denoted by $LMC$, $WAP$, $AP$, $SAP$, $D$, $MD$, and $SD$, respectively. We also write $GP$ for $MD \cap SD$, $LZ$ for $\{ f \in C(S) : f(st) = f(s) \text{ for all } s, t \in S \}$, and $RZ$ for $\{ f \in C(S) : f(st) = f(t) \text{ for all } s, t \in S \}$. Here, and also for other emerging spaces, when there is no risk of confusion, we have suppressed the letter $S$ from the notation. For ease of reference, we mention the next proposition which describes the universal mapping properties of these $m$-admissible algebras.

**Proposition 2.1.** See [1, Chap. 4] and [6, Thm. 3.4]. The $LMC$, $WAP$, $AP$, $SAP$, $D$, $MD$, $SD$, $GP$, $LZ$, and $RZ$-compactifications are universal with respect to the properties of being a (right topological) semigroup, a semitopological semigroup, a topological semigroup, a topological group, an inflation of a rectangular group, a left simple semigroup, a right simple semigroup, a group, a left zero semigroup, and a right zero semigroup, respectively.

### 3. The main results

To follow the main objective, we examine the properties of $AB$ and $BD$, where

$$AB = \{ f \in WAP : f(st) = f(ts), \text{ and } f(stu) = f(sut) \text{ for all } s, t, u \in S \}$$

(3.1)

and $BD$ consists of those $f \in LMC$ such that

$$\lim_{\alpha} \left( \lim_{\alpha} R_{t_\alpha} f \right)(s_\alpha) = \lim_{\alpha} f(s_\alpha);$$

$$\lim_{\alpha} \left( \lim_{\alpha} R_{s_\alpha} \left( \lim_{\alpha} R_{t_\alpha} f \right) \right)(s_\alpha) = \lim_{\alpha} \left( \lim_{\alpha} R_{t_\alpha} f \right)(s_\alpha);$$

$$\lim_{\alpha} R_{s_\alpha} \left( \lim_{\alpha} R_{s_\alpha} f \right) = \lim_{\alpha} R_{s_\alpha} f;$$

$$\lim_{\alpha} R_{s_\alpha} \left( \lim_{\alpha} R_{s_\alpha} \left( \lim_{\alpha} R_{t_\alpha} f \right) \right) = \lim_{\alpha} R_{s_\alpha} \left( \lim_{\alpha} R_{t_\alpha} f \right)$$

(3.2)

for all nets $\{ s_\alpha \}$ and $\{ t_\alpha \}$ in $S$ for which the relevant pointwise limits exist.
Also, we write $SL$ for $AB \cap BD$. The next lemma, which requires a routine proof, characterizes $AB$ and $BD$ in terms of the elements of $S^{WAP}$ and $S^{LMC}$, respectively.

**Lemma 3.1.** (i) A function $f \in WAP$ is in $AB$ if and only if $\mu \nu(f) = \nu \mu(f)$ and $T_{\mu \nu} f = T_{\nu \mu} f$ for all $\mu, \nu \in S^{WAP}$.

(ii) A function $f \in LMC$ is in $BD$ if and only if $\mu^2(f) = \mu(f), \mu^2 \nu(f) = \nu(f), T_{\mu \nu} f = T_{\mu} f, \text{and } T_{\mu^2} \nu \cdot f = T_{\nu} \mu \cdot f$ for all $\mu, \nu \in S^{LMC}$.

The following theorem states the main properties of $AB, BD,$ and $SL$.

**Theorem 3.2.** $AB, BD,$ and $SL$ are those $m$-admissible subalgebras of $C(S)$, whose corresponding compactifications of $S$ are universal with respect to the properties of being an abelian semigroup, a band, and a semilattice, respectively.

**Proof.** It is enough to prove the conclusion for $AB$ and $BD$. Using Lemma 3.1, the $m$-admissibility of $AB$ and $BD$ can be easily demonstrated, and also it follows that $S^{AB}$ and $S^{BD}$ are abelian and a band, respectively. Let $(\psi, X)$ be an abelian compactification of $S$, then $C(X) = AB(X)$ and so $\psi^*(C(X)) = \psi^*(AB(X)) \subseteq AB(S)$, where the latter inclusion can be easily verified. Thus, $(\epsilon, S^{AB})$ is the universal abelian compactification of $S$. Similarly, to see that $(\epsilon, S^{BD})$ is universal with respect to the property of being a band, it is enough to show that for any other band compactification $(\varphi, Z)$ of $S$, $\varphi^*(C(Z)) \subseteq BD(S)$. For this, let $\pi : (\epsilon, S^{LMC}) \rightarrow (\varphi, Z)$ be the canonical homomorphism whose existence is guaranteed by the universal property of $(\epsilon, S^{LMC})$. If $g \in C(Z)$, then $\varphi^*(g) \in LMC(S)$ and for all $\mu \in S^{LMC}$, $\mu^2(\varphi^*(g)) = g(\pi(\mu^2)) = g(\pi(\mu)) = \mu(\varphi^*(g))$. A similar argument shows that, for each $\nu \in S^{LMC}$, $\mu^2 \nu(\varphi^*(g)) = \mu \nu(\varphi^*(g))$, $T_{\mu^2} \nu(\varphi^*(g)) = T_{\mu} \varphi^*(g)$, and $T_{\mu^2} \nu \cdot (\varphi^*(g)) = T_{\mu \nu} \varphi^*(g)$. Now, Lemma 3.1 shows that $\varphi^*(g) \in BD(S)$, as required. \qed

It is trivial that $BD \subseteq BD_c$ (with the equality holding in the compact case), where

$$BD_c = \{ f \in C(S) : f(s^2) = f(s), f(st) = f(st^2), \text{and } f(st^2 u) = f(st u) \text{ for all } s, t, u \in S \}. \quad (3.3)$$

The joint continuity of the multiplication of $S^{AP}$ implies that $BD \cap AP = BD_c \cap AP$. Furthermore, $S^{SL}$ is a compact semitopological semilattice, so by Lawson's (joint continuity) theorem [8], $SL \subseteq AP$. Thus, $SL = AP \cap BD_c \cap AB$; more precisely:

**Proposition 3.3.** $SL = \{ f \in AP : f(s^2) = f(s), f(s^2 t) = f(st) = f(ts), \text{and } f(st^2 u) = f(st u) = f(st u) \text{ for all } s, t, u \in S \}$.

The universal properties of $(\epsilon, S^{BD})$ and $(\epsilon, S^{D})$ imply that $(\epsilon, S^{BD \cap D})$ is universal with respect to the property of being a rectangular band [1, Exercise 1.1.48]. Furthermore, since every such rectangular band is a topological semigroup, $BD \cap D \subseteq AP$ which implies that $BD \cap D = BD_c \cap D \cap AP$. On the other hand, an adaptation of Junghenn's ideas in the proof of Proposition 3.10 of [6] implies that $BD \cap D = (LZ \cup RZ) \cap (LZ \cup RZ)$, where $(LZ \cup RZ)$ is the $C^*$-subalgebra of $C(S)$ generated by $LZ \cup RZ$ and $LZ \cup RZ$ is the topological tensor product of $LZ$ and $RZ$; i.e., the completion in the least cross norm of the algebraic tensor product.
As a consequence of the universal properties of \((\varepsilon, S^{GP})\) and \((\varepsilon, S^{AB\cap GP})\), it is trivial that \((\varepsilon, S^{AB\cap GP})\) is the universal abelian group compactification of \(S\). Some other facts about \(AB \cap GP\) are collected in the next result. Also, see [3].

**Proposition 3.4.** (i) \(AB \cap MD = AB \cap GP = AB \cap SD = \{ f \in SAP : f(stu) = f(sut) \}, \) for all \(s,t,u \in S\).

(ii) \(AB \cap GP\) is the closed linear span of the set of all continuous characters of \(S\).

**Proof.** The facts that \(S^{AB \cap MD}\) and \(S^{AB \cap SD}\) are abelian groups and that \((\varepsilon, S^{AB \cap GP})\) is universal with respect to the property of being an abelian group imply that \(AB \cap MD = AB \cap GP = AB \cap SD \subseteq SAP\), where the latter containment is obtained from the Ellis’ (joint continuity) theorem [4]. Furthermore, the other condition in the definition of \(AB\), i.e., \(f(st) = f(ts)\) is automatically deduced from \(f(stu) = f(sut)\) and the fact that \(f \in SAP\). The observation that the dual mapping of \(\varepsilon\) from \(C(S^{AB \cap GP})\) onto \(AB \cap GP\) establishes a one-to-one correspondence between the continuous characters of \(S^{AB \cap GP}\) and those of \(S\) and using the Peter-Weyl theorem, [5, Thm. 22.17], for \(C(S^{AB \cap GP})\) imply that \(AB \cap GP\) is the closed linear span of the continuous characters of \(S\).

**Examples and Remarks 3.5.**

(i) For all right zero and left zero semigroups, it is simple to verify that \(AB = C\) (i.e., consists of the constant functions only) and that \(BD = C(S)\). Also, for all groups \(BD = C\).

(ii) Consider the discrete semigroup \(S = \{a, b, c, d\}\), with multiplication given by: \(a\) as a left identity, \(b\) and \(c\) be as left zeros, and \(ds = c\) for all \(s \in S\) (see [1, 1.1.7]). A direct computation shows that \(AB = \{ f \in C(S) : f(b) = f(c) = f(d)\}\) and \(BD = \{ f \in C(S) : f(c) = f(d)\}\).

(iii) Let \(S_3 = \langle a, b \mid a^3 = b^2 = (ab)^2 = 1 \rangle\) be the symmetric group of order 6. One may directly show that \(AB(S_3) = \{ f \in C(S_3) : f(1) = f(a) = f(a^2)\}\), and \(f(b) = f(ab) = f(a^2b)\). Of course, \(BD(S_3) = C\).

(iv) An inductive proof shows that a function \(f \in WAP\) lies in \(AB\) if and only if \(f(\text{each finite product of elements of } S) = f(\text{each re-ordering of it})\).

(v) Similar to what we have preceding to Proposition 3.3, using the Lawson’s theorem, [8], one may show that for abelian semigroups \(BD \cap WAP = SL = BD \cap AP\). Thus, for semilattices, \(SL = AP\).

(vi) The equality \(BD \cap MD = LZ\) can be easily demonstrated from the fact that all left simple bands are left zero semigroups. Similarly, \(BD \cap SD = RZ\). Also, we trivially have \(BD \cap GP = BD \cap SAP = C\).

(vii) The invariant mean on the abelian semigroup \(S^{AB}\) induces a unique invariant mean on \(AB\), where the uniqueness is obtained from the fact that the \(m\)-admissible subalgebras of \(WAP\) cannot have more than one invariant mean (see [1, Cor. 2.3.28, Exercice 4.2.7]). A similar statement holds for \(SL\) and \(AB \cap GP\). But \(BD\), in general, is not even left amenable. For example, for \(S = \{a, b, c, d\}\) as in part (ii), let \(f\) in \(BD\) be such that \(f(b) \neq f(c)\), then for each left invariant mean \(m\) on \(BD\), \(f(b) = m(Lbf) = m(Lcf) = f(c)\) and this contradicts the choice of \(f\).
(viii) It should be mentioned that $AB, SL, BD \cap D$, and $AB \cap GP$ are also admissible, i.e., they are invariant under $T_\mu$ for all $\mu$ in their duals [1, Cor. 4.2.7]. But we guess that $BD$ is not admissible in general. It would be desirable to investigate the inclusion $BD \subseteq WAP$.

(ix) Parallel to $BD$ and also $SL$ which are defined by right translates, we have the analogous spaces defined by left translates. It is a matter of fact that the left and right notations do not change the structure of $SL$ (see Proposition 3.3). A natural question that arises is whether they do not change $BD$. In our opinion, there is a close tie between the latter question and the inclusion $BD \subseteq WAP$. See (viii).

(x) It is obvious that the $SL$-compactification of the direct product of two semitopological semigroups is isomorphic (in the sense of [1, Sec. 5.2]) to the direct product of their $SL$-compactifcations. A similar fact holds for the $(AB \cap GP)$-compactification; (more generally, $(AB \cap GP)$-compactification, roughly speaking, passes through semidirect products. See [1, Lem. 5.2.3]).

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**References**


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